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Discrete-Time Distributed Optimization for Linear Uncertain Multi-Agent Systems

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Abstract— **The distributed optimization algorithm proposed by J. Wang and N. Elia in 2010 has been shown to achieve linear convergence for multi-agent systems with single-integrator dynamics. This paper extends their result, including the linear convergence rate, to a more complex scenario where the agents have heterogeneous multi-input multi-output linear dynamics and are subject to external disturbances and parametric uncertainties. Disturbances are dealt with via an internal-model-based control design, and the interaction among the tracking error dynamics, average dynamics, and dispersion dynamics is analyzed through a composite Lyapunov function and the cyclic small-gain theorem. The key is to ensure a small enough stepsize for the convergence of the proposed algorithm, which is similar to the condition for time-scale separation in singular perturbation theory.**

I. Introduction

Large-scale networked systems make a direct application of centralized control and optimization impractical due to limited communication and computational resources. In such cases, distributed control and optimization methods become attractive as each agent uses only local information in real-time decision making. Distributed optimization, where agents cooperate to minimize a global cost function made of a sum of local cost functions, is an important topic in this context [1].

Subgradient methods have been proposed to solve the distributed optimization problem, where each agent minimizes its local cost function and exchanges information with neighboring agents [2], and these methods usually employ a vanishing stepsize, which is inevitably associated with a degraded convergence rate. To overcome such limitation, Wang and Elia [3] introduced a proportional-integral control strategy with a fixed stepsize that not only accelerates convergence, but also exhibits robustness to additive noise. Recently, the robustness of the algorithm in the sense of input-to-state stability was established in [4] based on the time-scale separation property of the algorithm. Alternatively, *gradient tracking* has been introduced in distributed optimization to track the gradient of the global cost function and achieve faster convergence [5], [6].

The above-mentioned methods can be viewed as optimal coordination algorithms for multi-agent systems modeled as single integrators [1]. In practice, the agent dynamics can be more complex such as heterogeneous or/and uncertain systems, and the gradient information may be measured based only on the agent's current output [7]. This is the case, for instance, of *optimal steady-state control* problems [8], where a set of possibly heterogeneous systems has to be driven to a steady state optimizing a given performance metric. Most studies in this direction assume that the multi-agent systems are described by continuous-time models, e.g., [9]–[11]. Only a few papers considering complex agent dynamics were devoted to discrete-time multi-agent systems, which center around first-order and second-order systems [12] [13]. Discrete-time models are closely related to many practical sampled-data systems, and the results in continuous-time systems cannot be directly applied to discrete-time systems, e.g., the averaging algorithm calls for additional conditions on the stepsize for discrete-time models [14].

This paper deals with distributed optimization of multi-agent systems characterized by discrete-time linear uncertain dynamics. The paper delivers two main contributions. Firstly, a novel output regulation framework is proposed for heterogeneous discrete-time multi-agent systems, extending the class of systems considered in [11] and generalizing the convergence result of the Wang-Elia algorithm in [4] to more general systems. Secondly, the proposed strategy utilizes only partial information of the gradient of the objective function, gathered from the available measurements, since the analytic form of gradient functions usually cannot be obtained in the feedback-based optimization process [15]. This relaxation leads to the interaction between the reference generators and controlled agents, which is investigated by a composite Lyapunov function and the cyclic small-gain theorem [16]. As a consequence, the linear convergence to the global minimizer is established.

Notations. Throughout the paper, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues of a real symmetric matrix, respectively. $\sigma(\cdot)$ denotes the spectrum of a square matrix. A matrix is Schur if its eigenvalues lie inside the open unit disk. $X = \text{blockdiag}[X_1, X_2, \cdots, X_n]$ denotes the block diagonal concatenation of the matrices

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$X_1, X_2, \cdots, X_n$. $\text{vec}(A) = [a_1^\top, a_2^\top, \cdots, a_n^\top]^\top$ where a_i stands for the i th column of matrix A. $\otimes$ stands for the Kronecker product. I_n denotes the identity matrix with size n, and sometimes n is omitted when there is no confusion. $|\cdot|$ refers to the absolute value of a scalar, the Euclidean norm of a vector, and the induced 2-norm of a matrix. $\mathbf{1}_N \in \mathbb{R}^N$ denotes the N-dimensional vector $[1, \cdots, 1]^\top$. For any discrete-time signal $s : \mathbb{N} \to \mathbb{R}^n$, $^+$ denotes the time shift operator, i.e. $s^+(\cdot) = s(\cdot + 1)$. f^{-1} denotes the inverse function of f.

