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Robust finite-time stabilization of stochastic parabolic PDE systems via non-fragile spatial sampled-data control

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Abstract— **This paper addresses the robust finite-time stabilization (FTS) issue for stochastic parabolic PDE systems via non-fragile spatial sampled-data control scheme. First, a class of distributed parameter systems characterized by the delayed stochastic parabolic partial differential equation is developed for analyzing the effects of stochastic disturbance, structural uncertainty, and discrete delay on the system performance. Then, a non-fragile spatial sampled-data control scheme is established by setting sampling points in the spatial domain, which effectively saves communication resources and ensures that the closed-loop system maintains good performance when the controller is perturbed. Moreover, based on the partial differential equation theory, stochastic analysis approach, and the extended Wirtinger's inequality technique, several criteria are provided to ensure the robust FTS of stochastic parabolic PDE systems in the mean square sense. Lastly, a numerical example is provided to verify the feasibility of the suggested stabilization criteria and control scheme.**

Index Terms— **Robust finite-time stabilization, parabolic PDE systems, stochastic disturbance, structural uncertainty, non-fragile spatial sampled-data control.**

I. INTRODUCTION

In practice, many industrial processes are characterized both by their time evolution and spatial distribution, such as elastic vibration processes, chemical reaction processes, and heat conduction processes [1]. As an important class of distributed parameter systems, parabolic PDE systems, have received much attention for describing industrial processes in which the state space behaves as a spatiotemporal distribution more accurately and efficiently [2]. It is to be noticed that many practical systems are frequently perturbed by stochastic factors from the external environment during operation, while classical deterministic parabolic PDE systems cannot inscribe stochastic objects [3], [4]. Accordingly, stochastic parabolic PDE systems capable of describing stochastic disturbance have gradually become a hot research topic.

It is well established that stability is a fundamental issue in the performance analysis of stochastic parabolic PDE systems and a prerequisite for application in practical engineering [5]. In comparison to Lyapunov stability, which is employed to characterize the steady-state performance of the system, finite-time stability focuses on depicting the transient performance of the system [6], [7]. Finite-time stability, as the name suggests, is the state of the system under the given initial condition never exceeding a predetermined region within a specific time range. Contrary to traditional Lyapunov stability, finite-time stability has more practical significance in scientific research and engineering applications such as missile launch and aircraft trajectory control [8], [9]. Furthermore, uncertain parameters and discrete delay are inevitable in stochastic parabolic PDE systems due to factors such as measurement and modeling errors and the inherent communication time of signal transmission [10]. Considering that this structural uncertainty and discrete delay may cause the performance of the system to deteriorate or even become unstable, this paper explores the robust finite-time stabilization (FTS) of stochastic parabolic PDE systems with uncertain parameters and discrete delay.

To implement the robust FTS of stochastic parabolic PDE systems, a distributed controller was designed in [11]. Whereas, the suggested controller requires continuous updating of the control input. Different from continuous control, sampled-data control is a control scheme that intermittently updates the control input, effectively reducing the unnecessary waste of resources during signal transmission [12]. With the rapid development of information technology, many scholars have gradually favored the employment of sampled-data control to achieve the robust FTS of stochastic parabolic PDE systems [13]. It is worth highlighting that in the digital implementation of the controller, the control gain will inevitably be disturbed or offset due to inherent errors and equipment aging, thereby affecting the stable operation of the closed-loop system [14]. Consequently, how establishing a sampled-data control scheme that is insensitive to the uncertainty of the control gain to achieve the robust FTS of stochastic parabolic PDE systems is an essential and worthy problem to be studied.

Driven by the mentioned analysis, this paper concerns the problems of robust FTS and non-fragile control of stochastic parabolic PDE systems with uncertain parameters and discrete delay. The major work is listed below.

- A class of distributed parameter systems characterized by the delayed stochastic parabolic partial differential equation is constructed to handle the effects of practical factors such as stochastic disturbance, structural uncertainty, and discrete delay in a reasonable approximation. Different from [4], [7], [11], [13], this paper simultaneously considers the disturbances from the external environment and the internal system, and the established model is more universal and practical.

