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High-dimensional continuification control of large-scale multi-agent systems under limited sensing and perturbations

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***Abstract*— This paper investigates the robustness of a novel high-dimensional continuification control method for complex multi-agent systems. We begin by formulating a partial differential equation describing the spatio-temporal density dynamics of swarming agents. A stable control action for the density is then derived and validated under nominal conditions. Subsequently, we discretize this macroscopic strategy into actionable velocity inputs for the system's agents. Our analysis demonstrates the robustness of the approach beyond idealized assumptions of unlimited sensing and absence of perturbations.**

I. Introduction

The continuification-based control approach generates discrete microscopic control protocols through the continuum, macroscopic approximation of large-scale multi-agent systems [1], [2]. Within such an approach, the system dynamics is first described using a traditional agent-based framework, in the form of a large set of ordinary differential equations (ODEs). Next, a macroscopic description of the emergent behavior is derived, often as a small set of partial differential equations (PDEs) – usually in the form of mass conservation laws, under the assumption of an infinite number of agents. Although the control design is developed based on this macroscopic approximation of the system behavior, the macroscopic control action is ultimately discretized into actionable microscopic inputs, as illustrated in Fig. 1.

One of the most constraining assumptions of this approach is the premise that agents possess unlimited sensing capabilities, enabling them to influence one another even over long distances. Also, the potential existence of perturbations or disturbances that affect the system dynamics is frequently overlooked. In this paper, we expand upon the analysis conducted in [3], which focused on one-dimensional domains, to evaluate the robustness of the high-dimensional continuification control strategy elucidated in [4]. Our aim is to relax (i) the assumption of agents having unlimited access to knowledge on the other agents' dynamics and (ii) the assumption of absence of perturbations. Specifically, by considering appropriate conditions and utilizing the macroscopic system formulation, we establish analytical guarantees of semi-global and bounded stability under limited sensing capabilities and perturbations. Each scenario is rigorously validated through comprehensive simulations.

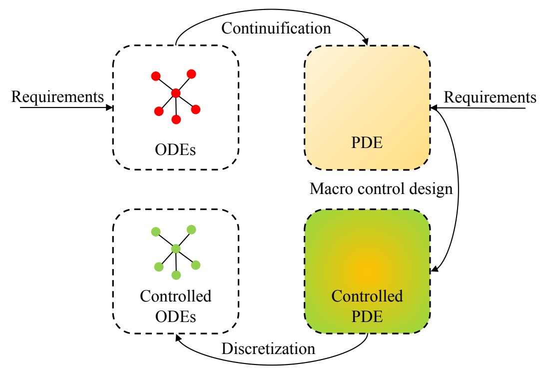

Fig. 1: Continuification control scheme (inspired by [2]).

The rest of the paper is organized as follows. In Section II, we provide useful mathematical notation; in Sections III, III-A and IV, we briefly recall the theoretical framework that is discussed in [4]; in Sections V and VI, we study the robustness of the solution. Analytical results are numerically validated in Section VII.

II. Mathematical preliminaries

Here, we give some mathematical notation that will be used throughout the paper. We define $\Omega := [-\pi, \pi]^d$, with $d = 1, 2, 3$ the periodic cube of side 2π. The case $d = 1$ corresponds to the unit circle, $d = 2$ to the periodic square, and $d = 3$ to the periodic cube. We refer to $\partial\Omega$ for indicating Ω's boundary.

Definition 1 (L^p-norm on Ω [5]) *Given a scalar function of $h : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, we define its L^p-norm as*

$$\|h(\cdot, t)\|_p := \left(\int_\Omega |h(\mathbf{x}, t)|^p \,\mathrm{d}\mathbf{x} \right)^{1/p}. \quad (1)$$

The case $p = \infty$ is defined as

$$\|h(\cdot, t)\|_\infty := \operatorname{ess\,sup}_\Omega |h(\mathbf{x}, t)|. \quad (2)$$

For brevity, we also denote these norms as $\|h\|_p$, without explicitly indicating their dependencies.

Notice that a vector-valued function is said to be L^p-bounded if all its components have a bounded L^p-norm.

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Lemma 1 (Holder's inequality [5]) *Given n L^p functions, f_i, with $i = 1, 2, \dots n$, we have*

$$\left\| \prod_{i=1}^{n} f_i \right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}, \text{ if } \sum_{i=1}^{n} \frac{1}{p_i} = 1. \tag{3}$$

For instance, if $n = 2$, $\|f_1 f_2\|_1 \leq \|f_1\|_2 \|f_2\|_2$ or $\|f_1 f_2\|_1 \leq \|f_1\|_1 \|f_2\|_\infty$.

Lemma 2 (Minkowsky inequality [5]) *Given two L^p functions, f and g, the following inequality holds:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \tag{4}$$

for $1 \leq p \leq \infty$.