II. Problem Formulation

In this section, we describe the class of multi-agent systems under concern and formulate the considered distributed optimization problem. Consider a group of N agents with agent i modeled by

$$x_i^+ = A_{wi} x_i + B_{wi} u_i + E_{wi} d_i \quad (1)$$

$$y_i = C_{wi} x_i + D_{wi} u_i + F_{wi} d_i \quad (2)$$

for $i = \{1, \cdots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^p$, and $u_i \in \mathbb{R}^{m_i}$ are the state, output, and input of agent i, respectively, and $d_i \in \mathbb{R}^{l_i}$ is the exogenous disturbance. $A_{wi} \in \mathbb{R}^{n_i \times n_i}$, $B_{wi} \in \mathbb{R}^{n_i \times m_i}$, $E_{wi} \in \mathbb{R}^{n_i \times l_i}$, $C_{wi} \in \mathbb{R}^{p \times n_i}$, $D_{wi} \in \mathbb{R}^{p \times m_i}$, and $F_{wi} \in \mathbb{R}^{p \times l_i}$ are uncertain matrices defined as $A_{wi} = A_i + \Delta A_i$, $B_{wi} = B_i + \Delta B_i$, $E_{wi} = C_i + \Delta E_i$, $C_{wi} = C_i + \Delta C_i$, $D_{wi} = D_i + \Delta D_i$, $F_{wi} = F_i + \Delta F_i$ where A_i, B_i, E_i, C_i, D_i, F_i are known matrices and ΔA_i, ΔB_i, ΔE_i, ΔC_i, ΔD_i, ΔF_i are unknown perturbations. Let

$$w_i = \text{vec}\left(\begin{bmatrix} \Delta A_i & \Delta B_i & \Delta E_i \\ \Delta C_i & \Delta D_i & \Delta F_i \end{bmatrix}\right)$$

and $w = [w_1^\top, w_2^\top, \cdots, w_N^\top]^\top$ represent the uncertainties of all agents. For each agent i, we suppose that the disturbance d_i satisfies the following autonomous equation

$$d_i^+ = S_i d_i \quad (3)$$

where $S_i \in \mathbb{R}^{l_i \times l_i}$ has no eigenvalues with modulus smaller than one. Moreover, we define a regulation error as

$$e_i = y_i - r_i \quad (4)$$

where $r_i \in \mathbb{R}^p$ is the reference signal. Let $d = [d_1^\top, \cdots, d_N^\top]^\top$, $e = [e_1^\top, \cdots, e_N^\top]^\top$, $r = [r_1^\top, \cdots, r_N^\top]^\top$, and $y = [y_1^\top, \cdots, y_N^\top]^\top$.

The information exchange topology among the agents is described by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ including a finite set of nodes $\mathcal{N} = \{1, \cdots, N\}$ corresponding to the agents, a finite set of edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{N \times N}$. Moreover, (i, j) denotes the edge from i to j and a_{ij} denotes the weight of the edge (j, i), which satisfies $a_{ij} > 0$ when $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ when $(j, i) \notin \mathcal{E}$. We say that $\mathcal{G}$ is connected if there exists a path between any two nodes, $\mathcal{A}$ is doubly-stochastic if $\mathcal{A}\mathbf{1}_N = \mathbf{1}_N$ and $\mathbf{1}_N^\top \mathcal{A} = \mathbf{1}_N^\top$, and $\mathcal{G}$ is undirected if $\mathcal{A} = \mathcal{A}^\top$. We make the following assumption on the graph.

Assumption 1: The digraph $\mathcal{G}$ is connected and $\mathcal{A}$ is doubly-stochastic.

Each agent i is associated with a local cost function $f_i : \mathbb{R}^p \to \mathbb{R}$ that is unknown to the other agents. The objective is to design distributed controllers such that

$$\lim_{k \to \infty} y_i(k) = y^* \quad (5)$$

for all $i \in \mathcal{N}$, where $y^* \in \mathbb{R}^p$ minimizes the sum of local cost functions $f = \sum_{i \in \mathcal{N}} f_i$. We make the following assumption on the local objective functions.

Assumption 2: For each $i \in \mathcal{N}$, f_i is continuously differentiable and there exists $l > 0$ such that

$$|\nabla f_i(a) - \nabla f_i(b)| \leq l |a - b| \quad (6)$$

for all $a, b \in \mathbb{R}^p$. Moreover, there exists $\mu > 0$ such that the global cost function f satisfies

$$(a - b)^\top (\nabla f(a) - \nabla f(b)) \geq \mu |a - b|^2 \quad (7)$$

for all $a, b \in \mathbb{R}^p$.

A continuously differentiable function satisfying (7) is said to be strongly convex, and thanks to (7), there exists a unique minimizer of f [17]. The considered problem is summarized as follows.

Problem 1: The distributed robust optimal output agreement problem is to find a distributed control system yielding a reference signal r_i and a control action u_i for each agent $i \in \mathcal{N}$, such that, for the multi-agent system (1), (2), and (3) with the network topology $\mathcal{G}$:

1) the closed-loop system is asymptotically stable at the origin when $w = 0$, $r = 0$, and $d = 0$;
2) there exists an open neighborhood W of $w = 0$ such that, for every $w \in W$, and every initial state $x_i(0)$, (5) is satisfied for all $i \in \mathcal{N}$.

III. Reference and Control Design

In this section, we present the design of distributed reference signals and robust tracking control for the multi-agent systems.

