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An Interval Impulsive Observer for Multi-Sensors Linear Systems with Delayed Measurements

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Abstract— **This work presents a novel approach to design an interval impulsive observer for a specific class of multi-sensor Linear Time-Invariant system with delayed output measurements with time-varying delays. The observer is located on a remote server and receives sensors data sporadically. The discrete-time delayed measurements are used to design the interval impulsive observer with an open-loop prediction output that helps the observer to catch up the measurement delay. We provide an observer design approach that leverages $\mathcal{L}_1$-gain input/output stability for the delayed observation error dynamics. These stability criteria are a set of algebraic inequalities that are solved via interval analysis. Additionally, we optimize sensor selection by using the predicted reduction of observation error width. An illustrative example is provided to support the theoretical framework, showcasing the practical implications of our approach.**

I. Introduction

The study of impulsive systems has a rich history dating back to the early days of modern control theory. As a significant type of hybrid system, impulsive systems exhibit both continuous and discontinuous dynamical behaviors. Due to this hybrid nature, they are particularly suited to modeling a wide range of real-world evolutionary processes where states change abruptly at specific moments. For further details, refer to [1]–[3] and the references provided therein. In many real-world systems, various factors such as economic constraints or technical limitations on measurement methods can result in not all state variables being accessible for analysis or state feedback control. To address this issue, observers theory has found increasing interest and applications across a wide range of fields, including attack detection [4], event-triggered impulsive control [5], and more. However, in many engineering scenarios, measurements are often collected at discrete-time instant, periodically or sporadically. This has led to the development of the impulsive observer, a concept first introduced by Raff and Allgower [6]. The estimation update is done in an impulsive manner using the measured outputs at discrete time. In real life, systems have to deal with constraints such as disturbances and delays. Time delays are a common occurrence in networked systems. They can have a notable impact on system performance, potentially leading to instability, oscillation, and other undesirable effects.

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Recently, there has been a great deal of interest among researchers in the stability analysis of time-delay systems (TDS), largely due to the challenges posed by communication channels [7], [8], switched systems [9], etc. In particular, Krasovskii and Razumikhin methods are widely used as research tools for analyzing the stability of TDS like in [10], [11]. In the first work, an observer has been designed for a system that is partially delayed in dynamics, but not in measurement. The class of considered systems has no disturbance nor measurement noise, but such conditions are difficult to achieve in practice. In the reference [11], the delay was considered on the measurement, but disturbances were neglected. In fact, since the delay already introduces uncertainty into the dynamics, it sometimes lead the authors to make strong restrictions for the study like neglecting disturbances. Interval estimators, which provide guaranteed interval estimations of the system state vector, offer an appropriate solution to this challenge. More recently, the $\mathcal{L}_p$ stability concept for hybrid systems is used to study the stability of TDS [12], where the $\mathcal{L}_p$ stability of a delayed neural network is investigated. The concept of $\mathcal{L}_p$ stability assesses how external disturbances affect the output, and it has been thoroughly explored in the literature over the years [13].

This paper deals with the design of an interval impulsive observer for a multi-sensor Linear Time-Invariant (LTI) system with sporadic, discrete-time delayed measurements with time-varying delays. The novelty is twofold: A novel interval observer design with delayed measurements, and a novel approach to select the most appropriate sensor to use. Before each measurement time-instant, our approach selects the sensor that will reduce most the width of the estimation uncertainty interval. This selection step optimizes the use of available resources and ensures optimum monitoring performance in complex dynamic environments. In network control or industrial systems, sensor selection plays a crucial role for optimum performance [14]. However, with the increasing complexity of systems and the need to minimize operational costs, it has become imperative to design intelligent and efficient sensor selection algorithms. As far as we know, there is no work in the literature that combines set-valued estimation and smart sensor selection.

II. Preliminaries

Definition 1: ($\mathcal{L}_p$ Stability.) Let us consider the hybrid system (1) consisting of two main parts: a differential equation that governs the continuous dynamics when the system flows, and a difference equation that governs the discrete

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dynamics when the system jumps:

$$\begin{cases} \dot{x} = f(x, d), & (x, d) \in \mathcal{C}, \\ x^+ = g(x, d), & (x, d) \in \mathcal{D}, \\ y = \varphi(x, d). \end{cases} \tag{1}$$

$\mathcal{C}$ and $\mathcal{D}$ are respectively flow and jump sets, x is the state vector, d the input vector, and y the output vector.

Remark 1: In the rest of this work, we use the hybrid time domains notation: $x \equiv x(t, j)$ for continuous time evolution, and $x^+ \equiv x(t_j, j)$ for discrete actualisation. For hybrid systems, the solutions are characterised by the continuous time t, and the jump index j [15].