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- Compared with the controller designed in [3], [4], [7], [11], [13], a non-fragile spatial sampled-data controller is presented, which effectively saves communication resources and reduces control costs in the spatial domain. Moreover, the closed-loop system has good resilience to external perturbation and parameter deviation. When the control gain changes within the allowable range, the non-fragile spatial sampled-data control scheme still maintains the robust FTS of stochastic parabolic PDE systems.
- In contrast to [3], [4], [7], [11], [13], an extended vector-valued Wirtinger's inequality is employed to address the second-order partial derivative term in stochastic parabolic PDE systems to further relax the constraints on the spatial domain. Then, a more general Lyapunov functional is developed that do not require the decision variable P to be a diagonal matrix. Additionally, several less conservative robust FTS conditions are derived by taking full advantage of the nonlinear function information.

Notations: Throughout this paper, $\mathbb{R}^{\hbar}$ and $\mathbb{R}^{\hbar_1\times\hbar_2}$ mean the $\hbar$-dimensional Euclidean space and the set of real matrices of order $\hbar_1 \times \hbar_2$, respectively. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are the maximum and minimum eigenvalues of the matrix respectively. $\mathrm{Tr}(\cdot)$ indicates the trace of the matrix. $\Xi(\cdot)$ is the mathematical expectation. In addition, $\|\varpi(\cdot,\epsilon)\|^2 = \int_{\hat{\sigma}}^{\check{\sigma}} \omega^T(\sigma,\epsilon)\omega(\sigma,\epsilon)d\sigma$.

II. Preliminaries

Consider the stochastic parabolic PDE systems with parameter uncertainties and discrete delay as follows

$$\begin{aligned}\mathrm{d}\varpi(\sigma,\epsilon) = &\Big[\tilde{D}\varpi_{\sigma\sigma}(\sigma,\epsilon) + \tilde{A}\varpi(\sigma,\epsilon) + \tilde{B}\zeta(\varpi(\sigma,\epsilon)) \\ &+\tilde{C}\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon))) + u(\sigma,\epsilon)\Big]\,\mathrm{d}\epsilon \\ &+ \Lambda(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon)))\mathrm{d}\vartheta(\epsilon), \qquad (1)\end{aligned}$$

where $\varpi(\sigma,\epsilon) \in \mathbb{R}^m$ means the state of the system, in which $\sigma \in [\hat{\sigma},\check{\sigma}]$ and $\epsilon \in [-\hat{\varphi},+\infty)$ denote the spatial and temporal variables, respectively. Then, $\varpi_{\sigma\sigma}(\sigma,\epsilon) = \frac{\partial^2\varpi(\sigma,\epsilon)}{\partial\sigma^2}$. $\varphi(\epsilon)$ stands for the transmission delay meeting $0 \le \varphi(\epsilon) \le \hat{\varphi}$ and $\dot{\varphi}(\epsilon) \le \check{\varphi} < 1$. $\vartheta(\epsilon)$ means the ν-dimensional standard Brownian motion. The boundary conditions and initial value are $\varpi(\hat{\sigma},\epsilon) = \varpi(\check{\sigma},\epsilon) = 0$ and $\varpi(\sigma,s) = \psi(\sigma,s)$, $(\sigma,s) \in [\hat{\sigma},\check{\sigma}] \times [-\hat{\varphi},0)$, respectively. $\zeta(\varpi) = (\zeta_1(\varpi_1),\zeta_2(\varpi_2),\ldots,\zeta_m(\varpi_m))^T \in \mathbb{R}^m$ is the nonlinear function. $\Lambda(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon))) \in \mathbb{R}^{m\times\nu}$ denotes the noise intensity matrix. $\tilde{D} = D + \Delta D(\sigma,\epsilon)$, $\tilde{A} = A + \Delta A(\sigma,\epsilon)$, $\tilde{B} = B + \Delta B(\sigma,\epsilon)$, and $\tilde{C} = C + \Delta C(\sigma,\epsilon)$, in which D, A, B, $C \in \mathbb{R}^{m\times m}$, and $\Delta D(\sigma,\epsilon)$, $\Delta A(\sigma,\epsilon)$, $\Delta B(\sigma,\epsilon)$, and $\Delta C(\sigma,\epsilon)$ indicate unknown matrices expressing spatiotemporal uncertainty. $u(\sigma,\epsilon)$ is the controller to be designed. Besides, system (1) satisfies the assumptions as follows.