A. Wang-Elia Algorithm

The Wang-Elia algorithm was proposed in [3] for the distributed optimization problem with single-integrator dynamics. We employ this algorithm with a small modification to generate the reference signals. For each agent $i \in \mathcal{N}$, define the system

$$\begin{aligned} r_i^+ &= r_i - \beta \sum_{j \in \mathcal{N}_i} a_{ij}(r_i - r_j + q_i - q_j) - \beta\alpha \nabla f_i(y_i) \\ q_i^+ &= q_i + \beta \sum_{j \in \mathcal{N}_i} a_{ij}(r_i - r_j) \end{aligned} \quad (8)$$

where $\alpha > 0$ is the stepsize and $\beta > 0$ is a consensus parameter, both of which will be designed in the next section; $\mathcal{N}_i = \{j \in \mathcal{N} | (j, i) \in \mathcal{E}\}$ denotes the set of all neighbors of agent i. Compared to the original Wang-Elia algorithm, the gradient is computed at y_i instead of r_i, since in many practical cases, only the real-time measurement of the gradient $\nabla f_i(y_i)$ is available [7]. Let $\Phi(y) = [\nabla f_1^\top(y_1), \cdots, \nabla f_N^\top(y_N)]^\top$, $L = I - \mathcal{A}$, and $\bar{L} = L \otimes I_p$.

Let $q = [q_1^\top, \cdots, q_N^\top]^\top$ and recall that $r = [r_1^\top, \cdots, r_N^\top]^\top$ denotes the composite reference signals, then, the state equations for all agents are:

$$\begin{aligned} r^+ &= (I - \beta\bar{L})r - \beta\bar{L}q - \alpha\beta\Phi(y) \\ q^+ &= q + \beta\bar{L}r. \end{aligned} \tag{9}$$

As $y = e + r$ by (4), the convergence of the Wang-Elia algorithm can be adversely affected by the tracking error e. In the following section, a robust tracking controller based on output regulation theory is introduced to deal with the tracking error e.

B. Robust Tracking Control

A design procedure of integral control based on the internal model principle is presented here assuming r is a constant. Specifically, the proposed design solution is based on the construction of [18]. For each agent $i \in \mathcal{N}$, let $\lambda^{\kappa_i} + \alpha_1\lambda^{(\kappa_i - 1)} + \cdots + \alpha_{\kappa_i - 1}\lambda + \alpha_{\kappa_i}$ be the minimal polynomial of $A_{i0} = \text{blockdiag}[I_p, S_i]$, which contains the factor $\lambda - 1$, and let

$$\Lambda_i = \left[\begin{array}{c|c} 0 & I_{\kappa_i - 1} \\ \hline -\alpha_{\kappa_i} & [-\alpha_{(\kappa_i-1)}, \cdots, -\alpha_1] \end{array}\right]$$

and $\rho_i = [0, 0, \cdots, 0, 1]^\top \in \mathbb{R}^{\kappa_i}$. Then, let $G_{i1} = \text{blockdiag}[\Lambda_{i1}, \Lambda_{i2}, \cdots, \Lambda_{ip}]$, with $\Lambda_{i1} = \Lambda_{i2} = \cdots = \Lambda_{ip} = \Lambda_i$, and $G_{i2} = \text{blockdiag}[\rho_{i1}, \rho_{i2}, \cdots, \rho_{ip}]$ with $\rho_{i1} = \rho_{i2} = \cdots = \rho_{ip} = \rho_i$. For each agent $i \in \mathcal{N}$, the dynamic state feedback controller takes on the form:

$$\begin{aligned} u_i &= K_{i1}x_i + K_{i2}z_i \\ z_i^+ &= G_{i1}z_i + G_{i2}e_i \end{aligned} \tag{10}$$

where $z_i \in \mathbb{R}^{p\kappa_i}$ and $K_{i1} \in \mathbb{R}^{m_i \times n_i}$, $K_{i2} \in \mathbb{R}^{m_i \times p\kappa_i}$ are chosen such that

$$A_{ic} = \begin{bmatrix} A_i & 0 \\ G_{i2}C_i & G_{i1} \end{bmatrix} + \begin{bmatrix} B_i \\ G_{i2}D_i \end{bmatrix} \begin{bmatrix} K_{i1} & K_{i2} \end{bmatrix}$$

is Schur. To guarantee the existence of (K_{i1}, K_{i2}), the following assumption is introduced.

Assumption 3: For each agent $i \in \mathcal{N}$, S_i has no eigenvalues with modulus smaller than one, the pair (A_i, B_i) is stabilizable, and

$$\text{rank} \begin{bmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{bmatrix} = n_i + p$$

for all $\lambda \in \sigma(S_i) \cup \{1\}$.

Lemma 1 ([19] Lemma 1.37): Under Assumption 3, for each agent $i \in \mathcal{N}$, the pair

$$\left(\begin{bmatrix} A_i & 0 \\ G_{i2}C & G_{i1} \end{bmatrix}, \begin{bmatrix} B \\ G_{i2}D \end{bmatrix} \right)$$

is stabilizable.

Remark 1: The problem of designing a robust tracking controller when the reference is produced by (9) is out of reach of canonical output regulation theory [18]–[20]. Indeed, (9) is not autonomous as it depends on the agents' outputs y. To overcome this issue, we proceed as follows. We first consider the ideal case in which a coordinator provides the optimal minimizer y^* as the reference signal. By the internal model principle [21], it is necessary to embed an integrator in the control loop to achieve asymptotic tracking as previously done. Next, we observe that [18] shows that integral control is robust for slow variations of the set-point. Finally, we notice that the Wang-Elia algorithm is characterized by a time-scale separation property when the stepsize α is sufficiently small [4] and that, as a result, the reference r can be assumed to be slowly time-varying. Therefore, the robustness result of [18] can be used, with the due adaptation, to study the stability of the composite system as shown in Section IV.