The $\mathcal{L}_p$-stability of system (1) can be analyzed using a storage function [13]. To this end, let us consider a positive semi-definite continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that :

$$0 \leq V(x) \leq c_2\|x\|^p, \forall (x, d) \in \mathcal{C}_x \cup \mathcal{D}_x, \tag{2}$$

$$\langle \nabla V(x), f(x, d) \rangle \leq -\gamma_{yf}\|\varphi(x, d)\|^p + \gamma_{df}\|d\|^p, \quad \forall (x, d) \in \mathcal{C}_x \tag{3}$$

$$V(g(x, d)) - V(x) \leq -\gamma_{yg}\|\varphi(x, d)\|^p + \gamma_{dg}\|d\|^p, \quad \forall (x, d) \in \mathcal{D}_x \tag{4}$$

where c_2, γ_{yf} and γ_{yg} are strictly positive constants and γ_{df} and γ_{dg} are non-negative ones. If conditions (2)-(4) are satisfied, $V(x)$ is a finite-gain $\mathcal{L}_p$ storage function for system (1). If such a function exists, system (1) is $\mathcal{L}_p$-stable and the $\mathcal{L}_p$-gain is upper bounded by $\gamma_p = \sqrt[p]{\gamma_d/\gamma_y}$ where $\gamma_d = \max\{\gamma_{df}, \gamma_{dg}\}$ and $\gamma_y = \max\{\gamma_{yf}, \gamma_{yg}\}$.

For matrices and vectors, inequalities should be understood element-wise. A matrix-vector product can be framed as follows: $A_+\underline{x} - A_-\overline{x} \leq Ax \leq A_+\overline{x} - A_-\underline{x}$, where A_+ and A_- are matrices are given by $A_+ = \max(A, 0)$, $A_- = A_+ - A$, where the $\max$ operator is applied element-wise. Then $|A| = A_+ + A_-$.

Lemma 1 ([16]): Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices such that $A \in [\underline{A}, \overline{A}] \geq 0$ is positive and bounded and B is an arbitrary bounded matrix $B \in [\underline{B}_+ - \underline{B}_-, \overline{B}_+ - \overline{B}_-]$. The matrix product AB can be bounded as:

$$\underline{AB}_+ - \overline{A}\underline{B}_- \leq AB \leq \overline{AB}_+ - \underline{A}\overline{B}_- \tag{5}$$

III. Problem Setup

We consider a multi-sensor cyber-physical system that can be modelled as a LTI system (see (6)). Digital operations, including state reconstruction, are performed on a remote server. The sensor-observer channel induces a variable transmission delay in the discrete-time measurements which are transmitted synchronously (same transmission instant and delay for all sensors) but sporadically (aperiodic sampling) as can be seen in Figure 1.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y_i(t_k - h(t_k)) = C_i x(t_k - h(t_k)) + F_i d(t_k - h(t_k)), \end{cases} \tag{6}$$
$$i \in \{1, \ldots, n\} \subset \mathbb{N}, h(t) \in \mathcal{H}$$

where $i \in \mathbb{N}$ is the sensor index, $t_k, k \in \mathbb{N}$ are sampling instants, $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, y_i \in \mathbb{R}^{n_y}$ are respectively the

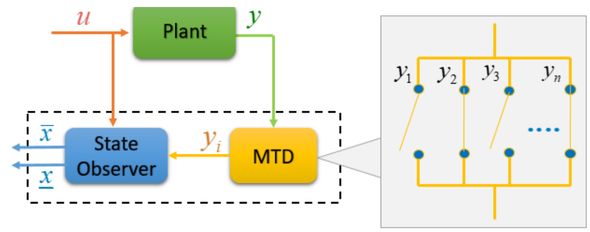

Fig. 1. The interval impulsive observer for a multi-sensor CPS with discrete, sporadic and delayed measurements.

state, the input and the output vectors with $t \in \mathbb{R}, k \in \mathbb{N}$, $t \in [t_k, t_{k+1})$; A, B, C_i, E, and F_i are known matrices. $h(t) \in \mathcal{H}$ is the bounded time-varying delay that satisfies: $0 \leq h_1 \leq h(t) \leq h_2$. The delay function $h(t)$ can be non-continuous with a non-zero minimum value.

Vector $d(t)$ gathers measurement noise and system disturbances with corresponding distribution matrices E and F_i.

Assumption 1: We assume that $\forall t$, $\underline{d} \leq d(t) \leq \overline{d}$, where $\underline{d}$ and $\overline{d}$ are known vectors.

Assumption 2: The inter-sampling times are bounded. Let τ_{min} and τ_{max} be two given real scalars ($\tau_{min} \leq \tau_{max}$), $\forall k \in \mathbb{N}$, $t_{k+1} - t_k \in [\tau_{min}, \tau_{max}]$.

In this paper, we develop a strategy based on the observation error width reduction to choose at each sampling instant the best sensor to use to perform state reconstruction while considering the delay. An interval impulsive observer for system with time-varying delayed outputs is constructed.

In the remainder of this work, we denote by $\overline{x}(t)$ and $\underline{x}(t)$ the upper and the lower estimate of the actual state vector $x(t)$ given by our interval impulsive observer such that

$$\forall t, \ \underline{x}(t) \leq x(t) \leq \overline{x}(t). \tag{7}$$

IV. The Interval Impulsive Observer for the Delayed System

The objective of this section is to derive the guaranteed enclosure (7) of the actual state of system (6).