We denote with subscripts t and x time and space partial derivatives. We indicate gradient as $\nabla(\cdot)$, divergence as $\nabla \cdot (\cdot)$, curl as $\nabla \times (\cdot)$, and Laplacian as $\nabla^2(\cdot)$.

We denote with " $*$ " the convolution operator. When referring to periodic domains and functions, the operator needs to be interpreted as a circular convolution [6]. We remark that the circular convolution is itself periodic. It can be shown [6] that we have

$$(f * g)_x(x) = (f_x * g)(x) = (f * g_x)(x). \tag{5}$$

Lemma 3 (Young's convolution inequality [5]) *Given two functions, $f \in L^p$ and $g \in L^q$, we have*

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q, \text{ if } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \tag{6}$$

where $1 \leq p, q, r \leq \infty$.

Lemma 4 (Comparison lemma [7]) *Given a scalar ODE $v_t = f(t, v)$, with $v(t_0) = v_0$, where f is continuous in t and locally Lipschitz in v, if a scalar function $u(t)$ fulfills the differential inequality*

$$u_t \leq f(t, u(t)), \;\; u(t_0) \leq v_0, \tag{7}$$

then,

$$u(t) \leq v(t), \;\; \forall\, t \geq t_0. \tag{8}$$

Lemma 5 (Chapter 1.2 of [8]) *Given a scalar function ψ, and a vector field $\mathbf{A}$, the following identity holds:*

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \nabla \psi \cdot \mathbf{A}. \tag{9}$$

Lemma 6 *For any function h that is periodic on $\partial\Omega$, we have*

$$\int_{\partial\Omega} h(\mathbf{x}) \cdot \hat{\mathbf{n}} \, \mathrm{d}\mathbf{x} = 0, \tag{10}$$

where $\hat{\mathbf{n}}$ is the is the outward pointing unit normal vector at each point on the boundary (by decomposing the integral on each side of the domain with the appropriate sign).

We denote by $\mathbb{n} = (n_1, \dots, n_d)$ the d-dimensional multi-index, consisting in the tuple of dimension d, with $n_i \in \mathbb{Z}$. Thus, $\mathbf{n} = [n_1, \dots, n_d]$ is the row vector associated with $\mathbb{n}$.

III. Model and Problem Statement

We consider N dynamical units moving in Ω. The agents' dynamics are modeled using the *kinematic assumption* [9], [10] (i.e., neglecting acceleration and considering a drag force proportional to the velocity). Specifically, we set

$$\dot{\mathbf{x}}_i = \sum_{k=1}^{N} \mathbf{f}\left(\{\mathbf{x}_i, \mathbf{x}_k\}\right) + \mathbf{u}_i, \quad i = 1, \dots, N, \tag{11}$$

where $\mathbf{x}_i \in \Omega$ is the i-th agent's position, and $\{\mathbf{x}_i, \mathbf{x}_k\}$ is the relative position between agent i and k, wrapped to have values in Ω (d-dimensional extension of what was defined in [1]), $\mathbf{f} : \Omega \to \mathbb{R}^d$ is a periodic velocity interaction kernel modeling pairwise interactions between the agents (repulsion, attraction, or a mix of the two at different ranges – see [4] for more details), and $\mathbf{u}_i$ is a velocity control input designed so as to fulfill some control problem.

A. Problem statement

The problem is that of selecting a set of control inputs $\mathbf{u}_i$ allowing the agents to organize into a desired macroscopic configuration on Ω. Specifically, given some desired periodic smooth density profile, $\rho^{\mathrm{d}}(\mathbf{x}, t)$, associated with the target agents' configuration, the problem can be reformulated as that of finding a set of control inputs $\mathbf{u}_i, \; i = 1, 2, \dots, N$ in (11) such that

$$\lim_{t \to \infty} \|\rho^{\mathrm{d}}(\cdot, t) - \rho(\cdot, t)\|_2 = 0, \tag{12}$$

for agents starting from any initial configuration.

IV. High-dimensional Continuification Control

In this section, we briefly recall the theoretical steps of the continuification-based control approach presented in [4].

A. Continuification

We recast the microscopic dynamics of the agents (11) as the mass balance equation [1], [10]

$$\rho_t(\mathbf{x}, t) + \nabla \cdot [\rho(\mathbf{x}, t) \mathbf{V}(\mathbf{x}, t)] = q(\mathbf{x}, t), \tag{13}$$

where

$$\mathbf{V}(\mathbf{x}, t) = \int_{\Omega} \mathbf{f}\left(\{\mathbf{x}, \mathbf{z}\}\right) \rho(\mathbf{z}, t) \, \mathrm{d}\mathbf{z} = (\mathbf{f} * \rho)(\mathbf{x}, t). \tag{14}$$

represents the characteristic velocity field encapsulating the interactions between the agents in the continuum. The scalar function q is the macroscopic control action.