IV. Main Results

In this section, we analyze the stability properties of the closed-loop systems under reference (8) and controller (10). First, the reference-tracking capability of (10) is characterized via a Lyapunov formulation considering the drift of r. Then, robustness of the distributed optimization process (8) is investigated. Finally, the two results are combined to establish the stability of the overall system.

A. Reference-Tracking Capability

For each agent $i \in \mathcal{N}$, as A_{ic} is Schur by the selection of K_{i1} and K_{i2}, there exists an open neighborhood W of $w = 0$ such that

$$A_{icw} = \begin{bmatrix} A_{wi} + B_{wi}K_{i1} & B_{wi}K_{i2} \\ G_{i2}(C_{wi} + D_{wi}K_{i1}) & G_{i1} + G_{i2}D_{wi}K_{i2} \end{bmatrix}$$

is Schur. Define $\hat{A}_i = A_{wi} + B_{wi}K_{i1}$, $\hat{B}_i = B_{wi}K_{i2}$, $\hat{C}_i = C_{wi} + D_{wi}K_{i1}$, $\hat{D}_i = D_{wi}K_{i2}$, $\hat{E}_i = [0, E_{wi}]$, and $\hat{F}_i = [-I_p, F_{wi}]$.

Lemma 2: Under Assumption 3, for each agent $i \in \mathcal{N}$, let X_i and Z_i be the solution of

$$\begin{aligned} X_i A_{i0} &= \hat{A}_i X_i + \hat{B}_i Z_i + \hat{E}_i \\ Z_i A_{i0} &= G_{i1} Z_i + G_{i2}(\hat{C}_i X_i + \hat{D}_i Z_i + \hat{F}_i) \end{aligned} \tag{11}$$

and define $[X_i^\top, Z_i^\top]^\top = [X_{ir}, X_{id}]$ with $X_{ir} \in \mathbb{R}^{(n_i + p\kappa_i) \times p}$, $v_i = [r_i^\top, d_i^\top]^\top$, $\tilde{x}_i = x_i - X_i v_i$, $\tilde{z}_i = z_i - Z_i v_i$, and $\eta_i = [\tilde{x}_i^\top, \tilde{z}_i^\top]^\top$. Then,

$$\eta_i^+ = A_{icw}\eta_i + X_{ir}(r_i - r_i^+) \tag{12}$$

$$e_i = [\hat{C}_i, \hat{D}_i]\eta_i. \tag{13}$$

Proof: The result is a direct application of Lemma 1.38 in [19]. ■

Define $\bar{A}_{cw} = \text{blockdiag}[A_{1cw}, A_{2cw}, \cdots, A_{Ncw}]$,

$$\bar{B}_{cw} = \text{blockdiag}[X_{1r}, X_{2r}, \cdots, X_{Nr}],$$

$$\bar{C}_{cw} = \text{blockdiag}[[\hat{C}_1, \hat{D}_1], [\hat{C}_2, \hat{D}_2], \cdots, [\hat{C}_N, \hat{D}_N]],$$

and $\eta = [\eta_1^\top, \eta_2^\top, \cdots, \eta_N^\top]^\top$. The tracking-error systems of all agents can be aggregated as

$$\eta^+ = \bar{A}_{cw}\eta + \bar{B}_{cw}(r - r^+) \tag{14}$$

$$e = \bar{C}_{cw}\eta. \tag{15}$$

The following proposition formalizes the tracking performance of the controller (10).

Proposition 1 ([22]): Under Assumption 3, consider the multi-agent systems with dynamics (14). Then, there exist a real symmetric and positive definite matrix P_1 and positive constants δ and θ such that the function $V_1(\eta) = \eta^\top P_1 \eta$ satisfies

$$V_1(\eta^+) - V_1(\eta) \le -\delta|\eta|^2 + \theta|r - r^+|^2 \tag{16}$$

for all $\eta \in \mathbb{R}^{\sum_{i=1}^N (n_i + p\kappa_i)}$.

B. Robustness of Wang-Elia Algorithm

It was proved in [4] that the Wang-Elia algorithm is robust to bounded disturbance. Here, we restate the result considering y^* as a vector, imposing additional conditions on β, and regarding $\tilde{\Phi}(r) - \tilde{\Phi}(y)$ as the perturbation.

1) Coordinate Transformation: Inspired by [4] and [10], choose $U \in \mathbb{R}^{N\times(N-1)}$ such that $P = [\mathbf{1}_N/\sqrt{N}, U]$ is an orthogonal matrix, and define $\bar{P} = P \otimes I_p$, which is an orthogonal matrix as $\bar{P}^{-1} = P^{-1} \otimes I_p = P^\top \otimes I_p = \bar{P}^\top$. Let $\bar{P} = [\bar{P}_m, \bar{P}_\perp]$ where $\bar{P}_m = (\mathbf{1}_N/\sqrt{N}) \otimes I_p$ and $\bar{P}_\perp = U \otimes I_p$, by which $r = \bar{P}_m \xi_m + \bar{P}_\perp \xi_\perp$ and $q = \bar{P}_m \phi_m + \bar{P}_\perp \phi_\perp$, where (ξ_m, ϕ_m) and $(\xi_\perp, \phi_\perp)$ correspond to the average and dispersion parts of (r, q), respectively [4]. Conversely, one can get $\xi_m = \bar{P}_m^\top r$, $\phi_m = \bar{P}_m^\top q$, $\xi_\perp = \bar{P}_\perp^\top r$, and $\phi_\perp = \bar{P}_\perp^\top q$. From (9), we obtain