Assumption 1: For any μ_1, $\mu_2 \in \mathbb{R}$ with $\mu_1 \neq \mu_2$, there are o_i^- and o_i^+ such that $\zeta_i(\cdot)$, $i = 1,2,\cdots,m$, satisfy

$$o_i^- \le \frac{\zeta_i(\mu_1) - \zeta_i(\mu_2)}{\mu_1 - \mu_2} \le o_i^+. \qquad (2)$$

Assumption 2: There are constants $\varrho_1 > 0$ and $\varrho_2 > 0$ such that noise intensity matrix $\Lambda(\epsilon,\eta_1,\eta_2)$ meets

$$\mathrm{Tr}(\Lambda^T(\epsilon,\eta_1,\eta_2)\Lambda(\epsilon,\eta_1,\eta_2)) \le \varrho_1\eta_1^T\eta_1 + \varrho_2\eta_2^T\eta_2. \qquad (3)$$

Assumption 3: There exist constant matrices $M \in \mathbb{R}^{m\times m_1}$, N_1, N_2, $N_3 \in \mathbb{R}^{m_2\times m}$, and unknown spatiotemporal function matrix $F(\sigma,\epsilon) \in \mathbb{R}^{m_1\times m_2}$ satisfying $F^T(\sigma,\epsilon)F(\sigma,\epsilon) \le I$, such that

$$[\Delta D, \Delta A, \Delta B, \Delta C] = MF(\sigma,\epsilon)[N_1,N_2,N_3,N_4]. \qquad (4)$$

In the following, we divide $[\hat{\sigma},\check{\sigma}]$ into n intervals by inserting $n-1$ points with $\hat{\sigma} = \sigma_0 < \sigma_1 < \ldots < \sigma_n = \check{\sigma}$. Then, there exists $\alpha > 0$ such that the sampling intervals satisfy $\sigma_{\iota+1} - \sigma_\iota \le \alpha$, $\iota = 0,1,\ldots,n-1$. The non-fragile spatial sampled-data control scheme is provided below

$$u(\sigma,\epsilon) = (K + \Delta K(\sigma_\iota,\epsilon))\varpi(\sigma_\iota,\epsilon), \qquad (5)$$

where K expresses the control gain matrix. Additionally, $\Delta K(\sigma_\iota,\epsilon) = EG(\sigma_\iota,\epsilon)H$ indicates the gain perturbation matrix, in which $E \in \mathbb{R}^{m\times m_3}$, $H \in \mathbb{R}^{m_4\times m}$, and $G(\sigma_\iota,\epsilon) \in \mathbb{R}^{m_3\times m_4}$ means the unknown spatiotemporal function matrix meeting $G^T(\sigma_\iota,\epsilon)G(\sigma_\iota,\epsilon) \le I$.

Then, we provide the necessary definition and lemma.

Definition 1: Given constants $0 < \delta_1 < \delta_2$ and $T > 0$, system (1) realizes robust FTS in the mean square sense with respect to (δ_1,δ_2,T), if there is a suitable control scheme $u(\sigma,\epsilon)$ such that for any $\epsilon \in [0,T]$,

$$\Xi\left(\sup_{s\in[-\hat{\varphi},0)} \|\varpi(\cdot,s)\|^2\right) \le \delta_1 \Rightarrow \Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right) \le \delta_2. \qquad (6)$$

Lemma 1 [15]: Let $\chi(\cdot) \in \mathbb{R}^m$ be an absolutely continuous function with a square-integrable derivative of order 1 and satisfy $\chi(\hat{\sigma}) = 0$ or $\chi(\check{\sigma}) = 0$. Then, for any $Z \ge 0$,

$$\int_{\hat{\sigma}}^{\check{\sigma}} \chi^T(\sigma)Z\chi(\sigma)d\sigma \le \frac{4(\check{\sigma}-\hat{\sigma})^2}{\pi^2}\int_{\hat{\sigma}}^{\check{\sigma}} \frac{d\chi^T(\sigma)}{d\sigma}Z\frac{d\chi(\sigma)}{d\sigma}d\sigma. \qquad (7)$$

III. Main results

We provide some conditions to ensure the robust FTS of the system in this section.

