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Direct Data-Driven Computation of Polytopic Robust Control Invariant Sets and State-Feedback Controllers

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Abstract— **This paper presents a direct data-driven approach for computing *robust control invariant* (RCI) sets and their associated state-feedback control laws for linear time-invariant systems affected by bounded disturbances. The proposed method utilizes a single state-input trajectory generated from the system, to compute a polytopic RCI set with a desired complexity and an invariance-inducing feedback controller, without the need to identify a model of the system. The problem is formulated in terms of a set of sufficient linear matrix inequality conditions that are then combined in a semi-definite program to maximize the volume of the RCI set while respecting the state and input constraints. We demonstrate through a numerical case study that the proposed data-driven approach can generate RCI sets that are of comparable size to those obtained by a model-based method in which exact knowledge of the system matrices is assumed.**

I. Introduction

Set invariance theory has received a significant attention over the years for constrained systems and stability analysis [6], [7]. A set is called *robust control invariant* (RCI) if from all initial states within the set, an admissible control input exists, which keeps the state trajectories within the set for all bounded disturbances acting on the system [6]. Several contributions have been proposed in the literature to compute the RCI set and its associated controllers given a model of the system, see, for *e.g.*, [8], [12], [15], [17], [21], [23]. These are *model-based* methods where the main underlying assumption is that a model of the true system is available. However, there are several challenges to obtain an accurate model of the system [16]. An inaccurate model can lead to loss of the invariance property as well as violation of constraints when operating in the closed-loop [24]. To overcome these limitations, recent developments have emphasized data-driven approaches. *Control-oriented* data-driven identification is proposed in [9], [20], consisting of concurrent model selection along with RCI set computation, which results in reduced conservatism. Alternatively, *direct* data-driven control approaches [2]–[5], synthesize robust controllers directly from the open-loop data, without the need for model identification.

The direct data-driven methods presented in [4], [5], compute a state-feedback controller from data to induce robust invariance in a given polyhedral set. However, these methods require that the set is fixed and known a priori. A recent work [24] offers a method for computing an invariant set as well as its associated feedback controller. This approach constructs an *ellipsoidal* invariant set. It should be noted that ellipsoidal sets are potentially more conservative than *polyhedral* sets, as the latter present several theoretical and practical advantages over the ellipsoids via flexible and arbitrarily complex representation [7], with the only drawback of scalability.

In this paper, we develop a direct data-driven approach to compute polytopic RCI sets and state-feedback controllers for unknown linear systems subject to bounded disturbances. We utilize a single state-input trajectory generated in open-loop and derive data-based sufficient linear matrix ineqality (LMI) conditions which can guarantee invariance and constraint satisfaction. The sufficient LMI conditions are then combined in a semi-definite program (SDP) to maximize the volume of the RCI set. The proposed approach can be seen as a data-driven counterpart to the model-based method presented in [13]. We point out that in [13], the exact model of the true system is assumed to be known, while the approach presented in this paper neither requires knowledge of the model nor any system identification step.

II. Notations and Preliminaries

An identity matrix of dimension n is denoted by I_n and e_i represent and its i-th column. A matrix of zeros with appropriate dimension is denoted as $\mathbf{0}$. The vector of ones with dimension m is denoted by $\mathbf{1}_m$. $X \succ 0\,(\succeq 0)$ denotes a positive (semi) definite matrix X. For compactness, in the text $*$'s will represent matrix entries that are uniquely identifiable from symmetry. Let $A \in \mathbb{R}^{m\times n}$ be a matrix written according to its n column vectors as $A = [\, a_1 \cdots a_n \,]$, we define vectorization of A as $\vec{A} \triangleq [\, a_1^\top \cdots a_n^\top \,]^\top$, which returns a vector of dimension $(mn\times 1)$, stacking the columns of A. For a finite set $\Theta_v = \{\theta^1, \theta^2, \ldots, \theta^r\}$ with $\theta^j \in \mathbb{R}^n$ for $j = 1, \ldots, r$, the convex-hull of Θ_v is given by, $\mathsf{conv}(\Theta_v) \triangleq \Big\{\theta \in \mathbb{R}^n : \theta = \sum_{j=1}^r \alpha_j \theta^j, \text{s.t } \sum_{j=1}^r \alpha_j = 1, \alpha_j \in [0, 1]\Big\}$. For matrices A and B of compatible dimensions, $A \otimes B$ denotes their Kronecker product. The following result will be used in the paper:

Lemma 1 (Vectorization): For matrices $A \in \mathbb{R}^{k\times l}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{k\times n}$, the matrix equation $ABC = D$ is equivalent to (see, [1, Ex. 10.18]),

$$(C^\top \otimes A)\vec{B} = \overrightarrow{ABC} = \vec{D} \tag{1}$$

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III. Problem Setting

A. Data-generating system and constraints

We consider the following discrete-time data-generating system

$$x(k+1) = Ax(k)+Bu(k)+w(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^n$ are the state, control input and the (additive) disturbance vectors at time k, respectively. The system matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are *unknown*. A state-input trajectory of $T+1$ samples $\{x(k), u(k)\}_{k=1}^{T+1}$ is generated from system (2). The generated data is represented with the following matrices,

$$X^+ \triangleq [x(2) \quad x(3) \quad \cdots \quad x(T+1)] \in \mathbb{R}^{n\times T}, \tag{3a}$$

$$X \triangleq [x(1) \quad x(2) \quad \cdots \quad x(T)] \in \mathbb{R}^{n\times T}, \tag{3b}$$

$$U \triangleq [u(1) \quad u(2) \quad \cdots \quad u(T)] \in \mathbb{R}^{m\times T}. \tag{3c}$$