$$\xi_m^+ = \xi_m - \alpha\beta \bar{P}_m^\top \Phi(y) \tag{17}$$

$$\begin{bmatrix} \xi_\perp^+ \\ \phi_\perp^+ \end{bmatrix} = A_\perp \begin{bmatrix} \xi_\perp \\ \phi_\perp \end{bmatrix} - \begin{bmatrix} \alpha\beta \bar{P}_\perp^\top \\ 0 \end{bmatrix} \Phi(y) \tag{18}$$

$$\phi_m^+ = \phi_m \tag{19}$$

where

$$A_\perp = \begin{bmatrix} I - \beta R & -\beta R \\ \beta R & I \end{bmatrix}$$

with $R = (U^\top L U) \otimes I_p$. As $y = e + r = e + \bar{P}_m \xi_m + \bar{P}_\perp \xi_\perp$, ϕ_m is independent of the other states and can be ignored without affecting the analysis of the other states. The equilibrium of the overall system is described as follows.

Lemma 3: Under Assumptions 1, 2, and 3, the system of (14), (17), and (18) admits a unique equilibrium, i.e., $\eta^* = 0$, $\xi_m^* = \sqrt{N} y^*$, $\xi_\perp^* = 0$, and $\phi_\perp^* = -\alpha R^{-1} \bar{P}_\perp^\top \Phi(\mathbf{1}_N \otimes y^*)$.

Proof: Let $r^* = \bar{P}_m \xi_m^* + \bar{P}_\perp \xi_\perp^*$. As ξ_m^* and $\xi_\perp^*$ are fixed points, it follows from (14) that $(I - \bar{A}_{cw})\eta^* = 0$ and $\eta^* = 0$ since $\bar{A}_{cw}$ is Schur by Assumption 3. Hence, $e = 0$ from (15) and $y = r^*$. From the dynamics of $\phi_\perp$ in (18), $R\xi_\perp^* = 0$. By Assumption 1, $LP = P\,\text{blockdiag}[0, U^\top L U]$. Thus, $\sigma(U^\top L U)$ contains all the eigenvalues of L except one eigenvalue 0. Since the digraph $\mathcal{G}$ is connected, by Perron-Frobenius Theorem [14], 0 is a simple eigenvalue of L. Hence, eigenvalues of $U^\top L U$ are non-zero and R is non-singular, which implies that $\xi_\perp^* = 0$. Hence $y = r^* = \bar{P}_m \xi_m^*$. From (17), $\sqrt{N} \bar{P}_m^\top \Phi(r^*) = \sum_{i\in N} \nabla f_i(\xi_m^*/\sqrt{N}) = 0$. By Assumption 2, there exists a unique minimizer y^* such that $\sum_{i\in N} \nabla f_i(y^*) = 0$, which implies that $\xi_m^* = \sqrt{N} y^*$. Therefore, $y = \sqrt{N} \bar{P}_m y^* = \mathbf{1}_N \otimes y^*$. Finally, $\phi_\perp^* = -\alpha R^{-1} \bar{P}_\perp^\top \Phi(\mathbf{1}_N \otimes y^*)$ can be obtained by (18). ∎

Remark 2: It can be seen that, at the equilibrium, $y = \bar{P}_m \xi_m^* = \mathbf{1}_N \otimes y^*$. Hence, if the equilibrium is asymptotically stable, the objective in (5) is achieved.

Define $\tilde{\xi}_m = \xi_m - \xi_m^*$ and $\tilde{\phi}_\perp = \phi_\perp - \phi_\perp^*$. The dynamics of (14), (17), and (18) can be rewritten as

$$\eta^+ = \bar{A}_{cw}\eta + \bar{B}_{cw}(r - r^+) \tag{20}$$

$$\tilde{\xi}_m^+ = \tilde{\xi}_m - \alpha\beta \bar{P}_m^\top \tilde{\Phi}(y) \tag{21}$$

$$\begin{bmatrix} \xi_\perp^+ \\ \tilde{\phi}_\perp^+ \end{bmatrix} = A_\perp \begin{bmatrix} \xi_\perp \\ \tilde{\phi}_\perp \end{bmatrix} - \begin{bmatrix} \alpha\beta \bar{P}_\perp^\top \\ 0 \end{bmatrix} \tilde{\Phi}(y) \tag{22}$$

where $r - r^+ = \bar{P}_m(\tilde{\xi}_m - \tilde{\xi}_m^+) + \bar{P}_\perp(\xi_\perp - \xi_\perp^+)$, $\tilde{\Phi}(y) = \Phi(y) - \Phi(\mathbf{1}_N \otimes y^*)$, and $y = \bar{P}_m \tilde{\xi}_m + \bar{P}_\perp \xi_\perp + \bar{P}_m \xi_m^* + \bar{C}_{cw}\eta$.

2) Robustness Result: The following result establishes that, under some conditions, it is possible to choose β such that $A_\perp$ is Schur.

Lemma 4: Under Assumption 1, when all eigenvalues of $\mathcal{A}$ are real, $A_\perp$ is Schur if and only if $0 < \beta < \frac{1}{2}$.