A. The Observer Structure

The proposed interval observer is an impulsive system. Before measurements become available, the optimal sensor is selected using the method outlined in Section IV-B, enhancing the accuracy of the state enclosure estimate (referred to as the **discrete update**). When the observer operates without measurements, the delay caused by the measurement is accounted for in the **continuous open-loop evolution**. The **output equation** of the impulsive observer acts as a predictor, compensating for the measurement delay. Since the measurement is delayed, the continuous evolution of the observer following the measurement is also delayed. Between two measurement instants $t \in [t_k, t_{k+1})$ the delay remains constant. For simplicity, we denote the delay at t_k as $h(t_k) = h$. By substituting inequalities (7) into (6), the **continuous evolution** of the observer bounds for each subsystem in the absence of measurements can be written as:

$$\begin{aligned}\dot{\underline{x}}(t-h) &= A_M\underline{x}(t-h) - A_N\overline{x}(t-h) \quad (8)\\ &\quad +Bu(t-h) + E_+\underline{d} - E_-\overline{d},\\ \dot{\overline{x}}(t-h) &= A_M\overline{x}(t-h) - A_N\underline{x}(t-h) \quad (9)\\ &\quad +Bu(t-h) + E_+\overline{d} - E_-\underline{d},\end{aligned}$$

with $A_M = D_A + (A - D_A)_+$ and $A_N = A_M - A$, where D_A is the diagonal of A. A_M and A_N are designed following Muller's existence theorem, also known as Internal Positive Realization [16], to ensure the Metzler property for

$$\overline{A} = \begin{bmatrix} A_M & A_N \\ A_N & A_M \end{bmatrix}. \quad (10)$$

Let's note the change of variable:

$$\tilde{t} = t - h, \quad \tilde{t}_k = t_k - h. \quad (11)$$

The equations (8)-(9) can be rewritten as follows:

$$\begin{cases}\dot{\underline{x}}(\tilde{t}) = A_M\underline{x}(\tilde{t}) - A_N\overline{x}(\tilde{t}) + Bu(\tilde{t}) + E_+\underline{d} - E_-\overline{d},\\ \dot{\overline{x}}(\tilde{t}) = A_M\overline{x}(\tilde{t}) - A_N\underline{x}(\tilde{t}) + Bu(\tilde{t}) + E_+\overline{d} - E_-\underline{d}.\end{cases} \quad (12)$$

We assume that the initial state interval satisfies:

$$\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0). \quad (13)$$

The **discrete update** is activated at measurement instants. Let the matrices $L_i \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, denote the observer gains associated with each sensors set $i \in \{1, \ldots n\}$:

$$\begin{aligned}x^+(\tilde{t}_k) &= x(\tilde{t}_k) + L_i\left(C_i x(\tilde{t}_k) + F_i d(\tilde{t}_k) - y_i(\tilde{t}_k)\right),\\ &= (I_n + L_iC_i)\, x(\tilde{t}_k) + L_iF_i d(\tilde{t}_k) - L_i y_i(\tilde{t}_k). \quad (14)\end{aligned}$$

At the correction instants, it's worth noting that $x^+ = x$ because $C_i x + F_i d - y_i = 0$. The evolution of the correction step is contingent on the selected sensors set i and can be expressed bounding (14) as: the discrete part of the observer is:

$$\begin{aligned}\underline{x}^+(\tilde{t}_k) &= (I_n + L_iC_i)_+\underline{x}(\tilde{t}_k) - (I_n + L_iC_i)_-\overline{x}(\tilde{t}_k)\\ &\quad +(L_iF_i)_+\underline{d} - (L_iF_i)_-\overline{d} - L_i y_i(\tilde{t}_k), \quad (15)\\ \overline{x}^+(\tilde{t}_k) &= (I_n + L_iC_i)_+\overline{x}(\tilde{t}_k) - (I_n + L_iC_i)_-\underline{x}(\tilde{t}_k)\\ &\quad +(L_iF_i)_+\overline{d} - (L_iF_i)_-\underline{d} - L_i y_i(\tilde{t}_k). \quad (16)\end{aligned}$$

The **output** of the impulsive observer, as defined in (17)-(18) and derived from (12), represents an interval predictor of the current state vector, based on its estimated delayed upper and lower bounds.

$$\begin{aligned}\bar{x}(t) &= \bar{x}\left(\tilde{t}\right) + \textstyle\int_{\tilde{t}}^{\tilde{t}+h}\left(A_M\bar{x}(l) - A_N\underline{x}(l) + Bu(l)\right.\\ &\quad \left.+E_+\bar{d} - E_-\underline{d}\right)\mathrm{d}l \quad (17)\\ \underline{x}(t) &= \underline{x}\left(\tilde{t}\right) + \textstyle\int_{\tilde{t}}^{\tilde{t}+h}\left(A_M\underline{x}(l) - A_N\bar{x}(l) + Bu(l)\right.\\ &\quad \left.+E_+\underline{d} - E_-\bar{d}\right)\mathrm{d}l. \quad (18)\end{aligned}$$

The equations (12), (15), (16), (17) and (18), are respectively the continuous dynamics, the discrete actualisation and the output prediction of the interval impulsive observer that we propose for the time-delay system (6).