The system (2) is subject to the following state, input and disturbance constraints, respectively:

$$\mathcal{X} \triangleq \{x : Hx \leq \mathbf{1}_{n_x}\}, \quad \mathcal{U} \triangleq \{u : Gu \leq \mathbf{1}_{n_u}\}, \tag{4a}$$

$$\mathcal{W} \triangleq \{w : -\mathbf{1}_{n_w} \leq Dw \leq \mathbf{1}_{n_w}\}, \tag{4b}$$

where $H \in \mathbb{R}^{n_x\times n}$, $G \in \mathbb{R}^{n_u\times m}$ and $D \in \mathbb{R}^{n_w\times n}$ are given matrices. The generated state samples in (3) are affected by bounded but *unknown* disturbance $w(k) \in \mathcal{W}$ for $k = 1, \ldots, T+1$.

B. Feasible model set and 'informative' data

We characterize a set of *feasible models* $M = \left[\, A \;\; B \,\right] \in \mathbb{R}^{n\times(n+m)}$, which are compatible with the measured data X^+, X, U and the disturbance set $\mathcal{W}$, defined as follows

$$\mathcal{M} \triangleq \left\{M : x(k+1) - M \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{W}, k = 1, \ldots, T\right\}. \tag{5}$$

Using the definitions of data matrices in (3) and disturbance set $\mathcal{W}$ in (4), the feasible model set $\mathcal{M}$ is represented as,

$$\mathcal{M} \triangleq \left\{M : -\bar{\mathbf{1}} \leq DX^+ - DM \begin{bmatrix} X \\ U \end{bmatrix} \leq \bar{\mathbf{1}}\right\}, \tag{6}$$

with $\bar{\mathbf{1}} \triangleq \left[\, \mathbf{1}_{n_w} \; \mathbf{1}_{n_w} \cdots \; \mathbf{1}_{n_w} \,\right] \in \mathbb{R}^{n_w\times T}$.

Proposition 1 ([5], [24]): The feasible model set $\mathcal{M}$ in (6) is a bounded polyhedron if and only if $\operatorname{rank}\left(\begin{bmatrix} X \\ U \end{bmatrix}\right) = n+m$ and D has a full column rank.

The above proposition relates to the "informative" data and *persistency of excitation* conditions [10]. The full row rank of $\begin{bmatrix} X \\ U \end{bmatrix}$ can be checked from the data, if this condition is not satisfied, the set $\mathcal{M}$ is unbounded which makes it difficult to find a feasible controller and RCI set for all $M \in \mathcal{M}$.

C. RCI set definition and invariance inducing controller

Let us consider a static state feedback control law

$$u(k) = \mathbf{K}x(k), \tag{7}$$

where $\mathbf{K} \in \mathbb{R}^{m\times n}$ is a feedback gain matrix. The resulting closed-loop dynamics for a feasible model $M \in \mathcal{M}$ (using (5) and (7)) is

$$x^+ = M \begin{bmatrix} I \\ \mathbf{K} \end{bmatrix} x + w, \tag{8}$$

where the k dependence is dropped and $x(k+1)$ is denoted as x^+ for convenience.

Let us consider the following polytopic set[1]

$$\mathcal{C} \triangleq \left\{x \in \mathbb{R}^n : -\mathbf{1}_{n_p} \leq P\mathbf{W}^{-1}x \leq \mathbf{1}_{n_p}\right\}, \tag{9}$$

where $P \in \mathbb{R}^{n_p\times n}$, $\mathbf{W} \in \mathbb{R}^{n\times n}$.

The set $\mathcal{C}$ is referred to as *robustly invariant* for the system (8), if the following condition is satisfied:

$$x \in \mathcal{C} \;\Rightarrow\; x^+ \in \mathcal{C}, \; \forall w \in \mathcal{W}, \;\; \forall M \in \mathcal{M}. \tag{10}$$

The set $\mathcal{C}$ has to satisfy the state and input constraints, this implies $\mathcal{C} \subseteq \mathcal{X}$ and $K\mathcal{C} \subseteq \mathcal{U}$, which can be further expressed as

$$x \in \mathcal{C} \;\Rightarrow\; x \in \mathcal{X}, \tag{11}$$

$$x \in \mathcal{C} \;\Rightarrow\; u = \mathbf{K}x \in \mathcal{U}. \tag{12}$$

The problem considered in this paper is formalized as follows:

Problem 1: Given data matrices (X^+, X, U) defined in (3), the constraints sets (4) and a fixed matrix P, find the matrix $\mathbf{W}$ defining the invariant set $\mathcal{C}$ in (9) and a feedback controller gain $\mathbf{K}$ such that: (i) The invariance condition (10) holds; (ii) All elements of the set $\mathcal{C}$ satisfy the state and input constraints (11) and (12), respectively.