Proof: For any $\lambda \in \sigma(\mathcal{A})$ with $-1 \le \lambda < 1$, then, following the proof of Lemma 3, the corresponding eigenvalue of βR is $\beta(1-\lambda)$. Since $A_\perp = I - A_c \otimes \beta R$ with

$$A_c = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

and $\sigma(A_c) = \{\frac{1+\sqrt{3}j}{2}, \frac{1-\sqrt{3}j}{2}\}$ where j denotes the unit imaginary number, the corresponding eigenvalue z of $A_\perp$ satisfies $z = 1 - \beta(1-\lambda)\frac{1\pm\sqrt{3}j}{2}$, and $|z| = \sqrt{\beta(1-\lambda)(\beta(1-\lambda)-1)+1}$. As $-1 \le \lambda < 1$, $|z| < 1$ is equivalent to $0 < \beta < 1/(1-\lambda)$, for each $-1 \le \lambda < 1$. Therefore, $0 < \beta < \frac{1}{2}$ guarantees $|z| < 1$. Conversely, assuming $\beta \ge \frac{1}{2}$, there exists a digraph with two nodes without self loops, such that one eigenvalue $\lambda = -1$, which implies that $A_\perp$ is not Schur as $\beta \ge 1/(1-\lambda)$. ∎

Remark 3: When $\mathcal{G}$ is undirected, $\mathcal{A}$ is symmetric and has only real eigenvalues [23], so Lemma 4 can be used and the condition is the same as the one in [4] to render $A_\perp$ Schur, but its necessity was not pointed out. One may wonder if similar conditions can be derived when only Assumption 1 holds; however, this is impossible as there exists a cycle digraph with 6 nodes, where one eigenvalue of its adjacency matrix is $\lambda = \frac{1-\sqrt{3}j}{2}$ [14] and, correspondingly, one eigenvalue z of $A_\perp$ satisfies $|z| > 1$ for any $\beta > 0$, which implies that $A_\perp$ is not Schur. We do not restrict the graph to be undirected, but given a digraph $\mathcal{G}$ satisfying Assumption 1, β should be chosen such that $A_\perp$ is Schur.

System (21) and (22) can be regarded as a perturbed system with $\chi = \tilde{\Phi}(r) - \tilde{\Phi}(y)$ as the perturbation. The robustness of the Wang-Elia algorithm to any bounded perturbation has been investigated in [4] for $p = 1$ when $A_\perp$ is Schur. For completeness of the paper, we restate the robustness result about χ for $p \in \mathbb{N}_+$, and the proof is similar to that of [4], hence omitted. Let $\psi = [\xi_\perp^\top, \tilde{\phi}_\perp^\top]^\top$.

Proposition 2: Under Assumptions 1 and 2, suppose there exists $\beta > 0$ such that $A_\perp$ is Schur. Then, there exist a positive constant $\bar{\alpha}$ and a real symmetric positive definite matrix P_2 such that, for each $\alpha \in (0, \bar{\alpha}]$, there exist $c_0, c_1(\alpha), c_2(\alpha), c_3(\alpha) > 0$, such that

$$\Delta V_m \leq -\frac{3}{2}\alpha c_0 |\tilde{\xi}_m|^2 + c_1(\alpha)|\psi|^2 + c_2(\alpha)|\chi|^2 \quad (23)$$

$$\Delta V_\perp \leq -|\psi|^2 + \alpha c_0 |\tilde{\xi}_m|^2 + c_3(\alpha)|\chi|^2 \quad (24)$$

where $\Delta V_m = V_m(\tilde{\xi}_m^+) - V_m(\tilde{\xi}_m)$ with $V_m(\tilde{\xi}_m) = \tilde{\xi}_m^\top \tilde{\xi}_m$ and $\Delta V_\perp = V_\perp(\psi^+) - V_\perp(\psi)$ with $V_\perp(\psi) = \psi^\top P_2 \psi$.

C. Stability of the Overall System

We are ready to state the stability result on the overall system in closed-loop with the tracking controller and the distributed reference generator. Let $\zeta = [\eta^\top, \tilde{\xi}_m^\top, \psi^\top]^\top$ and let ζ_k denote the state ζ at time step k.

Theorem 1: Under Assumptions 1, 2, and 3, suppose there exists $\beta > 0$ such that $A_\perp$ is Schur. Then, there exist positive constants c and α^*, and for each $\alpha \in (0, \alpha^*]$, there exists $\rho \in [0, 1)$, such that the state ζ of system (20), (21), and (22) satisfies

$$|\zeta_k| \leq c\rho^k |\zeta_0| \quad (25)$$

for all $k \in \mathbb{N}$.