We require periodicity of ρ on $\partial\Omega$ $\forall t \in \mathbb{R}_{\geq 0}$ and that $\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x})$. We remark that $\mathbf{V}$ is periodic by construction, as it comes from a circular convolution. Thus, for the periodicity of the density it is enough to ensure $\left(\int_{\Omega} \rho(\mathbf{x}, t) \, \mathrm{d}\mathbf{x}\right)_t = 0$, when $q = 0$ (using the divergence theorem and the periodicity of the flux).

B. Macroscopic Control Design

We assume the desired density profile, $\rho^{\mathrm{d}}(\mathbf{x},t)$, obeys to the mass conservation law

$$\rho_t^{\mathrm{d}}(\mathbf{x},t) + \nabla \cdot \left[\rho^{\mathrm{d}}(\mathbf{x},t)\mathbf{V}^{\mathrm{d}}(\mathbf{x},t)\right] = 0, \tag{15}$$

where

$$\mathbf{V}^{\mathrm{d}}(\mathbf{x},t) = \int_{\Omega} \mathbf{f}\left(\{\mathbf{x},\mathbf{z}\}\right)\rho^{\mathrm{d}}(\mathbf{z},t)\,\mathrm{d}\mathbf{z} = (\mathbf{f} * \rho^{\mathrm{d}})(\mathbf{x},t). \tag{16}$$

Periodic boundary conditions and initial condition for (15) are set similarly to those of (13). Furthermore, we define the error function $e(\mathbf{x},t) := \rho^{\mathrm{d}}(\mathbf{x},t) - \rho(\mathbf{x},t)$.

Theorem 1 (Macroscopic convergence) *Choosing*

$$q(\mathbf{x},t) = K_{\mathrm{p}}e(\mathbf{x},t) - \nabla \cdot \left[e(\mathbf{x},t)\mathbf{V}^{\mathrm{d}}(\mathbf{x},t)\right] - \nabla \cdot \left[\rho(\mathbf{x},t)\mathbf{V}^{\mathrm{e}}(\mathbf{x},t)\right], \tag{17}$$

where K_{p} is a positive control gain and $\mathbf{V}^{\mathrm{e}}(\mathbf{x},t) = (\mathbf{f} * e)(\mathbf{x},t)$, the error dynamics globally asymptotically converges to 0

$$\lim_{t\to\infty} e(\mathbf{x},t) = 0 \quad \forall\, e(\mathbf{x},0). \tag{18}$$

Proof: See Theorem 1 in [4]. ■

C. Discretization and Microscopic control

In order to dicretize the macroscopic control action q, we first recast the macroscopic controlled model as

$$\rho_t(\mathbf{x},t) + \nabla \cdot \left[\rho(\mathbf{x},t)\left(\mathbf{V}(\mathbf{x},t) + \mathbf{U}(\mathbf{x},t)\right)\right] = 0, \tag{19}$$

where $\mathbf{U}$ is a controlled velocity field, that incorporates the control action. Equation (19) is equivalent to (13), if

$$\nabla \cdot \left[\rho(\mathbf{x},t)\mathbf{U}(\mathbf{x},t)\right] = -q(\mathbf{x},t). \tag{20}$$

In contrast to the case of $d = 1$ discussed in [1], equation (20) is insufficient to uniquely determine $\mathbf{U}$ from q since it represents only a scalar relationship. Hence, we define the flux $\mathbf{w}(\mathbf{x},t) := \rho(\mathbf{x},t)\mathbf{U}(\mathbf{x},t)$, and close the problem by adding an extra differential constraint on the curl of $\mathbf{w}$. Namely, we consider the set of equations

$$\begin{cases} \nabla \cdot \mathbf{w}(\mathbf{x},t) = -q(\mathbf{x},t) \\ \nabla \times \mathbf{w}(\mathbf{x},t) = 0 \end{cases} \tag{21}$$

For problem (21) to be well-posed, we require $\mathbf{w}$ to be periodic on $\partial\Omega$. We remark that the choice of closing the problem using the irrotationality condition is arbitrary, and other closures can be considered.

Similarly to the typical approach used in electrostatic [8], since Ω is simply connected and $\nabla\times\mathbf{w} = 0$, we can express $\mathbf{w}$ using the scalar potential φ. Specifically, we pose $\mathbf{w}(\mathbf{x},t) = -\nabla\varphi(\mathbf{x},t)$. Plugging this into the divergence relation in (21), we recast (21) as the Poisson equation

$$\nabla^2\varphi(\mathbf{x},t) = q(\mathbf{x},t). \tag{22}$$

Problem (22) is characterized by the periodicity of $\nabla\varphi(\mathbf{x},t)$ on $\partial\Omega$. Thus, (22), together with its boundary conditions, defines φ up to a constant C. Since we are interested in computing $\mathbf{w} = -\nabla\varphi$, the value of C is irrelevant. We solve the Poisson problem (22) in Ω by expanding φ as a Fourier series. Specifically, we get