B. Sensor selection

One of the objectives of this work is to provide the most accurate estimation of the actual state vector of system (6) despite the presence of time-varying delays in the available discrete-time measurements. To achieve this, we propose a smart sensor selection algorithm that identifies the most appropriate sensor as the one that maximally reduces the size of the uncertainty interval for the reconstructed state vector. Since the measurement data acts in the past, we propose to use the delayed state observer (15)-(16) to build the selection criterion. Let us define the width of the uncertainty interval as follows:

$$\delta(\tilde{t}) = \overline{x}(\tilde{t}) - \underline{x}(\tilde{t}). \quad (19)$$

From equations (15) and (16) one can compute the width of the observation error after a discrete update using sensor of index i:

$$\delta_i^+(\tilde{t}_k) = |I + L_iC_i|\,\delta(\tilde{t}_k) + |L_iF_i|\left(\bar{d} - \underline{d}\right). \quad (20)$$

We can now define a performance criterion $\kappa(i, \tilde{t}_k)$ for each sensor, from the width of the uncertainty interval after an update with any sensor of index $i \in \{1, ..., n\}$ at time $\tilde{t}_k$, as follows:

$$(i, \tilde{t}_k) \mapsto \kappa(i, \tilde{t}_k) = \|\delta_i^+(\tilde{t}_k)\|_1 \equiv 1_n^\top \delta_i^+(\tilde{t}_k) \quad (21)$$

It important to note that one can compute the size (20) before actually performing the measurement. Although the next update used delayed measurements, the sensor selection occurs in the present. The index i^* of the best sensor to use for next update is then given by $i^* = \arg\min_i \kappa(i, \tilde{t}_k)$. Finally, the impulsive observer uses sensor of index i^* for discrete update.

C. The Observation Error Structure

The upper and lower error bounds of the observation error are defined by:

$$\begin{cases}\underline{e}\left(\tilde{t}, j\right) = x\left(\tilde{t}, j\right) - \underline{x}\left(\tilde{t}, j\right)\\ \overline{e}\left(\tilde{t}, j\right) = \overline{x}\left(\tilde{t}, j\right) - x\left(\tilde{t}, j\right).\end{cases} \quad (22)$$

The extended error vector is: $\xi = \left[\underline{e}^\top, \ \overline{e}^\top\right]^\top$. The state variable of the hybrid error dynamics is defined as:

$$z = \left[\xi^\top, \tau, h\right]^\top. \quad (23)$$

Equations, (12), (15)-(16) and (22)-(23) allow us to define the hybrid system that models the error dynamics of the switched-impulsive observer:

$$\begin{cases}\dot{z} = f(z, \psi), & z \in \mathcal{C}_\xi\\ z^+ = g_i(z, \psi), & z \in \mathcal{D}_\xi\\ y = \varphi(z, \psi),\end{cases} \quad (24)$$

where the output $y(t, j)$ is given by function $\varphi(z, \psi)$ introduced in (32), ψ defined in (29) and where

$$\begin{aligned}f(z, \psi) &= \left[\left(\overline{A}\xi + \tilde{E}\psi\right)^\top, \ -1, \ 0\right]^\top \quad (25)\\ g_i(z, \psi) &= \left[\left(\Gamma_i(L_i)\,\xi + \tilde{F}_i(L_i)\,\psi\right)^\top, \ \mu, \ h\right]^\top \quad (26)\end{aligned}$$

$i \in \{1, \dots n\} \subset \mathbb{N}, \quad \mu \in [\tau_{\min}, \tau_{\max}], \quad h \in [h_1, h_2].$

The different matrices in (25) and (26) are given by:

$$\overline{A} = \begin{bmatrix} A_M & A_N \\ A_N & A_M \end{bmatrix}, \tilde{E} = \begin{bmatrix} E_+ & E_- \\ E_- & E_+ \end{bmatrix}, \tag{27}$$

$$\tilde{F}_i(L_i) = \begin{bmatrix} (L_iF_i)_+ & (L_iF_i)_- \\ (L_iF_i)_- & (L_iF_i)_+ \end{bmatrix},$$

$$\Gamma_i(L_i) = \begin{bmatrix} (I_n + L_iC_i)_+ & (I_n + L_iC_i)_- \\ (I_n + L_iC_i)_- & (I_n + L_iC_i)_+ \end{bmatrix}, \tag{28}$$

$$\psi(t,j) = \left[\left(d - \underline{d}\right)^\top,\ \left(\overline{d} - d\right)^\top\right]^\top. \tag{29}$$

The scalars $\mu \in \left[\tau_{\min}, \tau_{\max}\right]$ and $h \in \left[h_1, h_2\right]$ represent respectively the value of the timer τ after the jump and the delay; and $\mathcal{C}_\xi$ and $\mathcal{D}_\xi$ are defined as the flow and jump sets, respectively:

$$\mathcal{C}_\xi = \left\{z = (\xi, \tau, h) \in \mathbb{R}^{2n_x} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : \tau \in [0, \tau_{\max}]\right\},$$
$$\mathcal{D}_\xi = \left\{z = (\xi, \tau, h) \in \mathbb{R}^{2n_x} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : \tau = 0\right\}.$$

Now let's build the function $\varphi(z, \psi)$, which is an open-loop propagation of $\xi(\tilde{t}, j)$ over the period h.

Assumption 3: The Metzler matrix $\bar{A}$ is non-singular.