Theorem 1: Assume that Assumptions 1-3 hold, system (1) is robust FTS via controller (5) in the mean square sense with respect to (δ_1,δ_2,T) if there exist positive definite matrices P, Q, $R \in \mathbb{R}^{m\times m}$, m-dimensional positive definite diagonal matrices $\beth_1$, $\beth_2$, $\beth_3$, an identity matrix I with appropriate dimensions, and positive numbers β and γ such that

$$P \le \beta I,\ \varepsilon\delta_1 e^{\gamma T} \le \delta_2\lambda_{\min}(P), \qquad (8)$$

$$\Phi_1=\begin{bmatrix}\Sigma_{11} & \daleth_{12} & \daleth_{13} & \daleth_{14} & 3PM & 2PE\\ * & \daleth_{22} & \daleth_{23} & \daleth_{24} & 0 & 0\\ * & * & \daleth_{33} & \daleth_{34} & 0 & 0\\ * & * & * & \daleth_{44} & 0 & 0\\ * & * & * & * & -3I & 0\\ * & * & * & * & * & -2I\end{bmatrix}<0, \tag{9}$$

$$\Phi_2=\begin{bmatrix}\Upsilon_{11} & PM & \frac{2\alpha}{\pi}\Gamma^T\\ * & -I & 0\\ * & * & -I\end{bmatrix}<0, \tag{10}$$

where $\Sigma_{11}=\mathrm{Sym}\{PA+\Gamma-\Pi_1^T(\beth_1+\beth_3)\Pi_2\}+N_2^TN_2+H^TH+(\varrho_1\beta+1)I+Q-\gamma P$, $\daleth_{11}=\Sigma_{11}+3PMM^TP+2PEE^TP$, $\daleth_{12}=2\Pi_1^T\beth_3\Pi_2$, $\daleth_{13}=PB+(\Pi_1+\Pi_2)^T(\beth_1+\beth_3)$, $\daleth_{14}=PC-(\Pi_1+\Pi_2)^T\beth_3$, $\daleth_{22}=\varrho_2\beta I-\bar{\varphi}Q-\mathrm{Sym}\{\Pi_1^T(\beth_2+\beth_3)\Pi_2\}$, $\daleth_{23}=-(\Pi_1+\Pi_2)^T\beth_3$, $\daleth_{24}=(\Pi_1+\Pi_2)^T(\beth_2+\beth_3)$, $\daleth_{33}=N_3^TN_3+R-2\beth_1-2\beth_3$, $\daleth_{34}=2\beth_3$, $\daleth_{44}=N_4^TN_4-\bar{\varphi}R-2\beth_2-2\beth_3$, $\Upsilon_{11}=\frac{4\alpha^2}{\pi^2}H^TH+N_1^TN_1-\mathrm{Sym}\{PD\}$, $\Gamma=PK$, $\bar{\varphi}=1-\check{\varphi}$, $\Pi_1=\mathrm{diag}\{o_1^-,o_2^-,\ldots,o_m^-\}$, $\Pi_2=\mathrm{diag}\{o_1^+,o_2^+,\ldots,o_m^+\}$, and $\varepsilon=\lambda_{\max}(P)+\hat{\varphi}e^{\gamma\hat{\varphi}}(\lambda_{\max}(Q)+\lambda_{\max}(R)\lambda_{\max}(\Pi_2^T\Pi_2))$.