$$\varphi(\mathbf{x}) = \sum_{\mathbb{m}\in\mathbb{Z}^d} \gamma_{\mathbb{m}}\, \mathrm{e}^{\boldsymbol{j}\mathbf{m}\cdot\mathbf{x}} + C, \tag{23}$$

where, $\gamma_{\mathbf{m}}$ is the $\mathbb{m}$-th Fourier coefficient, j is the imaginary unit, and $\mathbf{x}$ is assumed to be a column. Given this expression for the potential, we write its Laplacian as

$$\nabla^2\varphi(\mathbf{x}) = \sum_{\mathbb{m}\in\mathbb{Z}^d} \gamma_{\mathbb{m}} \|\mathbf{m}\|^2 \mathrm{e}^{\boldsymbol{j}\mathbf{m}\cdot\mathbf{x}}. \tag{24}$$

Next, we can apply Fourier series to q, resulting in

$$q(\mathbf{x}) = \sum_{\mathbb{m}\in\mathbb{Z}^d} c_{\mathbb{m}}\, \mathrm{e}^{\boldsymbol{j}\mathbf{m}\cdot\mathbf{x}}, \tag{25}$$

where, since at time t the function q is known, we can also express the coefficients as

$$c_{\mathbb{m}} = \frac{1}{(2\pi)^d}\int_{\Omega} q(\mathbf{x})\mathrm{e}^{-\boldsymbol{j}\mathbf{m}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}. \tag{26}$$

Then, recalling (22), we express the coefficients of the Fourier series of the potential φ as

$$\gamma_{\mathbb{m}} = -\frac{c_{\mathbb{m}}}{\|\mathbf{m}\|^2}. \tag{27}$$

For a practical algorithmic implementation, when computing φ, we approximate it truncating the summation after some large M. Then, $\mathbf{w} = -\nabla\varphi$ and, $\mathbf{U} = \mathbf{w}/\rho$.

Finally, we compute the microscopic control inputs for the discrete set of agents by spatially sampling $\mathbf{U}(\mathbf{x},t)$, that is

$$\mathbf{u}_i(t) = \mathbf{U}(\mathbf{x}_i,t), \quad i = 1,2,\ldots,N. \tag{28}$$

Remark *The macroscopic velocity field $\mathbf{U}$ is well-defined only when $\rho \neq 0$. As $\mathbf{U}$ will be sampled at the agents locations, i.e. where the density is different from 0, we know $\mathbf{U}$ is well defined where it is needed. Moreover, for implementation, we will finally estimate the density starting from the agents positions with a Gaussian estimation kernel, making it always different from 0.*

V. Robustness to limited sensing

The macroscopic control law we propose in (17) is based on the non-local convolution term $\mathbf{V}^{\mathrm{e}}$. For computing such a control action, agents need to know e everywhere in Ω, meaning that they need to posses sensing capabilities to cover the whole set Ω.

Here, we relax this unrealistic assumption, considering agents only possess a limited sensing radius Δ, i.e., they can only measure e in a neighborhood of radius Δ located about their positions. We model such a case by considering a modified interaction kernel defined as

$$\hat{\mathbf{f}}(\mathbf{z}) = \begin{cases} \mathbf{f}(\mathbf{z}) & \text{if } \|\mathbf{z}\|_2 \le \Delta \\ \mathbf{0} & \text{otherwise} \end{cases}. \tag{29}$$

In this scenario, the macroscopic control law takes the form

$$\hat{q}(\mathbf{x},t) = K_\mathrm{p} e(\mathbf{x},t) - \nabla\cdot\left[e(\mathbf{x},t)\mathbf{V}^\mathrm{d}(\mathbf{x},t)\right] - \nabla\cdot\left[\rho(\mathbf{x},t)\hat{\mathbf{V}}^\mathrm{e}(\mathbf{x},t)\right], \quad (30)$$

where $\hat{\mathbf{V}}^\mathrm{e} = (\hat{\mathbf{f}} * e)$. Under control action (30), the error system dynamics may be written as

$$e_t(\mathbf{x},t) = -K_\mathrm{p} e(\mathbf{x},t) + \nabla\cdot\left[\rho^\mathrm{d}(\mathbf{x},t)\tilde{\mathbf{V}}(\mathbf{x},t)\right] - \nabla\cdot\left[e(\mathbf{x},t)\tilde{\mathbf{V}}(\mathbf{x},t)\right], \quad (31)$$

where $\tilde{\mathbf{V}} = (\mathbf{g} * e)$ and $\mathbf{g} = \hat{\mathbf{f}} - \mathbf{f}$.

Now, we provide some lemmas, that will be used for studying the stability properties of the perturbed error system (31).

Lemma 7 *The following inequality holds:*

$$\|\nabla\cdot\tilde{\mathbf{V}}\|_\infty \le \|e\|_2 \sum_{i=1}^{d} \|g_{i,x_i}\|_2,$$

where g_{i,x_i} is the x_i-derivative of the i-th component of $\mathbf{g}$.