Let us denote $\Omega(h) = \overline{A}^{-1}(\mathrm{e}^{\overline{A}h} - I)\tilde{E}$, and

$$\mathcal{I} = \begin{bmatrix} I_{n_d \times n_d} & I_{n_d \times n_d} \\ I_{n_d \times n_d} & I_{n_d \times n_d} \end{bmatrix}. \tag{30}$$

The worst-case disturbance scenario is

$$\bar{\psi} = \left[\overline{d}^\top - \underline{d}^\top,\ \overline{d}^\top - \underline{d}^\top\right]^\top = \mathcal{I}\psi; \tag{31}$$

then, an over-approximation of the actual dynamics of $\xi(t,j)$ is given by:

$$\varphi(z, \psi) = \mathrm{e}^{\bar{A}h}\xi(\tilde{t}, j) + \Omega(h)\bar{\psi}. \tag{32}$$

V. Stability analysis of the observer error dynamics

This section deals with the stability analysis of the observation error hybrid dynamics described in (24). The aim is to establish stability conditions of the delayed dynamics with respect to the predicted output, using the theory of $\mathcal{L}_1$ stability for hybrid systems introduced in Definition 1. The observation error dynamics (24) represents an impulsive system with both continuous and discrete modes, and an output. In this work, the stability analysis is conducted using a distinct gain matrix L_i for each sensor, while employing a single Lyapunov function. As previously mentioned, the value of the timer $\tau \in (0, \tau_{\max}]$ at the jump is denoted by $\mu \in [\tau_{\min}, \tau_{\max}]$. Let $L_i \in \mathbb{R}^{n_x \times n_y}$, $i \in \{1, \dots, n\}$, be the observation gain matrices, and $\lambda \in \mathbb{R}^{2n_x}_{>0}$ a positive vector.

Theorem 1: Consider the observation error dynamics (24). For given matrices $L_i \in \mathbb{R}^{n_x \times n_y}$, $i \in \{1, \dots n\}$, if there exist a vector $\lambda \in \mathbb{R}^{2n_x}_{>0}$, positive scalars λ_0, c_1, $\gamma_{yg}, \gamma_{dg}, \gamma_{df}$, and γ_{yf} satisfying inequalities (33) for all $\tau \in [0, \tau_{\max}]$, $\mu \in [\tau_{\min}, \tau_{\max}]$, and $h \in [h_1, h_2]$:

$$\begin{cases} \lambda^\top \mathrm{e}^{\lambda_0\tau}(\bar{A} - \lambda_0 I) + \gamma_{yf}1^\top_{2n} & \leq 0^\top_{2n}, \\ \lambda^\top \mathrm{e}^{\lambda_0\tau}\left(\mathrm{e}^{\bar{A}h}\tilde{E} - (\bar{A} - \lambda_0 I)\Omega(h)\mathcal{I}\right) - \gamma_{df}1^\top_{2n_d} & \leq 0^\top_{2n_d}, \\ \lambda^\top\left(\mathrm{e}^{\lambda_0\mu}\mathrm{e}^{\bar{A}h}\Gamma_i(L_i)\mathrm{e}^{-\bar{A}h} - I\right) + \gamma_{yg}1^\top_{2n} & \leq 0^\top_{2n}, \\ \lambda^\top \mathrm{e}^{\lambda_0\mu}\mathrm{e}^{\bar{A}h}\left(\tilde{F}_i - \Gamma_i(L_i)\mathrm{e}^{-\bar{A}h}\Omega(h)\mathcal{I}\right) & \\ \quad + \lambda^\top\Omega(h)\mathcal{I} - \gamma_{dg}1^\top_{2n_d} & \leq 0^\top_{2n_d}. \end{cases} \tag{33}$$

Then, the systems (12), (15), (16), (17) and (18) form a finite $\mathcal{L}_1$-gain interval observer for system (6). Furthermore, the impulsive dynamics of the observation error (24) is $\mathcal{L}_1$-gain stable from $\psi(t)$ to $\xi(t,j)$ with $\mathcal{L}_1$ gain $\gamma_1 = \max\{\gamma_{df}, \gamma_{dg}\}/\min\{\gamma_{yf}, \gamma_{yg}\}$.

Proof: The proof of Theorem 1 relative to the stability of the error dynamics (24) is divided into two steps:

Step 1: Non-negativity of the observation error: Let the ordering condition (13) be true. Knowing that $\overline{A}$ is a Metzler matrix, $\tilde{E}$ and ψ are both non-negative, the continuous dynamics of (24) is non-negative for all $\tilde{t}$ in $\left[\left(\tilde{t}_j, j\right), \left(\tilde{t}_{j+1}, j\right)\right]$. Now, the relations (22) and the non-negativity of $\Gamma_i(L)$ ensure the non-negativity of the error dynamics at reset time instants. Consequently, the non-negativity of the observation error is guaranteed by construction: $\xi(0,0) \geq 0 \Rightarrow \forall\left(\tilde{t}, j\right) \in \mathrm{dom}\xi\ \xi\left(\tilde{t}, j\right) \geq 0$.

Step 2 : $\mathcal{L}_1$ stability of the observation error: The stability of the augmented error dynamics (24) is tied to the conditions (2)-(4), used to assess $\mathcal{L}_1$ stability. Now, let us define the set Θ which contains the origin space values of the hybrid system (24):

$$\Theta = \{z = \left[\xi^\top, \tau, h\right]^\top \in \mathbb{R}^{2n_x} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}| \quad \xi = 0,\ \tau \in [0, \tau_{\max}],\ h \in [h_1, h_2]\}. \tag{34}$$