Proof: Construct the Lyapunov functional as below

$$\mathscr{V}(\epsilon)=\sum_{\ell=1}^{3}\mathscr{V}_\ell(\epsilon), \tag{11}$$

where

$$\mathscr{V}_1(\epsilon)=\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\varpi(\sigma,\epsilon)d\sigma,$$
$$\mathscr{V}_2(\epsilon)=\int_{\hat{\sigma}}^{\check{\sigma}}\int_{\epsilon-\varphi(\epsilon)}^{\epsilon}e^{\gamma(\epsilon-r)}\varpi^T(\sigma,r)Q\varpi(\sigma,r)drd\sigma,$$
$$\mathscr{V}_3(\epsilon)=\int_{\hat{\sigma}}^{\check{\sigma}}\int_{\epsilon-\varphi(\epsilon)}^{\epsilon}e^{\gamma(\epsilon-r)}\zeta^T(\varpi(\sigma,r))R\zeta(\varpi(\sigma,r))drd\sigma.$$

By utilizing the Itô's formula,

$$\begin{aligned}\mathcal{L}\mathscr{V}_1(\epsilon)=&\,2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\Big[\tilde{D}\varpi_{\sigma\sigma}(\sigma,\epsilon)+(\tilde{A}+\tilde{K})\varpi(\sigma,\epsilon)\\&+\tilde{B}\zeta(\varpi(\sigma,\epsilon))+\tilde{C}\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon)))\Big]d\sigma\\&+\int_{\hat{\sigma}}^{\check{\sigma}}Tr\Big(\Lambda^T(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon)))P\\&\times\Lambda(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon)))\Big)d\sigma\\&+2\sum_{\iota=0}^{n-1}\int_{\sigma_\iota}^{\sigma_{\iota+1}}\varpi^T(\sigma,\epsilon)P\tilde{K}\bar{\varpi}(\sigma,\epsilon)d\sigma,\end{aligned} \tag{12}$$

where $\tilde{K}=K+\Delta K(\sigma_\iota,\epsilon)$ and $\bar{\varpi}(\sigma,\epsilon)=\varpi(\sigma_\iota,\epsilon)-\varpi(\sigma,\epsilon)$.

Furthermore, we easily get

$$\begin{aligned}\mathcal{L}\mathscr{V}_2(\epsilon)\leq&\,\gamma\mathscr{V}_2(\epsilon)+\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)Q\varpi(\sigma,\epsilon)d\sigma\\&-\bar{\varphi}\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon-\varphi(\epsilon))Q\varpi(\sigma,\epsilon-\varphi(\epsilon))d\sigma,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\mathcal{L}\mathscr{V}_3(\epsilon)\leq&\,\gamma\mathscr{V}_3(\epsilon)+\int_{\hat{\sigma}}^{\check{\sigma}}\zeta^T(\varpi(\sigma,\epsilon))R\zeta(\varpi(\sigma,t))d\sigma\\&-\bar{\varphi}\int_{\hat{\sigma}}^{\check{\sigma}}\zeta^T(\varpi(\sigma,\epsilon-\varphi(\epsilon)))R\\&\times\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon)))d\sigma.\end{aligned} \tag{14}$$

It follows from the boundary conditions of system (1) that

$$\begin{aligned}&2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\tilde{D}\varpi_{\sigma\sigma}(\sigma,\epsilon)d\sigma\\&=2\varpi^T(\sigma,\epsilon)P\tilde{D}\varpi_\sigma(\sigma,\epsilon)\Big|_{\sigma=\hat{\sigma}}^{\sigma=\check{\sigma}}\\&\quad-2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi_\sigma^T(\sigma,\epsilon)P\tilde{D}\varpi_\sigma(\sigma,\epsilon)d\sigma\\&\leq\int_{\hat{\sigma}}^{\check{\sigma}}\varpi_\sigma^T(\sigma,\epsilon)(PMM^TP+N_1^TN_1-2PD)\varpi_\sigma(\sigma,\epsilon)d\sigma.\end{aligned} \tag{15}$$

Considering that $L_1^TL_2+L_2^TL_1\leq L_1^TL_1+L_2^TL_2$ holds for any suitable dimensional matrices L_1 and L_2, we derive