Proof: Expanding $\nabla\cdot\tilde{\mathbf{V}}$ into its components (recalling the definition of convolution derivative in Section II), and using the Minkowsky inequality (see Lemma 2), we can write

$$\|\nabla\cdot\tilde{\mathbf{V}}\|_\infty = \left\|\sum_{i=1}^{d}(g_{i,x_i} * e)\right\|_\infty \le \sum_{i=1}^{d}\|(g_{i,x_i} * e)\|_\infty, \quad (32)$$

Using Young's convolution inequality, we construct the bound

$$\|\nabla\cdot\tilde{\mathbf{V}}\|_\infty \le \|e\|_2 \sum_{i=1}^{d} \|g_{i,x_i}\|_2, \quad (33)$$

proving the lemma. ∎

Lemma 8 *If $\nabla\rho^\mathrm{d} \in L^2$, i.e. $\|\rho^\mathrm{d}_{x_i}\|_2 \le M_i$, for some constants M_i and $i = 1, 2, 3$, then*

$$\|e\nabla\rho^\mathrm{d}\cdot\hat{\mathbf{V}}\|_1 \le \|e\|_2^2 \sum_{i=1}^{d} M_i\|g_i\|_2,$$

where g_i is the i-th component of $\mathbf{g}$.

Proof: By expanding $\nabla\rho^\mathrm{d}\cdot\hat{\mathbf{V}}$, we get

$$\|e\nabla\rho^\mathrm{d}\cdot\hat{\mathbf{V}}\|_1 = \left\|e\sum_{i=1}^{d}\rho^\mathrm{d}_{x_i}\tilde{V}_i\right\|_1 = \left\|e\sum_{i=1}^{d}\rho^\mathrm{d}_{x_i}(g_i * e)\right\|_1. \quad (34)$$

Then, applying Minkowsky (see Lemma 2) and the Holder (see Lemma 1) inequalities, we establish

$$\left\|e\sum_{i=1}^{d}\rho^\mathrm{d}_{x_i}(g_i * e)\right\|_1 \le \sum_{i=1}^{d}\left\|e\rho^\mathrm{d}_{x_i}(g_i * e)\right\|_1 \le \le \sum_{i=1}^{d}\|e\|_2\|\rho^\mathrm{d}_{x_i}\|_2\|(g_i * e)\|_\infty. \quad (35)$$

Finally, applying the Young's covolution inequality, we have

$$\sum_{i=1}^{d}\|e\|_2\|\rho^\mathrm{d}_{x_i}\|_2\|(g_i * e)\|_\infty \le \sum_{i=1}^{d}\|e\|_2^2\|\rho^\mathrm{d}_{x_i}\|_2\|g_i\|_2, \quad (36)$$

which, thanks to the L^2-boundedness of $\nabla\rho^\mathrm{d}$ is equivalent to

$$\sum_{i=1}^{d}\|e\|_2^2\|\rho^\mathrm{d}_{x_i}\|_2\|g_i\|_2 \le \|e\|_2^2\sum_{i=1}^{d}M_i\|g_i\|_2 \quad (37)$$

Comparing (34) and (37) yields the claim. ∎

Theorem 2 (Semiglobal stability with limited sensing) *If ρ_d and $\nabla\rho_\mathrm{d} \in L^2$, control strategy* (30) *achieves semiglobal stabilization of error dynamics* (31)*, so that, for any initial condition in the compact set $\|e(\cdot,0)\| < \gamma$, choosing K_p sufficiently large ensures the error to converge asymptotically to 0.*

Proof: We choose $\|e\|_2^2$ as a candidate Lyapunov function. Then, taking into account (31), we write (omitting explicit dependencies for simplicity)

$$(\|e\|_2^2)_t = \int_\Omega 2ee_t\,\mathrm{d}\mathbf{x} = -2K_\mathrm{p}\|e\|_2^2 + 2\int_\Omega e\nabla\cdot(\rho^\mathrm{d}\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} - 2\int_\Omega e\nabla\cdot(e\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x}. \quad (38)$$

This relation may be rewritten as

$$(\|e\|_2^2)_t = \int_\Omega 2ee_t\,\mathrm{d}\mathbf{x} = -2K_\mathrm{p}\|e\|_2^2 + 2\int_\Omega e\nabla\cdot(\rho^\mathrm{d}\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} - \int_\Omega e^2\nabla\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x}, \quad (39)$$

where, applying Lemma 5, the divergence theorem, and Lemma 6, we establish

$$\begin{aligned} 2\int_\Omega e\nabla\cdot(e\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} &= 2\int_\Omega \nabla\cdot(e^2\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} - 2\int_\Omega \nabla e\cdot(e\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} \\ &= 2\int_{\partial\Omega} e^2\tilde{\mathbf{V}}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathbf{x} - 2\int_\Omega \nabla e\cdot(e\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} = -2\int_\Omega \nabla e\cdot(e\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} \\ &= -\int_\Omega \nabla(e^2)\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x} = -\int_\Omega \nabla\cdot(e^2\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x} + \int_\Omega e^2\nabla\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x} \\ &= -\int_{\partial\Omega} e^2\tilde{\mathbf{V}}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathbf{x} + \int_\Omega e^2\nabla\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x} = \int_\Omega e^2\nabla\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x}. \end{aligned} \quad (40)$$