We aim at maximizing the volume of set $\mathcal{C}$ solving Problem 1.

IV. Tractable formulations for System Constraints and Invariance Condition

In this section, we present a convenient coordinate transformation [13] such that state and control input constraints (11)-(12) are expressed as affine inequalities, while the invariance condition (10) is expressed as a set of LMIs.

A. System constraints

Let us consider the following state transformation

$$\theta = W^{-1}x \Leftrightarrow x = W\theta. \tag{13}$$

This allows us to express the set $\mathcal{C}$ in (9) as

$$\mathcal{C} \triangleq \{\mathbf{W}\theta \in \mathbb{R}^n : \theta \in \Theta\}, \tag{14}$$

where Θ is a symmetric set defined as follows:

$$\Theta \triangleq \left\{\theta \in \mathbb{R}^n : -\mathbf{1}_{n_p} \leq P\theta \leq \mathbf{1}_{n_p}\right\}. \tag{15}$$

Note that in the θ-state-space, the candidate invariant set Θ is a *known* symmetric set around the origin. The corresponding polytopic set $\mathcal{C}$ in the x-state-space will be completely determined by the choice of $\mathbf{W}$, which we aim to compute. As P is a known matrix, the symmetric set Θ can be expressed as the convex hull of the finitely many *known* vertices $\left\{\theta^1, \ldots, \theta^{2\sigma}\right\}$:

$$\Theta = \mathsf{conv}\left(\left\{\theta^1, \ldots, \theta^{2\sigma}\right\}\right), \tag{16}$$

where σ is some known positive integer determined by the choice of P. We now express the state and input inequality constraints (4) in the θ-state-space by using the transformation (13). Satisfaction of these inequalities constraints

[1] We have assumed that $\mathbf{W}$ is invertible, which would be later guaranteed by the LMI conditions for invariance. Choice of P is discussed in [14].

at the vertices $\{\theta^j\}_{j=1}^{2\sigma}$ ensures that they are satisfied over the whole set Θ as well. Therefore, we can write the state constraints (11) in terms of $\mathbf{W}$ as follows:

$$H\mathbf{W}\theta \leq \mathbf{1}_{n_x}, \forall \theta \in \Theta \;\Leftrightarrow\; H\mathbf{W}\theta^j \leq \mathbf{1}_{n_x},\; j=1,\ldots,2\sigma. \tag{17}$$

In order to express the control input constraints in terms of $\mathbf{W}$, let us consider a new matrix variable as follows:

$$\mathbf{N} \triangleq \mathbf{K}\mathbf{W} \;\Leftrightarrow\; \mathbf{K} = \mathbf{N}\mathbf{W}^{-1}. \tag{18}$$

The control input constraints in (12) are then given as

$$G\mathbf{N}\theta \leq \mathbf{1}_{n_u}, \forall \theta \in \Theta \;\Leftrightarrow\; G\mathbf{N}\theta^j \leq \mathbf{1}_{n_u},\; j=1,\ldots,2\sigma. \tag{19}$$

The system constraints (17) and (19) are affine and are identified by $n_x \times 2\sigma$ and $n_u \times 2\sigma$ scalar inequalities.

B. Invariance conditions in the transformed state-space

Let us express the system dynamics in the θ-state-space. Using (13), the closed-loop dynamics (8) can be written as

$$\mathbf{W}\theta^+ = M \begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix} \theta + w, \tag{20}$$

for a feasible model $M \in \mathcal{M}$ and $w \in \mathcal{W}$.

We now state two equivalent invariance conditions in the θ state-space based on the closed-loop dynamics (20).

Lemma 2: If the set Θ in (15) is robustly invariant for system (20) then the following two statements are equivalent:

(i) for all $\theta \in \Theta$, $\forall (w,M) \in (\mathcal{W},\mathcal{M})$,

$$\theta^+ = \left(\mathbf{W}^{-1}M\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\theta + \mathbf{W}^{-1}w\right) \in \Theta \tag{21}$$

(ii) for each vertex $\theta^j, j=1,\ldots,2\sigma$ of the set Θ, and $\forall (w,M) \in (\mathcal{W},\mathcal{M})$,

$$\theta^{j^+} = \left(\mathbf{W}^{-1}M\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\theta^j + \mathbf{W}^{-1}w\right) \in \Theta \tag{22}$$

Proof: Since for each vertex θ^j, it holds that $\theta^j \in \Theta$, it can be easily seen that $(i) \Rightarrow (ii)$. Let us now prove the converse statement, *i.e.*, $(ii) \Rightarrow (i)$. From (16), any $\theta \in \Theta$ can be expressed as a convex combination of the vertices,

$$\theta = \sum_{j=1}^{2\sigma}\alpha_j\theta^j, \quad \sum_{j=1}^{2\sigma}\alpha_j = 1, \quad \alpha_j \in [0,1]. \tag{23}$$