Proof: First we quantify $\Delta V_1 = V_1(\eta^+) - V_1(\eta)$. It can be seen that $|r - r^+| = |\bar{P}_m(\tilde{\xi}_m - \tilde{\xi}_m^+) + \bar{P}_\perp(\xi_\perp - \xi_\perp^+)| \leq |\tilde{\xi}_m - \tilde{\xi}_m^+| + |\psi - \psi^+|$ as $|\bar{P}_m| = |\bar{P}_\perp| = 1$. Besides, from (21) and (22), $|\tilde{\xi}_m - \tilde{\xi}_m^+| \leq \alpha\beta|\tilde{\Phi}(y)|$, $|\psi - \psi^+| \leq \sigma_0|\psi| + \alpha\beta|\tilde{\Phi}(y)|$ with $\sigma_0 = |I - A_\perp|$, and $|\tilde{\Phi}(y)| \leq l(|\tilde{\xi}_m| + |\psi| + |\bar{C}_{cw}||\eta|)$. Then, $|r - r^+| \leq 2\alpha\beta l(|\tilde{\xi}_m| + |\psi| + |\bar{C}_{cw}||\eta|) + \sigma_0|\psi|$. By (16), $\Delta V_1 \leq (-\delta + 12\theta\alpha^2\beta^2|\bar{C}_{cw}|^2)|\eta|^2 + 12\theta\alpha^2\beta^2 l^2|\tilde{\xi}_m|^2 + 6\theta(\sigma_0^2 + 4\alpha^2\beta^2 l^2)|\psi|^2$. Since $\theta > 0$, when $\alpha \leq \alpha_5$ with $\alpha_5 = \sqrt{\delta}/\sqrt{24\theta\beta^2|\bar{C}_{cw}|^2}$,

$$\Delta V_1 \leq -\frac{\delta}{2}|\eta|^2 + 12\theta\alpha^2\beta^2 l^2|\tilde{\xi}_m|^2 + 6\theta(\sigma_0^2 + 4\alpha^2\beta^2 l^2)|\psi|^2. \quad (26)$$

Considering the composite Lyapunov function $V(\zeta) = \sigma_1 V_1(\eta) + V_m(\tilde{\xi}_m) + V_\perp(\psi)$ with $0 < \sigma_1 \leq 1/(24\theta\sigma_0^2)$, by (23), (24), (26), and $|\chi| \leq l|\bar{C}_{cw}||\eta|$,

$$V(\zeta^+) - V(\zeta) \leq -\frac{\sigma_1\delta}{4}|\eta|^2 - \frac{\alpha c_0}{4}|\tilde{\xi}_m| - \frac{1}{8}|\psi|^2 \quad (27)$$

when $\alpha \leq \min\{\alpha_6, \alpha_7, \alpha_8, \alpha_9\}$ where $\alpha_6 = c_1^{-1}(\frac{1}{2})$, $\alpha_7 = c_0/(48\sigma_1\theta\beta^2 l^2)$, $\alpha_8 = 1/(\sqrt{192\sigma_1\theta\beta^2 l^2})$ and $\alpha_9 = (c_2 + c_3)^{-1}(\sigma_1\delta/4l^2|\bar{C}_{cw}|^2)$. Define $\nu = \min\{\frac{\alpha c_0}{4}, \frac{1}{8}, \frac{\sigma_1\delta}{4}\}$, $\gamma_1 = \min\{1, \lambda_{\min}(P_2), \sigma_1\lambda_{\min}(P_1)\}$, and $\gamma_2 = \max\{1, \lambda_{\max}(P_2), \sigma_1\lambda_{\max}(P_1)\}$. Then, $\gamma_1|\zeta|^2 \leq V(\zeta) \leq \gamma_2|\zeta|^2$ and $V(\zeta^+) \leq (1 - v/\gamma_2)V(\zeta)$. If follows that (25) holds by letting $c = \sqrt{\gamma_2/\gamma_1}$, $\rho = \sqrt{1 - v/\gamma_2}$ and $\alpha^* = \min\{\bar{\alpha}, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}$. ■

Remark 4: As $y - \mathbf{1}_N \otimes y^* = \bar{C}_{cw}\eta + \bar{P}_m\tilde{\xi}_m + \bar{P}_\perp\xi_\perp$, by (25), the output converges to the global minimizer with a linear convergence rate [17]. In addition, the condition on the stepsize α is a reminiscence of the condition for time-scale separation in singular perturbation [24].

Besides, apart from constructing the composite Lyapunov function, the asymptotic stability of the origin can also be proved by the cyclic small-gain theorem [25] [16], and

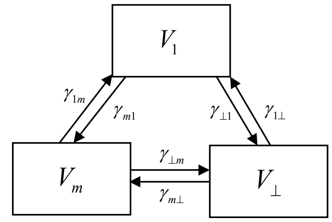

Fig. 1. Gain digraph of subsystems η, $\tilde{\xi}_m$, and ψ.

similar condition on stepsize α can be derived by enforcing the small-gain conditions, i.e., $\gamma_{1m}\gamma_{m1} < 1$, $\gamma_{1\perp}\gamma_{\perp 1} < 1$, $\gamma_{m\perp}\gamma_{\perp m} < 1$, $\gamma_{1m}\gamma_{m\perp}\gamma_{\perp 1} < 1$, and $\gamma_{1\perp}\gamma_{\perp m}\gamma_{m1} < 1$ with the asymptotic gains $(\gamma_{m\perp}, \gamma_{m1})$, $(\gamma_{\perp m}, \gamma_{\perp 1})$, and $(\gamma_{1m}, \gamma_{1\perp})$ from (23), (24), and (26), respectively, as shown in Fig. 1. By Remark 2, Problem 1 can be solved by Theorem 1 as follows.

Corollary 1: Under Assumptions 1, 2, and 3, let β be chosen such that $A_\perp$ is Schur and $\alpha \in (0, \alpha^*]$. Then, Problem 1 is solved by (8) and (10) for each agent $i \in \mathcal{N}$.