We can provide bounds for the last two terms of (39), namely

$$\begin{aligned} \left|\int_\Omega e\nabla\cdot(\rho^\mathrm{d}\tilde{\mathbf{V}})\,\mathrm{d}\mathbf{x}\right| &\le \int_\Omega \left|e\nabla\cdot(\rho^\mathrm{d}\tilde{\mathbf{V}})\right|\,\mathrm{d}\mathbf{x} \\ &= \|e\nabla\cdot(\rho^\mathrm{d}\tilde{\mathbf{V}})\|_1 = \|e\rho^\mathrm{d}\nabla\cdot\tilde{\mathbf{V}} + e\nabla\rho^\mathrm{d}\cdot\tilde{\mathbf{V}}\|_1 \le \\ &\le \|e\rho^\mathrm{d}\nabla\cdot\tilde{\mathbf{V}}\|_1 + \|e\nabla\rho^\mathrm{d}\cdot\tilde{\mathbf{V}}\|_1 \le \|e\|_2\|\rho^\mathrm{d}\|_2\|\nabla\cdot\tilde{\mathbf{V}}\|_\infty \\ &+ \|e\|_2^2\sum_{i=1}^{d}M_i\|g_i\|_2 \le \|e\|_2^2\left(\sum_{i=1}^{d}L\|g_{i,x_i}\|_2 + M_i\|g_i\|_2\right), \end{aligned} \quad (41)$$

$$\left|\int_\Omega e^2\nabla\cdot\tilde{\mathbf{V}}\,\mathrm{d}\mathbf{x}\right| \le \int_\Omega \left|e^2\nabla\cdot\tilde{\mathbf{V}}\right|\,\mathrm{d}\mathbf{x} = \|e^2\nabla\cdot\tilde{\mathbf{V}}\|_1 \le$$

$$\le \|e\|_2^2\|\nabla\cdot\tilde{\mathbf{V}}\|_\infty \le \|e\|_2^3\sum_{i=1}^d \|g_{i,x_i}\|_2, \quad (42)$$

where L is a positive constant bounding $\|\rho^{\mathrm{d}}\|_2$, and we used Lemma 7 and 8, as well as the Holder's inequality. Ultimately, we establish that

$$(\|e\|_2^2)_t \le (-2K_{\mathrm{p}} + F + G\|e\|_2)\|e\|_2^2, \quad (43)$$

where

$$F = 2\sum_{i=1}^d L\|g_{i,x_i}\|_2 + M_i\|g_i\|_2, \quad (44)$$

$$G = \sum_{i=1}^d \|g_{i,x_i}\|_2. \quad (45)$$

Choosing $K_{\mathrm{p}} > (F+G\gamma)/2$, the error asymptotically converges to 0, as it ensures γ is in the basin of attraction of the origin. ■

VI. Structural Perturbations

Here, we assume our system to be perturbed by spatio-temporal disturbances. In particular, we study

$$\rho_t(\mathbf{x},t) + \nabla\cdot[\rho(\mathbf{x},t)\,(\mathbf{V}(\mathbf{x},t)+\mathbf{W}(\mathbf{x},t))] = q(\mathbf{x},t), \quad (46)$$

where $\mathbf{W}$ is a perturbing velocity field. Further, we hypothesize (i) $\mathbf{W}$ to be periodic on $\partial\Omega$, (ii) components of $\mathbf{W}$ to be L^∞ bounded by some positive constants $\bar{W}_i$ $(i=1,2,3)$, and (iii) $\|\nabla\cdot\mathbf{W}\|_\infty \le \widehat{W}$. In such a scenario, the error dynamics become

$$e_t(\mathbf{x},t) = -K_{\mathrm{p}}e(\mathbf{x},t) + \nabla\cdot[\rho^{\mathrm{d}}(\mathbf{x},t)\mathbf{W}(\mathbf{x},t)] - \nabla\cdot[e(\mathbf{x},t)\mathbf{W}(\mathbf{x},t)]. \quad (47)$$

Theorem 3 (Bounded stability with perturbations) *In the presence of a bounded spatio-temporal disturbance* $\mathbf{W}$, *and if* $\|\rho^{\mathrm{d}}\|_2 \le L$ *and* $\|\rho^{\mathrm{d}}_{x_i}\| \le M_i$ $(i=1,\dots,d)$, *there exists a threshold value* $\kappa > 0$, *such that for* $K_{\mathrm{p}} > \kappa$, *the dynamics of* $\|e\|_2^2$ *remains bounded. Specifically,*

$$\lim_{t\to\infty}\sup\|e(\cdot,t)\|_2 \le \frac{H}{2K_{\mathrm{p}}-\widehat{W}}, \quad (48)$$

with $H = 2\left(L\widehat{W} + \sum_{i=1}^d M_i\bar{W}_i\right)$.