Then, based on the closed-loop dynamics (20) we get,

$$\begin{aligned}\theta^+ &= \mathbf{W}^{-1}M\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\left(\sum_{j=1}^{2\sigma}\alpha_j\theta^j\right) + \mathbf{W}^{-1}w\\ &= \sum_{j=1}^{2\sigma}\alpha_j \underbrace{\left(\mathbf{W}^{-1}M\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\theta^j + \mathbf{W}^{-1}w\right)}_{\theta^{j^+}}\end{aligned} \tag{24}$$

We know that $\theta^{j^+} \in \Theta, \forall (w,M) \in (\mathcal{W},\mathcal{M})$ according to (22). Since θ^+ in (24) is obtained as a convex combination of θ^{j^+} and as the set Θ is convex, it necessarily follows that $\theta^+ \in \Theta\ \forall (w,M) \in (\mathcal{W},\mathcal{M})$, thus proving $(ii) \Rightarrow (i)$. ■

In the rest of the paper, we will consider condition (22) for robust invariance of the set Θ.

C. Data-based LMI condition for invariance

We will now state and prove a data-based sufficient condition to render the set Θ invariant with an associated state-feedback controller. Let us first define the following matrix $Z \in \mathbb{R}^{Tn_w \times n(n+m)}$ and a vector $d \in \mathbb{R}^{Tn_w}$, which are constructed from the given state-input data (3) and a known disturbance set matrix D in (4).

$$Z \triangleq \left(\begin{bmatrix}X\\ U\end{bmatrix}^\top \otimes D\right),\; d \triangleq \begin{bmatrix}Dx(2)\\ \vdots\\ Dx(T+1)\end{bmatrix} \tag{25}$$

The following theorem states the data-based sufficient LMI feasibility condition for invariance and control.

Theorem 3 (Data-based LMI for invariance): Given data matrices (X^+, X, U) and a fixed matrix $P \in \mathbb{R}^{n_p \times n}$, if there exists $\mathbf{W} \in \mathbb{R}^{n \times n}$, $\mathbf{N} \in \mathbb{R}^{m \times n}$, and the variables $\{\boldsymbol{\phi}_{ij} \in \mathbb{R}_+, \boldsymbol{\Lambda}_{ij} \in \mathbb{D}_+^{Tn_w}, \boldsymbol{\Gamma}_{ij} \in \mathbb{D}_+^{n_w}\}$ that satisfy the following LMIs for $i=1,\ldots,n_p$ and $j=1,\ldots,2\sigma$,

$$\begin{bmatrix}\boldsymbol{r}_{ij} & -d^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathbf{0}\\ * & Z^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathcal{G}^\top(\mathbf{W},\mathbf{N},\theta^j)\\ * & * & D^\top\boldsymbol{\Gamma}_{ij}D & I_n\\ * & * & * & \mathbf{W}+\mathbf{W}^\top-\boldsymbol{\phi}_{i,j}P^\top e_i e_i^\top P\end{bmatrix} \succcurlyeq 0, \tag{26}$$

where,

$$\boldsymbol{r}_{ij} \triangleq \boldsymbol{\phi}_{ij} - \mathbf{1}^\top\boldsymbol{\Lambda}_{ij}\mathbf{1} - \mathbf{1}_{n_w}^\top\boldsymbol{\Gamma}_{ij}\mathbf{1}_{n_w} + d^\top\boldsymbol{\Lambda}_{ij}d \quad \in \mathbb{R}, \tag{27a}$$

$$\mathcal{G}\left(\mathbf{W},\mathbf{N},\theta^j\right) \triangleq \left(\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\theta^j\right)^\top \otimes I_n \quad \in \mathbb{R}^{n \times n(n+m)}, \tag{27b}$$

then, the state feedback controller gain is obtained as $\mathbf{K} = \mathbf{N}\mathbf{W}^{-1}$ which renders the set $\mathcal{C}$ in (14) robust invariant.

Proof: We first rewrite the feasible model set $\mathcal{M}$ in (6) for vector $\overrightarrow{M} \in \mathbb{R}^{n(n+m)}$ as follows,

$$\mathcal{M} \triangleq \left\{\overrightarrow{M} : -\mathbf{1}_{Tn_w} + d \leq Z\overrightarrow{M} \leq \mathbf{1}_{Tn_w} + d\right\}, \tag{28}$$

where we have used the identity (1) to rewrite the inequalities in (6) in a vector form and substituted Z, d as defined in (25).

Similarly, using the identity (1), we rewrite the closed-loop dynamics (20) at the vertex θ^j as follows,

$$\mathbf{W}\theta^{j^+} = \underbrace{\left(\left(\begin{bmatrix}\mathbf{W}\\ \mathbf{N}\end{bmatrix}\theta^j\right)^\top \otimes I_n\right)}_{\mathcal{G}(\mathbf{W},\mathbf{N},\theta^j)}\overrightarrow{M} + w. \tag{29}$$

From (15) the invariance condition in (22) can be written as, for all $i=1,\ldots,n_p,\; j=1,\ldots,2\sigma$,

$$1 - (e_i^\top P\theta^{j^+})^2 \geq 0,\; \forall w \in \mathcal{W},\; \forall \overrightarrow{M} \in \mathcal{M}, \tag{30}$$

where e_i is the i-th column vector of the identity matrix I_{n_p}.