V. Simulation Results

In this section, we illustrate our proposed method for a distributed robust optimal output agreement problem. The experiment is carried out using Matlab R2020a in a laptop with Windows 10 operating system and AMD Ryzen 7 5800H processor.

We consider a linear multi-agent system with $N = 30$ agents modeled by (1) and (2), and the related matrices A_i, B_i are sampled from a double integrator with the sampling time drawn from a uniform distribution $U[0.1, 1]$; $C_i = [1, 0]$, $D_i = 0$, and $F_i = 0$ for $i \in \{1, 2, \cdots, 30\}$. Besides, let $E_1 = 0.1I_2$, $E_{11} = 0.2I_2$, $E_{21} = 0.15I_2$, and for exosystem (3),

$$S_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad S_{11} = \begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix},$$

and

$$S_{21} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}.$$

All eigenvalues of E_1, E_{11}, and E_{21} are with modulus one and d_1, d_{11}, and d_{21} are sinusoidal signals. The corresponding internal models are obtained with

$$G_{i1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(1 + 2\cos\tau_i) & 1 + 2\cos\tau_i \end{bmatrix}$$

where $\tau_1 = \pi/4$, $\tau_{11} = \pi/3$, and $\tau_{21} = \pi/6$ and the minimum polynomial of G_{i1} contains the factor $\lambda - 1$, for $i \in \{1, 11, 21\}$. For the other agents, $E_i = E_1, S_i = S_1, G_{i1} = G_{11}$ if $i \leq 10$, $E_i = E_{11}, S_i = S_{11}, G_{i1} = G_{11,1}$ if $10 < i \leq 20$, and $E_i = E_{21}, S_i = S_{21}, G_{i1} = G_{21,1}$ if $20 < i \leq 30$. K_{i1} and K_{i2} are decided by placing the eigenvalues of A_{ic} at $[0.25 + 0.25j, 0.25 - 0.25j, -0.2 + 0.5j, -0.2 - 0.5j, 0.25]$ for $i \in \{1, 2, \cdots, 30\}$. The agents communicate with each other via a two-dimensional undirected grid graph with 5×6 nodes

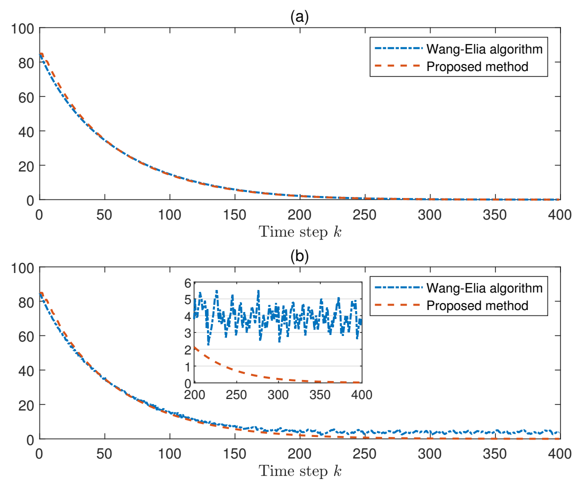

Fig. 2. (a) Plot of $|y - \mathbf{1}_N \otimes y^*|$ when agent dynamics is without uncertainties; (b) Plot of $|y - \mathbf{1}_N \otimes y^*|$ when partial agent dynamics is with parametric uncertainties.

and the corresponding adjacency matrix satisfies $\mathcal{A}^\top = \mathcal{A}$ and $\mathcal{A}\mathbf{1}_{30} = \mathbf{1}_{30}$, from which Assumption 1 holds. The cost functions are chosen to be quadratic, i.e., $f_i(y) = (y - i)^2$, for $i \in \{1, 2, \cdots, 30\}$. The unique minimizer of the global cost function is $31/2$. The stepsize α is chosen to be 0.025 and the consensus parameter β is 0.4 by Lemma 4. The initial states of x_i, r_i and q_i are zeros, and the initial d_i is drawn from a uniform distribution $U[-1, 1]$, for $i \in \{1, 2, \cdots, 30\}$. Without parametric uncertainties, Wang-Elia algorithm can be applied by transforming the state equations into single-integrator forms via appropriate feedback linearization techniques [24]. The comparison between our proposed method and Wang-Elia algorithm, with the same parameter settings, is shown in Fig. 2 (a). It can be seen that both methods can make the output of each agent converge to the global minimizer with similar convergence rates.

However, when there are unknown perturbations with

$$\Delta A_i = \begin{bmatrix} 0 & 0.05 \\ 0.01 & 0 \end{bmatrix}, \Delta B_i = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

for $i \in \{2, 12, 14, 16, 18, 21, 22, 25, 26, 30\}$, as shown in Fig. 2 (b), Wang-Elia algorithm cannot achieve the same convergence result due to the unknown uncertainties in agent dynamics, but our proposed method overcomes the obstacles and still guarantees the convergence in the presence of the uncertainties.

VI. Conclusions

In this paper, the distributed optimization problem for multi-agent systems described by general discrete-time, linear, and uncertain models has been studied. An internal-model-based controller and a reference generator derived by the Wang-Elia algorithm are developed for each agent to achieve robust optimal output agreement with guaranteed linear convergence. Simulation results show that, compared with the Wang-Elia algorithm, the proposed method can achieve the optimal agreement for the complex multi-agent systems, even in the presence of parametric uncertainties.

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