$$\begin{aligned}&2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\Big[(\tilde{A}+\tilde{K})\varpi(\sigma,\epsilon)+\tilde{B}\zeta(\varpi(\sigma,\epsilon))\Big]d\sigma\\&\leq2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\Big[(A+K)\varpi(\sigma,\epsilon)+B\zeta(\varpi(\sigma,\epsilon))\Big]d\sigma\\&\quad+\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)\Big[2PMM^TP+PEE^TP\Big]\varpi(\sigma,\epsilon)d\sigma\\&\quad+\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)\Big[N_2^TN_2+H^TH\Big]\varpi(\sigma,\epsilon)d\sigma\\&\quad+\int_{\hat{\sigma}}^{\check{\sigma}}\zeta^T(\varpi(\sigma,\epsilon))N_3^TN_3\zeta(\varpi(\sigma,\epsilon))d\sigma.\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}&2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)P\tilde{C}\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon)))d\sigma\\&\leq2\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)PC\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon)))d\sigma\\&\quad+\int_{\hat{\sigma}}^{\check{\sigma}}\varpi^T(\sigma,\epsilon)PMM^TP\varpi(\sigma,\epsilon)d\sigma\\&\quad+\int_{\hat{\sigma}}^{\check{\sigma}}\zeta^T(\varpi(\sigma,\epsilon-\varphi(\epsilon)))N_4^TN_4\zeta(\varpi(\sigma,\epsilon-\varphi(\epsilon)))d\sigma.\end{aligned} \tag{17}$$

From Assumption 2, we have

$$\begin{aligned}&\int_{\hat{\sigma}}^{\check{\sigma}}Tr\Big(\Lambda^T(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon)))P\\&\quad\times\Lambda(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon)))\Big)d\sigma\\&\leq\varrho_1\beta\varpi^T(\sigma,\epsilon)\varpi(\sigma,\epsilon)\\&\quad+\varrho_2\beta\varpi^T(\sigma,\epsilon-\varphi(\epsilon)))\varpi(\sigma,\epsilon-\varphi(\epsilon))).\end{aligned} \tag{18}$$

According to Lemma 1, we have

$$
\begin{aligned}
& 2\sum_{\iota=0}^{n-1}\int_{\sigma_\iota}^{\sigma_{\iota+1}} \varpi^T(\sigma,\epsilon)P\tilde{K}\bar{\varpi}(\sigma,\epsilon)d\sigma \\
\leq & \int_{\hat{\sigma}}^{\check{\sigma}} \varpi^T(\sigma,\epsilon)(I+PEE^TP)\varpi(\sigma,\epsilon)d\sigma \\
& + \sum_{\iota=0}^{n-1}\int_{\sigma_\iota}^{\sigma_{\iota+1}} \bar{\varpi}^T(\sigma,\epsilon)(K^TPPK+H^TH)\bar{\varpi}(\sigma,\epsilon)d\sigma \\
\leq & \int_{\hat{\sigma}}^{\check{\sigma}} \varpi^T(\sigma,\epsilon)(I+PEE^TP)\varpi(\sigma,\epsilon)d\sigma \\
& + \frac{4\alpha^2}{\pi^2}\int_{\hat{\sigma}}^{\check{\sigma}} \varpi_\sigma^T(\sigma,\epsilon)(K^TPPK+H^TH)\varpi_\sigma(\sigma,\epsilon)d\sigma.
\end{aligned} \tag{19}
$$

Then, we let $\Omega_1(\sigma,z) = \zeta(\varpi(\sigma,z)) - \Pi_1\varpi(\sigma,z)$, $\Omega_2(\sigma,z) = \Pi_2\varpi(\sigma,z) - \zeta(\varpi(\sigma,z))$, $\Omega_3(\sigma,z_1,z_2) = (\zeta(\varpi(\sigma,z_1))-\zeta(\varpi(\sigma,z_2)))-\Pi_1(\varpi(\sigma,z_1)-\varpi(\sigma,z_2))$, and $\Omega_4(\sigma,z_1,z_2) = \Pi_2(\varpi(\sigma,z_1)-\varpi(\sigma,z_2)) - (\zeta(\varpi(\sigma,z_1)) - \zeta(\varpi(\sigma,z_2)))$. From Assumption 1, for any diagonal matrices $\beth_\ell \geq 0$, $\ell = 1,2,3$, we obtain

$$
2\int_{\hat{\sigma}}^{\check{\sigma}} \Omega_1(\sigma,\epsilon)\beth_1\Omega_2(\sigma,\epsilon)d\sigma \geq 0, \tag{20}
$$

$$
2\int_{\hat{\sigma}}^{\check{\sigma}} \Omega_1(\sigma,\epsilon-\varphi(\epsilon))\beth_2\Omega_2(\sigma,\epsilon-\varphi(\epsilon))d\sigma \geq 0, \tag{21}
$$

$$
2\int_{\hat{\sigma}}^{\check{\sigma}} \Omega_3(\sigma,\epsilon,\epsilon-\varphi(\epsilon))\beth_3\Omega_4(\sigma,\epsilon,\epsilon-\varphi(\epsilon))d\sigma \geq 0. \tag{22}
$$