We now multiply (30) by a positive scalar variable $\boldsymbol{\phi}_{ij} > 0$ and lower bound the left hand side by a term that is known to be non-negative for all $w \in \mathcal{W},\ \overrightarrow{M} \in \mathcal{M}$ (S-procedure [22]).

In this way, we obtain a sufficient condition for invariance as follows,

$$\begin{aligned}\boldsymbol{\phi}_{ij}(1-(e_i^\top P\theta^{j+})^2) \geq\; & 2\left(\theta^{j+}\right)^\top \underbrace{\left(\mathcal{G}(\mathbf{W},\mathbf{N},\theta^j)\overrightarrow{M}+w-\mathbf{W}\theta^{j+}\right)}_{0} \\ & + \underbrace{\left((\mathbf{1}+d)-Z\overrightarrow{M}\right)^\top \boldsymbol{\Lambda}_{ij}\left((\mathbf{1}-d)+Z\overrightarrow{M}\right)}_{\geq 0} \\ & + \underbrace{(\mathbf{1}+Dw)^\top \boldsymbol{\Gamma}_{ij}(\mathbf{1}-Dw)}_{\geq 0}, \end{aligned} \tag{31}$$

with $\boldsymbol{\Lambda}_{ij} \in \mathbb{D}_+^{Tn_w}, \boldsymbol{\Gamma}_{ij} \in \mathbb{D}_+^{n_w}$, being diagonal matrices having non-negative entries. Based on (29) and the set definitions $\mathcal{W}$, $\mathcal{M}$ in (4), (28) respectively, it is straightforward to verify that the right hand side of (31) is nonnegative. A sufficient invariance condition is obtained by re-arranging (31) into the following quadratic form,

$$\varkappa^\top \mathcal{P}_{ij}(\mathbf{W},\mathbf{N},\boldsymbol{\Lambda}_{ij},\boldsymbol{\Gamma}_{ij},\boldsymbol{\phi}_{ij})\varkappa \succcurlyeq 0,\ \forall \varkappa, \tag{32}$$

where $\varkappa^\top = \begin{bmatrix}1 & \overrightarrow{M}^\top & w^\top & -(\theta^{j+})^\top\end{bmatrix}$ and $\mathcal{P}_{ij}$ is a symmetric matrix given by the left hand side of (26). The invariance condition (22) holds if $\mathcal{P}_{ij} \succcurlyeq 0$. ■

D. Dilated data-based LMI condition for invariance

In this subsection, we derive a set of modified data-based LMI conditions for invariance. These LMIs have additional matrix variables and are potentially less conservative than those introduced in Theorem 3. We will now state the following dilated sufficient LMI conditions for invariance.

Theorem 4 (Dilated LMI conditions for invariance): Given data matrices (X^+, X, U) and a fixed matrix $P \in \mathbb{R}^{n_p \times n}$, if there exists $\mathbf{W} \in \mathbb{R}^{n\times n}$, $\mathbf{N} \in \mathbb{R}^{m \times n}$, and variables $\{\boldsymbol{\phi}_{ij} \in \mathbb{R}_+, \boldsymbol{\Lambda}_{ij} \in \mathbb{D}_+^{Tn_w}, \boldsymbol{\Gamma}_{ij} \in \mathbb{D}_+^{n_w}, \boldsymbol{X}_{ij}, \mathbf{V}_{i,j} \in \mathbb{R}^{n\times n}\}$ that satisfy the following LMIs for $i = 1,\ldots,n_p$ and $j = 1,\ldots,2\sigma$,

$$\begin{bmatrix}\mathbf{W}^\top+\mathbf{W}-\boldsymbol{X}_{ij} & \boldsymbol{\phi}_{ij}P^\top e_i \\ \boldsymbol{\phi}_{ij}e_i^\top P & \boldsymbol{\phi}_{ij}\end{bmatrix} \succcurlyeq 0. \tag{33}$$

$$\begin{bmatrix}\boldsymbol{r}_{ij} & -d^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & Z^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathcal{G}^\top\left(\mathbf{W},\mathbf{N},\theta^j\right) & \mathbf{0} \\ * & * & D^\top\boldsymbol{\Gamma}_{ij}D & I_n & \mathbf{0} \\ * & * & * & \mathbf{V}_{ij}+\mathbf{V}_{ij}^\top & \mathbf{V}_{ij}^\top \\ * & * & * & * & \boldsymbol{X}_{ij}\end{bmatrix} \succcurlyeq 0, \tag{34}$$

where, $\boldsymbol{r}_{ij}$, $\mathcal{G}(\mathbf{W},\mathbf{N},\theta^j)$ are as defined in (27a), (27b), then, the state feedback controller gain is obtained as $\mathbf{K} = \mathbf{N}\mathbf{W}^{-1}$ which renders the set $\mathcal{C}$ in (14) robust invariant.