Proof: We write the dynamics of $\|e\|_2^2$ as

$$(\|e\|_2^2)_t = 2\int_\Omega ee_t\,\mathrm{d}\mathbf{x} = -2K_{\mathrm{p}}\|e\|_2^2 + 2\int_\Omega e\nabla\cdot(\rho^{\mathrm{d}}\mathbf{W})\,\mathrm{d}\mathbf{x} - 2\int_\Omega e\nabla\cdot(e\mathbf{W})\,\mathrm{d}\mathbf{x}. \quad (49)$$

Similarly to the proof of Theorem 2 (specifically (39) and (40)), we can rewrite (49) as

$$(\|e\|_2^2)_t = 2\int_\Omega ee_t\,\mathrm{d}\mathbf{x} = -2K_{\mathrm{p}}\|e\|_2^2 + 2\int_\Omega e\nabla\cdot(\rho^{\mathrm{d}}\mathbf{W})\,\mathrm{d}\mathbf{x} - \int_\Omega e^2\nabla\cdot\mathbf{W}\,\mathrm{d}\mathbf{x}. \quad (50)$$

Similarly to (41) and (42) in the proof of Theorem 2, we can give the bounds

$$\left|\int_\Omega e\nabla\cdot(\rho^{\mathrm{d}}\mathbf{W})\,\mathrm{d}\mathbf{x}\right| \le \int_\Omega \left|e\nabla\cdot(\rho^{\mathrm{d}}\mathbf{W})\right|\,\mathrm{d}\mathbf{x}$$
$$= \|e\nabla\cdot(\rho^{\mathrm{d}}\mathbf{W})\|_1 = \|e\rho^{\mathrm{d}}\nabla\cdot\mathbf{W} + e\nabla\rho^{\mathrm{d}}\cdot\mathbf{W}\|_1 \le$$
$$\le \|e\rho^{\mathrm{d}}\nabla\cdot\mathbf{W}\|_1 + \|e\nabla\rho^{\mathrm{d}}\cdot\mathbf{W}\|_1 \le \|e\|_2\|\rho^{\mathrm{d}}\|_2\|\nabla\cdot\mathbf{W}\|_\infty +$$
$$+ \|e\|_2\sum_{i=1}^d \|\rho^{\mathrm{d}}_{x_i}\|_2\|W_i\|_\infty \le \frac{H}{2}\|e\|_2, \quad (51)$$

$$\left|\int_\Omega e^2\nabla\cdot\mathbf{W}\,\mathrm{d}\mathbf{x}\right| \le \int_\Omega \left|e^2\nabla\cdot\mathbf{W}\,\mathrm{d}\mathbf{x}\right| = \left\|e^2\nabla\cdot\mathbf{W}\right\|_1 \le$$
$$\le \|e\|_2\|e\|_2\|\nabla\cdot\mathbf{W}\|_\infty \le \widehat{W}\|e\|_2^2 \quad (52)$$

Then, setting $\eta = \|e\|_2^2$, we establish

$$\eta_t \le -A\eta + H\sqrt{\eta}, \quad (53)$$

where H is given in the theorem statement, and $A = 2K_{\mathrm{p}} - \widehat{W}$. If we assume A to be positive, i.e., $2K_{\mathrm{p}} > \widehat{W}$, the bounding field is exhibiting a global asymptotically stable equilibrium point at H^2/A^2. Then, thanks to the Lemma 4, (48) is recovered. Hence, if $K_{\mathrm{p}} > \kappa > \widehat{W}/2$, $\|e\|_2$ remains bounded by H/A. ■

Remark *Similarly to [3] (Section VI), the bounded steady-state error may be reduced through a spatially distributed integral action.*

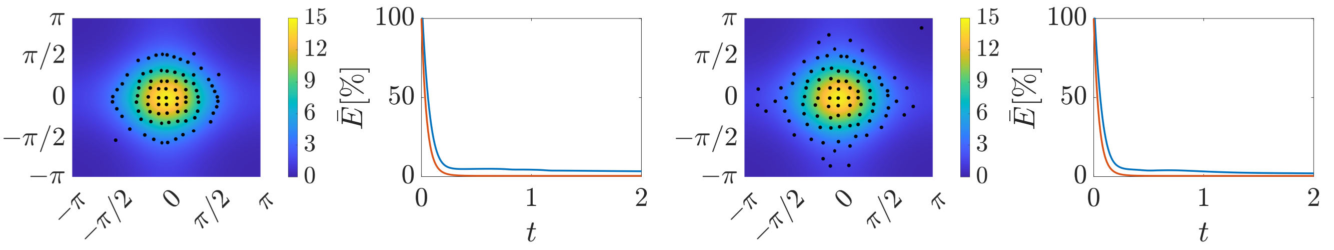
VII. Numerical validation

For numerical validation, we adopt the configuration presented in [4]. In this setup, in the absence of control mechanisms, agents engage through a repulsive periodic kernel, leading to their dispersal across the domain towards a uniform density. The target density configuration to be achieved is a 2D von Mises distribution with its center at the domain's origin. For details on this setup, the reader is directed to [4].