The norm $\|z\|_\Theta$ to the set Θ is defined by:

$$\|z\|_\Theta = \|z\|_1 = \|\xi\|_1, \tag{35}$$

where the notation $\|.\|_1$ denote the $\mathcal{L}_1$-norm. To study the stability of the hybrid system (24), we have to verify conditions (2)-(4) for all i in $\{1, ...n\}$. There are as many impulsive conditions as there are sensors. We first analyze the continuous component of the error dynamics. Let us define $V(z)$ as the continuous Lyapunov function which will be used in the proof:

$$V(z) = \lambda^\top \mathrm{e}^{\lambda_0\tau}\mathrm{e}^{\bar{A}h}\xi(\tilde{t}, j), \tag{36}$$

where λ is a positive vector and λ_0 a positive scalar. The design of the storage function $V(z)$ is inspired by the work of Nešić et al. [13] where a reverse average dwell time is considered. The continuous Lyapunov function $V(z)$ is non-negative because λ and ξ are positive vectors, and matrix $\overline{A}$ is Metzler. Furthermore, it is bounded by a positive scalar c_1 as follows:

$$0 \leq V(z) \leq c_1 \|z\|_\Theta, \forall z \in \mathcal{C}_\xi \cup \mathcal{D}_\xi,$$

where

$$c_1 = \max(\max_{[\tau],[h]}(\lambda^\top \mathrm{e}^{\lambda_0\tau}\mathrm{e}^{\bar{A}h})),$$

with $[\tau]=[0,\tau_{max}]$ and $[h]=[h_1,h_2]$. From the expression of the Lyapunov function (36), we can deduce

$$\nabla V(\xi,\tau,h)=\left[\lambda^\top e^{\lambda_0\tau}e^{\overline{A}h},\lambda^\top\lambda_0 e^{\lambda_0\tau}e^{\overline{A}h}\xi,\lambda^\top e^{\lambda_0\tau}\overline{A}e^{\overline{A}h}\xi\right]^\top \tag{37}$$

Applying (37) on (3), using (25) and (32), and recalling that $\overline{A}$ and $e^{\overline{A}h}$ commute, we have:

$$\begin{aligned}\langle\nabla V(z),f(z,\psi)\rangle &= \lambda^\top e^{\lambda_0\tau}(\overline{A}-\lambda_0 I)\varphi\\ &+\lambda^\top e^{\lambda_0\tau}\left(e^{\overline{A}h}\tilde{E}-(\overline{A}-\lambda_0 I)\Omega(h)\mathcal{I}\right)\psi.\end{aligned} \tag{38}$$

Considering (35), knowing that $\|\varphi\|_1=1_{2n}^\top\varphi$, $\|\psi\|_1=1_{2n_d}^\top\psi$ and by designing an upper bound of $\mathcal{L}_1$-gain of the operator $\psi\to\varphi$, we have the relation:

$$\begin{aligned}\lambda^\top e^{\lambda_0\tau}(\overline{A}-\lambda_0 I)\varphi+\lambda^\top e^{\lambda_0\tau}\left(e^{\overline{A}h}\tilde{E}-(\overline{A}-\lambda_0 I)\Omega(h)\mathcal{I}\right)\psi\\ \le-\gamma_{yf}1_{2n}^\top\varphi+\gamma_{df}1_{2n_d}^\top\psi.\end{aligned}$$

This inequality can be rewritten as follows:

$$\begin{bmatrix}\lambda^\top e^{\lambda_0\tau}(\overline{A}-\lambda_0 I)+\gamma_{yf}1_{2n}^\top\\ \lambda^\top e^{\lambda_0\tau}\left(e^{\overline{A}h}\tilde{E}-(\overline{A}-\lambda_0 I)\Omega(h)\mathcal{I}\right)\\ -\gamma_{df}1_{2n_d}^\top\end{bmatrix}^\top\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\le 0. \tag{39}$$

By applying (4) to the discrete part of the impulsive system, we have:

$$\begin{aligned}V(g_i(z,\psi))-V(z)&=\lambda^\top\left(e^{\lambda_0\mu}e^{\overline{A}h}\Gamma_i(L_i)e^{-\overline{A}h}-I\right)\varphi\\ +\lambda^\top&\left(e^{\lambda_0\mu}e^{\overline{A}h}\left(\tilde{F}_i\psi-\Gamma_i(L_i)e^{-\overline{A}h}\Omega(h)\overline{\psi}\right)+\Omega(h)\overline{\psi}\right)\end{aligned}$$

This is satisfied if

$$\begin{bmatrix}\lambda^\top\left(e^{-\lambda_0\mu}e^{\overline{A}h}\Gamma_i(L_i)e^{-\overline{A}h}-I\right)+\gamma_{yg}1_{2n}^\top\\ \lambda^\top e^{\lambda_0\mu}e^{\overline{A}h}\left(\tilde{F}_i-\Gamma_i(L_i)e^{-\overline{A}h}\Omega(h)\mathcal{I}\right)\\ +\lambda^\top\Omega(h)\mathcal{I}-\gamma_{dg}1_{2n_d}^\top\end{bmatrix}^\top\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\le 0 \tag{40}$$

Eq. (39)-(40) must be satisfied for every φ and ψ. Knowing that $\xi\ge 0$ and $\psi\ge 0$, the inequalities (39) and (40) reduce to (33). Considering the assertions made in Proposition 1 of the work of Nešić et al. in 2013 [13], Eq. (39) and (40) are sufficient conditions for $\mathcal{L}_1$ stability of (24). This concludes the proof. ∎