Proof: Let us introduce new matrix variables $\mathbf{V}_{ij} \in \mathbb{R}^{n\times n}$ and signals $\xi_{ij} = \mathbf{V}_{ij}^{-1}\mathbf{W}\theta^{j+}$, for $i = 1,\ldots,n_p$ and $j = 1,\ldots,2\sigma$. From the dynamics (29) we obtain,

$$\mathcal{G}\left(\mathbf{W},\mathbf{N},\theta^j\right)\overrightarrow{M}+w-\mathbf{V}_{ij}\xi_{ij} = 0. \tag{35}$$

The sufficient condition in (31) is now expressed in the new introduced variables as follows:

$$\begin{aligned}\boldsymbol{\phi}_{ij}(1-(e_i^\top P\mathbf{W}^{-1}\mathbf{V}_{ij}\xi_{ij})^2) \geq\; & 2\xi_{ij}^\top \underbrace{\left(\mathcal{G}(\mathbf{W},\mathbf{N},\theta^j)\overrightarrow{M}+w-\mathbf{V}_{ij}\xi_{ij}\right)}_{0} \\ & + \underbrace{\left((\mathbf{1}+d)-Z\overrightarrow{M}\right)^\top \boldsymbol{\Lambda}_{ij}\left((\mathbf{1}-d)+Z\overrightarrow{M}\right)}_{\geq 0} \\ & + \underbrace{(\mathbf{1}+Dw)^\top \boldsymbol{\Gamma}_{ij}(\mathbf{1}-Dw)}_{\geq 0}. \end{aligned} \tag{36}$$

As described in the previous subsection, a sufficient condition for invariance is obtained by re-arranging (36) into the following quadratic form:

$$\varkappa^\top \mathcal{P}_{ij}(\mathbf{W},\mathbf{N},\boldsymbol{\Lambda}_{ij},\boldsymbol{\Gamma}_{ij},\boldsymbol{\phi}_{ij},\mathbf{V}_{ij})\varkappa \succcurlyeq 0,\ \forall \varkappa, \tag{37}$$

where $\varkappa^\top = \begin{bmatrix}1 & \overrightarrow{M}^\top & w^\top & -\xi_{ij}^\top\end{bmatrix}$ and $\mathcal{P}_{ij}$ is a symmetric matrix. The invariance condition thus holds if $\mathcal{P}_{ij} \succcurlyeq 0$, *i.e.*,

$$\begin{bmatrix}\boldsymbol{r}_{ij} & -d^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathbf{0} \\ * & Z^\top\boldsymbol{\Lambda}_{ij}Z & \mathbf{0} & \mathcal{G}^\top\left(\mathbf{W},\mathbf{N},\theta^j\right) \\ * & * & D^\top\boldsymbol{\Gamma}_{ij}D & I_n \\ * & * & * & \mathbf{V}_{ij}+\mathbf{V}_{ij}^\top-\mathbf{V}_{ij}^\top\mathcal{L}_{ij}\mathbf{V}_{ij}\end{bmatrix} \succcurlyeq 0 \tag{38}$$

where $\mathcal{L}_i \triangleq \boldsymbol{\phi}_{ij}\mathbf{W}^{-\top}P^\top e_i e_i^\top P\mathbf{W}^{-1}$ and $\boldsymbol{r}_{ij}$, $\mathcal{G}(\mathbf{W},\mathbf{N},\theta^j)$ are as defined in (27a), (27b) respectively. Note that the block $(4,4)$ in (38) has a nonlinear dependence on $\boldsymbol{\phi}_{i,j}, \mathbf{V}_{i,j}$ and $\mathbf{W}$, which is resolved by introducing a new matrix variable $\boldsymbol{X}_{ij} = \boldsymbol{X}_{ij}^\top \succ 0$ such that $\boldsymbol{X}_{ij}^{-1}-\mathcal{L}_{ij}\succ 0$. Then by applying Schur complement followed by a congruence transform, eq. (38) can be rewritten as the LMI in (34), see [18, Theorem 4] for the detailed proof. ■

V. Computation of RCI set with volume maximization

In this section, we present algorithms to maximize the volume of the RCI set, combining state, input constraints and LMI invariance conditions derived in the previous section in a SDP problem.

A. One-step algorithm:

For a fixed P, volume of the invariant set $\mathcal{C}$ in (9) is proportional to the determinant $|\det(\mathbf{W})|$ [14]. Moreover, the RCI set is required to satisfy the state constraints (17), control input constraints (19) as well as data-based LMI conditions for invariance (33)-(34) (or (26)). Under these constraints, we formulate a determinant maximization problem. Thus, Problem 1 is feasible if the following SDP program has a feasible solution,

Algorithm 1:

$$\begin{aligned} \max_{\boldsymbol{Z}_{\mathrm{SDP}}} \quad & \log\det(\mathbf{W}) \\ \text{subject to:} \quad & \mathbf{W} = \mathbf{W}^\top, \\ & (17), (19), \qquad \text{(state-input constraints)} \\ & (33)-(34) \text{ (or (26))} \qquad \text{(invariance LMIs)} \end{aligned} \tag{39}$$

where the optimization variables are $\boldsymbol{Z}_{\rm SDP} \triangleq (\mathbf{W}, \mathbf{N}, \boldsymbol{X}_{ij}, \mathbf{V}_{ij}, \boldsymbol{\phi}_{ij}, \boldsymbol{\Lambda}_{ij}, \boldsymbol{\Gamma}_{ij})$ for $i = 1, \dots, n_p, j = 1, \dots, 2\sigma$. The symmetry condition $\mathbf{W} = \mathbf{W}^\top$ is imposed to make the objective function $\log\det(\mathbf{W})$ concave.