We consider a sample of 100 agents starting from a constant density profile, and, for each trial, we run both a discrete and a continuous simulation. This means that, in every scenario, we numerically integrate both (11) and its continuified version (13), allowing us to understand how well the continuum approximation holds. For the discrete trials, we use forward Euler with $\Delta t = 0.001$, and, for the computation of spatial functions involved in the definition of $\mathbf{u}_i$, we discretize Ω using a mesh of 50×50 cells. We remark that agents are not constrained to move on this mesh, that is merely used for evaluating spatial functions. For the numerical integration of the continuified problem, we use a Lax-Friedrichs finite volumes scheme [11] with the same Δt and spatial mesh of the discrete simulations.

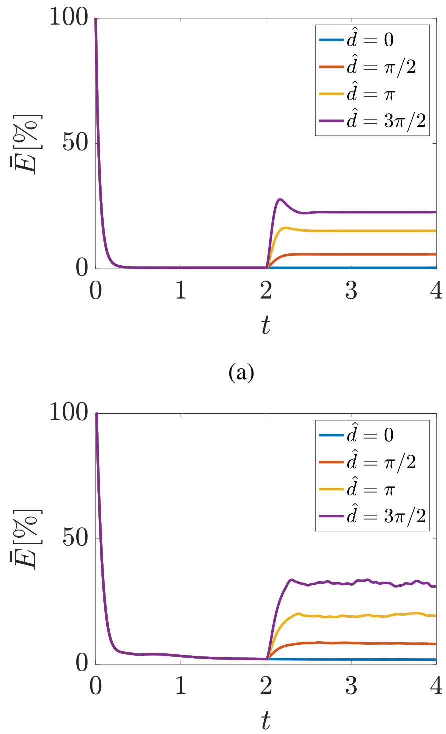
The performance of the trials is assessed using the normalized percentage error

$$\bar{E}(t) = \frac{\|e(\cdot,t)\|_2^2}{\max_t\|e(\cdot,t)\|_2^2}100. \quad (54)$$



(a) (b) (c) (d)

Fig. 2: Robustness to limited sensing ($K_\mathrm{p} = 100$, (a, b) $\Delta = 0.1\pi$, (c, d) $\Delta = \pi$). (a, c) Final displacement of the system on top of the desired density; (b, d) time evolution of the percentage error for the discrete (blue) and continuous case (orange).



(a)

(b)

Fig. 3: Robustness to perturbations ($K_\mathrm{p} = 100$): percentage error in time for (a) a continuous trial, (b) a discrete trial.

a) Robustness to limited sensing: to validate the stability result of Theorem 2, we fix $K_\mathrm{p} = 100$. When running a trial of 200 time steps, we obtain the results in Fig. 2a, 2b choosing $\Delta = 0.1\pi$ (i.e., agents have a sensing radius of 10% of the domain), and those in Fig. 2c, 2d with $\Delta = \pi$ (i.e., unlimited sensing). This choice of K_p ensures the performance is independent of the sensing capabilities of the agents. In the discrete trials, we observe a non-zero steady-state error. This is due to the finite-size effect of assuming a swarm of 100 agents. This residual error is slightly worse in the case of limited sensing.

b) Robustness to perturbations: to numerically assess robustness to perturbations, we consider a step disturbance of amplitude $\hat{d}$ on both the x and y direction coming at half of the trial, that is $\mathbf{W}(x,t) = \hat{d}\,[\mathrm{step}(t - tf/2), \mathrm{step}(t - tf/2)]^T$. In this case, for a trial of 400 time steps, we observe the results in Fig. 3a for the continuous case and Fig. 3b for the discrete one. For both scenarios, we observe that, when the perturbation is active, the error settles to a bounded value, confirming findings in Theorem 3. We also remark that the error settles well below the theoretical estimate of Theorem 3, for example, when $\hat{d} = 3\pi/2$, $H/A \approx 0.8$, while $\|e(\cdot, t_\mathrm{f})\|_2 \approx 0.1$.

VIII. Conclusions

Upon recalling the theoretical framework presented in [4], we analytically assessed the robustness of the control solution with respect to limited sensing capabilities and perturbations. We demonstrated that, out of the nominal condition, stability can still be preserved. Future work aims at the distributed and experimental implementation of the proposed control solution, eventually highlighting links with the optimal transport Literature.

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