VI. SYNTHESIS METHOD

In this subsection, our attention is directed towards the design of the observer gains $L_i, i\in\{1,\dots,n\}$ that meet the criteria outlined in Theorem 1. Considering the stability conditions of the error dynamics as shown in (33), we aim to simplify the synthesis problem by introducing a change of variables using non-negative realizations of gain-dependant matrices. Let us denote $G_i=I+L_iC_i$, and define G_{i+} and $-G_{i-}$ respectively the positive and negative parts of G_i. One has $|G_i|=G_{i+}+G_{i-}$. Then, note that for any couple of matrices $G_{1,i}$, $G_{2,i}\in\mathbb{R}_{\ge 0}^{n\times n}$, that satisfy $G_i=G_{1,i}-G_{2,i}$, there exists a corresponding matrix $\Delta\in\mathbb{R}_{\ge 0}^{n\times n}$ such that $G_i=(G_{i+}+\Delta)-(G_{i-}+\Delta)$. Therefore, $\Gamma_i(L_i)$ can be rewritten as:

$$\Gamma_i(L_i)=\Gamma(G_{1,i},G_{2,i})=\begin{bmatrix}G_{1,i}&G_{2,i}\\ G_{2,i}&G_{1,i}\end{bmatrix}. \tag{41}$$

The same principle is applied to the matrix $\tilde{F}_i(L_i)$, with $R_{1,i}-R_{2,i}=L_iF_i$. The matrices $G_{1,i}$, $G_{2,i}$, $R_{1,i}$, and $R_{2,i}$ are intermediate matrices that allow to compute the observer gains L_i. Given that τ, μ, and h belong to known intervals, we propose a resolution approach using interval analysis [17]. In fact, $\forall a\in[a_1,a_2], a_2\le 0\Rightarrow a\le 0$. For a function f, the notation "$\sup(f([x]))$" (resp. "$\inf(f([x]))$") denotes the component-wise upper bound (resp. the lower bound) of $f([x])$, the image of the box $[x]$ by mapping f.

Let us define the matrices

$$\begin{aligned}\overline{\Omega}&=\sup\left(\overline{A}^{-1}(e^{\overline{A}[h]}-I)\tilde{E}\right),\\ \overline{\Xi}_h=\sup\left(e^{\overline{A}[h]}\right)&,\underline{\Xi}_h=\inf\left(e^{\overline{A}[h]}\right),\\ \overline{\Xi}_{-h}=\sup\left(e^{-\overline{A}[h]}\right)&,\underline{\Xi}_{-h}=\inf\left(e^{-\overline{A}[h]}\right).\end{aligned} \tag{42}$$

Lemma 2: Per the comparison theorems for monotone dynamical systems, if A is a Metzler matrix, then e^{At} is a non-negative matrix for all $t\ge 0$.

Using Lemma 2, interval analysis and the positivity of the matrices $\tilde{E}$, $\Gamma(L_i)$ and the vector λ we can rewrite the stability constraints. For example, for the case of the fourth inequality of the stability conditions (33), we have:

$$\begin{aligned}\forall h\in[h],\forall\mu\in[\mu],\ &-\lambda^\top e^{\lambda_0\mu}e^{\overline{A}h}\Gamma e^{-\overline{A}h}\Omega(h)\mathcal{I}\\ &+\lambda^\top\Omega(h)\mathcal{I}+\lambda^\top e^{\lambda_0\mu}e^{\overline{A}h}\tilde{F}_i\le\gamma_{dg}1_{2n_d}^\top\end{aligned} \tag{43}$$

To satisfy (43), it suffices to have

$$\begin{aligned}&\sup\left(-\lambda^\top e^{\lambda_0[\mu]}e^{\overline{A}[h]}\Gamma_i e^{-\overline{A}[h]}\Omega([h])\mathcal{I}\right)\\ &+\sup\left(\lambda^\top\Omega([h])\mathcal{I}\right)+\sup\left(\lambda^\top e^{\lambda_0[\mu]}e^{\overline{A}[h]}\tilde{F}_i\right)\le\gamma_{dg}1_{2n}^T\\ &\iff-\lambda^\top\inf\left(e^{\lambda_0[\mu]}e^{\overline{A}[h]}\Gamma_i e^{-\overline{A}[h]}\Omega([h])\right)\mathcal{I}\\ &\qquad+\lambda^\top\overline{\Omega}\mathcal{I}+\lambda^\top e^{\lambda_0\tau_{\max}}\overline{\Xi}_h\tilde{F}_i\le\gamma_{dg}1_{2n_d}^\top\end{aligned}$$

To compute the term $\Lambda_2=\inf\left(e^{\lambda_0[\mu]}e^{\overline{A}[h]}\Gamma_i e^{-\overline{A}[h]}\Omega\right)$, we use the Lemma 1 knowing that $e^{\lambda_0[\mu]}e^{\overline{A}[h]}\Gamma_i$ is a positive matrix. Then,

$$\begin{aligned}\Lambda_{2,i}&=e^{\lambda_0\tau_{\min}}\inf\left(e^{\overline{A}[h]}\right)\Gamma_i\left(\inf\left(e^{-\overline{A}[h]}\Omega\right)\right)_+\\ &-e^{\lambda_0\tau_{\max}}\sup\left(e^{\overline{A}[h]}\right)\Gamma_i\left(\inf\left(e^{-\overline{A}[h]}\Omega\right)\right)_-.\end{aligned} \tag{44}$$