B. Iterative algorithm:

We now present an *iterative* volume maximization scheme to compute the RCI set. In this approach, the SDP (39) is solved with an iterative procedure such that the solution obtained at the q-th iteration is utilized in the problem to be solved at the $(q+1)$-th iteration in order to reduce conservatism. In such iterative scheme, $\mathbf{W}$ is not required to be symmetric and the conservatism introduced due to the linearization can be reduced. Let W^q and X_{ij}^q denote the values of the variables $\mathbf{W}$, $\boldsymbol{X}_{ij}$ obtained at the q-th iteration. In order to ensure that at each iteration the volume of the RCI set increases, *i.e.*, $|\det(W^{q+1})| \geq |\det(W^q)|$, we impose the following,

$$\mathbf{W}^\top W^q + (W^q)^\top \mathbf{W} - (W^q)^\top W^q \succcurlyeq \mathbf{W}_{\rm obj} \succ 0, \quad (40)$$

where $\mathbf{W}_{\rm obj} = \mathbf{W}_{\rm obj}^\top \in \mathbb{R}^{n\times n}$ is the new symmetric matrix variable. Moreover, the non-linearity can be written as,

$$\mathbf{W}^\top \boldsymbol{X}_{ij}^{-1} \mathbf{W} \succcurlyeq \mathbf{W}^\top Z_{ij}^q + (Z_{ij}^q)^\top \mathbf{W} - (Z_{ij}^q)^\top \boldsymbol{X}_{ij} Z_{ij}^q, \quad (41)$$

and the $(1,1)$-block in (33) is replaced with the right hand side of (41) at the q-th iteration as follows,

$$\begin{bmatrix} \mathbf{W}^\top Z_{ij}^q + (Z_{ij}^q)^\top \mathbf{W} - (Z_{ij}^q)^\top \boldsymbol{X}_{ij} Z_{ij}^q & \boldsymbol{\phi}_{ij} P^\top e_i \\ \boldsymbol{\phi}_{ij} e_i^\top P & \boldsymbol{\phi}_{ij} \end{bmatrix} \succcurlyeq 0. \quad (42)$$

For brevity, we omit the detailed proof of the iterative algorithm. The reader is referred to [14], [18] for the details.

The iterative algorithm is summarized as follows:

Algorithm 2: q-th iteration:

$$\begin{array}{lll} \max\limits_{\boldsymbol{Z}_{\rm SDP}} & \log\det(\mathbf{W}_{\rm obj}) & \\ \text{subject to:} & (40), & \\ & (17), (19), & \text{(state-input constraints)} \\ & (34)\ (42), & \text{(invariance LMIs)} \end{array} \quad (43)$$

where the optimization variables are $\boldsymbol{Z}_{\rm SDP} \triangleq (\mathbf{W}, \mathbf{N}, \boldsymbol{X}_{ij}, \mathbf{V}_{ij}, \boldsymbol{\phi}_{ij}, \boldsymbol{\Lambda}_{ij}, \boldsymbol{\Gamma}_{ij}, \mathbf{W}_{\rm obj})$ for $i = 1, \dots, n_p, j = 1, \dots, 2\sigma$.

VI. Numerical Example

We demonstrate the effectiveness of the proposed approach via a numerical case study. All algorithms have been implemented in the `Python` environment using `cvxpy` package [11] utilizing `MOSEK` [19] to solve the SDP programs.

We consider an open-loop unstable double integrator system having dynamics described as in (2) with

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(k) + w(k). \quad (44)$$

Note that the system matrices (A, B) are *unknown*, but they are only used to gather the data. A single state-input trajectory of $T = 20$ samples is gathered (see, [18, Fig. 1]) by exciting the system (44) with inputs uniformly distributed in $[-2, 2]$. The data satisfies the rank conditions given in Proposition 1. The disturbance w is assumed to take values in the bound $[-0.1,\ 0.1]$. The state constraints are $(x_1, x_2) \in [-2, 2] \times [-2,\ 2]$ and the input constraints are $u \in [-2, 2]$.