Finally a sufficient condition to ensure that (43) is satisfied is as follows, where $\Lambda_{2,i}$ is defined in (44):

$$\lambda^\top(\overline{\Omega}-\Lambda_{2,i})\mathcal{I}+\lambda^\top e^{\lambda_0\tau_{max}}\overline{\Xi}_h\tilde{F}_i\le\gamma_{dg}1_{2n}^\top. \tag{45}$$

Let us introduce the following matrix:

$$\begin{aligned}\Lambda_{1,i}&=e^{\lambda_0\tau_{\min}}\inf\left(e^{\overline{A}[h]}\right)\Gamma_i\left(\inf\left(e^{-\overline{A}[h]}\right)\right)_+\\ &-e^{\lambda_0\tau_{\max}}\sup\left(e^{\overline{A}[h]}\right)\Gamma_i\left(\inf\left(e^{-\overline{A}[h]}\right)\right)_-.\end{aligned} \tag{46}$$

Using the same principle, we can rewrite the stability constraints of (33). The stability conditions become:

$$
\forall i \in \{1, ..., n\},
\begin{cases}
\lambda^\top \mathrm{e}^{\lambda_0 \tau_{\max}} (\overline{A} - \lambda_0 I) + \gamma_{yf} 1_{2n}^\top \leq 0_{2n}^\top, \\
\lambda^\top \mathrm{e}^{\lambda_0 \tau_{\max}} \left[\Xi_h \tilde{E} - (\overline{A} - \lambda_0 I) \overline{\Omega} \mathcal{I} \right] - \gamma_{df} 1_{2n_d}^\top \leq 0_{2n_d}^\top, \\
\lambda^\top \Lambda_{1,i} - \lambda^\top + \gamma_{yg} 1_{2n}^\top \leq 0_{2n}^\top, \\
\lambda^\top (\overline{\Omega} - \Lambda_{2,i}) \mathcal{I} + \lambda^\top \mathrm{e}^{\lambda_0 \tau_{max}} \Xi_h \tilde{F}_i - \gamma_{dg} 1_{2n_d}^\top \leq 0_{2n_d}^\top
\end{cases}
\tag{47}
$$

In order to minimize the impact of the uncertainty on the ouput (estimated state), we have chosen in the synthesis algorithm, to minimize the objective function: $\gamma_{df} + \gamma_{dg} - \gamma_{yf} - \gamma_{yg}$.

VII. Illustrative Example

A 4-dimensional multi-output system is considered as a case study. The inputs are the multi-periodic forcing terms $u = [f_1, f_2]$ with $f_1 = 14\,(1 + 2\sin(10t) + \cos(40t))$, $f_2 = 10\,(2\sin(15t) + \sin(30t))$. The measurement data are given by three sensors. The delay varies randomly at each measurement time instant, but remains bounded, $h \in [h_1, h_2]$. The matrices describing system dynamics are:

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.3 & -5.6 & 1.6 & 2.3 \\ 0 & 0 & 0 & 1 \\ 1 & 1.4 & -2 & -2.6 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1.6 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
E = \begin{bmatrix} 0.1 & -0.2 \\ -0.7 & 0.6 \\ 0.2 & -0.2 \\ -0.5 & 0.6 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix},
$$

$$
C_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

$$
F_1 = \begin{bmatrix} 0.6 & -0.8 \\ -0.4 & 0.5 \end{bmatrix}, F_2 = \begin{bmatrix} 0.2 & -0.03 \\ -0.4 & 0.6 \end{bmatrix},
$$

$$
F_3 = \begin{bmatrix} 0.16 & -0.23 \\ 0.1 & 0.42 \end{bmatrix}.
$$

$d(t)$ is a 2-dimensional vector bounded by $\overline{d}$ and $-\overline{d}$ with $\overline{d} = [0.5, 0.5]$. State reconstruction is now performed with (Fig. 2-left) and without (Fig. 2-right) sensor selection strategy. It can be seen that the error obtained in steady state without the selection strategy is approximately twice as large as the one obtained with the selection strategy.

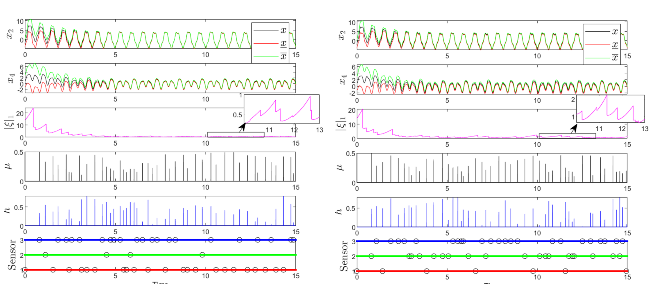

Fig. 2. Actual state variables (x_2, x_4) with the estimated bounds, inter-measurement time μ, delay h, $\|\xi\|_1$, smart choice on sensor (left) and random choice (right).

VIII. Conclusion

The presented interval impulsive observer offers a promising solution for handling time-varying delays and sporadic sensor data transmission in multi-sensor systems. By leveraging open-loop predictions and optimizing sensor choices based on the predicted reduction of the observation error width, our approach enhances system stability and performance. This research may contribute to the advancement of observer design for multi-sensor systems with time delays, offering a valuable tool for various applications in engineering and beyond.

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