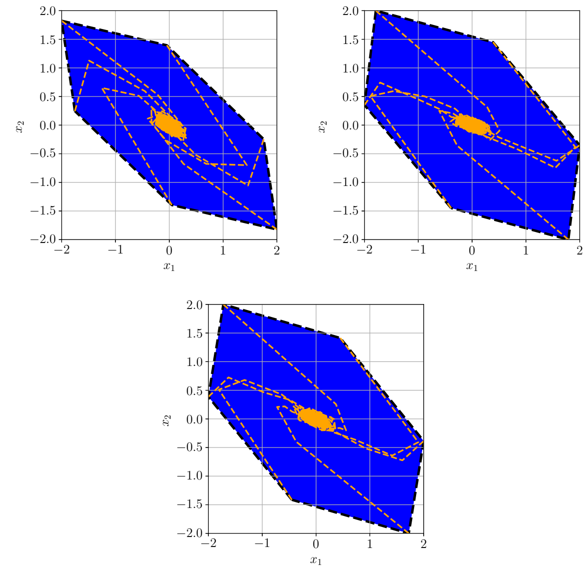

Fig. 1: Top left: DD 1-step algorithm (39), Top right: DD iterative approach (43), Bottom: Model-based approach [13].

Comparison between data-driven (DD) approaches and a model-based (MB) method: We compare the proposed DD algorithms with a model-based approach [13]. In the MB approach, exact values of the system matrices (A, B) are assumed to be known. The complexity of the RCI sets is selected as $n_p = 3$ by choosing $P = \begin{bmatrix} 10 & 10 \\ 10 & 0 \\ 1 & 11 \end{bmatrix}$. The RCI sets and the associated state-feedback control laws are computed by running one-step **Algorithm 1** solving (39) and **Algorithm 2** solving (43) iteratively for 5 iterations with dilated LMI conditions. We also compute the RCI set and control law based on dilated LMI conditions given in the model-based method [13]. The resulting RCI sets matrices and the state-feedback gains are obtained as follows:

$$\begin{bmatrix} \mathbf{W} \\ \hline \mathbf{K} \end{bmatrix} = \left[\begin{array}{cc} 17.54 & -2.46 \\ -2.46 & 15.77 \\ \hline -0.71 & -1.45 \end{array}\right], \text{ (Data-driven: 1-step)}$$

$$\begin{bmatrix} \mathbf{W} \\ \hline \mathbf{K} \end{bmatrix} = \left[\begin{array}{cc} 20.00 & 2.11 \\ -3.51 & 16.48 \\ \hline -0.38 & -1.21 \end{array}\right], \text{ (Data-driven: iterative)}$$

$$\begin{bmatrix} \mathbf{W} \\ \hline \mathbf{K} \end{bmatrix} = \left[\begin{array}{cc} 20.00 & 2.77 \\ -3.87 & 16.13 \\ \hline -0.41 & -1.18 \end{array}\right] \text{ (Model-based)}$$

The obtained RCI sets are depicted in Fig. 1. It can be observed that the proposed direct data-driven approach is able to generate RCI sets which are of comparable volume to

TABLE I: Comparison between DD and MB algorithms.

	DD: 1-step	DD: iterative	MB [13]
Volume of $\mathcal{C}$	8.31	9.86	9.50

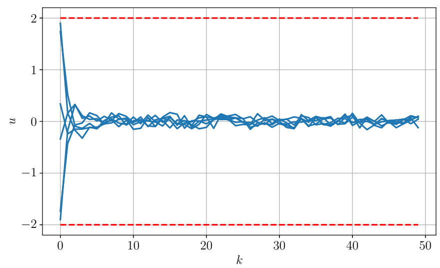

Fig. 2: Control input $u = \mathbf{K}x$ trajectories for the computed state-feedback gain (blue) and input constraints (dashed-red).

those obtained with the model-based method. The main advantage is that explicit knowledge of model matrices (A, B) is not required, thus avoiding an additional identification step. The corresponding volumes of the RCI sets are reported in Table I, which shows that iterative **Algorithm 2** with dilated LMI conditions generates relatively larger size RCI sets than those computed with the one-step **Algorithm 1**, which indicate that **Algorithm 2** is indeed less conservative for this example. Furthermore, Fig. 1 also shows closed-loop state trajectories starting from each vertex of the RCI set. These trajectories are obtained by simulating the true system in closed-loop with the state-feedback controller $u = \mathbf{K}x$. During the closed-loop simulation, a random disturbance uniformly distributed in the interval $[-0.1, 0.1]$ is acting on the system at each time instance. The figure shows that the approach guarantees robust invariance in the presence of a bounded but unknown disturbance while respecting the state-constraints. The corresponding input trajectories computed with the iterative data-driven algorithm are shown in Fig. 2. The figure shows that the input constraints are also satisfied. Further analyses on the effect of choosing different P matrices corresponding to different complexities of the polytope is presented in the report [18].

VII. CONCLUSIONS

We proposed a direct data-driven approach to compute a full complexity polytopic RCI set and an associated linear state-feedback control law. In the proposed algorithm neither the model of the system is required to be known nor any identification step is necessary. The algorithm is robust w.r.t. a set of all feasible models compatible with the available state-input data and satisfying the disturbance bounds. The direct data-driven approach is able to generate RCI sets with sizes that are comparable to that of an approach in which exact system knowledge is assumed. As a future work, the proposed approach can be extended to generate RCI sets and controllers for a more general class of systems, *e.g.*, linear parameter-varying and non-linear systems.

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