Combining (12) to (22) gives

$$
\begin{aligned}
\mathcal{L}\mathscr{V}(\epsilon) \leq & \gamma\mathscr{V}(\epsilon) + \int_{\hat{\sigma}}^{\check{\sigma}} \xi^T(\sigma,\epsilon)\Psi_1\xi(\sigma,\epsilon)d\sigma \\
& + \int_{\hat{\sigma}}^{\check{\sigma}} \varpi_\sigma^T(\sigma,\epsilon)\Psi_2\varpi_\sigma(\sigma,\epsilon)d\sigma,
\end{aligned} \tag{23}
$$

where $\Psi_1 = (\daleth_{\imath\jmath})_{4\times4}$, $\Psi_2 = \frac{4\alpha^2}{\pi^2}(H^TH + \Gamma^T\Gamma) + PMM^TP + N_1^TN_1 - \mathrm{Sym}\{PD\}$, and $\xi(\sigma,\epsilon) = [\varpi^T(\sigma,\epsilon), \varpi^T(\sigma,\epsilon - \varphi(\epsilon)), \zeta^T(\varpi(\sigma,\epsilon)), \zeta^T(\varpi(\sigma,\epsilon - \varphi(\epsilon)))]^T$.

By virtue of the Schur complement, $\Phi_1 < 0$ and $\Phi_2 < 0$ are equivalent to $\Psi_1 < 0$ and $\Psi_2 < 0$, respectively. Besides, integrating (17) from 0 to ϵ deduces

$$
\Xi(\mathscr{V}(\epsilon)) \leq \Xi(\mathscr{V}(0)) + \gamma\int_0^\epsilon \Xi(\mathscr{V}(r))dr. \tag{24}
$$

Based on the Gronwall inequality, we derive

$$
\Xi(\mathscr{V}(\epsilon)) \leq e^{\gamma\epsilon}\Xi(\mathscr{V}(0)) \leq \varepsilon e^{\gamma\epsilon}\Xi\left(\sup_{s\in[-\hat{\varphi},0)}\|\varpi(\cdot,s)\|^2\right), \tag{25}
$$

where $\varepsilon = \lambda_{\max}(P) + \hat{\varphi}e^{\gamma\hat{\varphi}}(\lambda_{\max}(Q) + \lambda_{\max}(\Pi_2^TR\Pi_2))$.

Obviously, it can be gained from (25) that

$$
\Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right) \leq \frac{\Xi(\mathscr{V}(\epsilon))}{\lambda_{\min}(P)} \leq \frac{\varepsilon\delta_1 e^{\gamma T}}{\lambda_{\min}(P)} \leq \delta_2. \tag{26}
$$

Thence, according to Definition 1, system (1) is robust FTS in the mean square sense with respect to (δ_1,δ_2,T).

Moreover, when the structural uncertainty and transmission delay of the system and the non-fragility of the controller are not taken into account, $\zeta(\cdot) = 0$, and $\Lambda(\epsilon,\varpi(\sigma,\epsilon),\varpi(\sigma,\epsilon-\varphi(\epsilon))) = Y\varpi(\sigma,\epsilon)$ with $Y \in \mathbb{R}^{m\times m}$, system (1) becomes the stochastic parabolic PDE systems studied in [13] as follows

$$
\begin{aligned}
\mathrm{d}\varpi(\sigma,\epsilon) = & [D\varpi_{\sigma\sigma}(\sigma,\epsilon) + A\varpi(\sigma,\epsilon) + K\varpi(\sigma_\iota,\epsilon)]\,\mathrm{d}\epsilon \\
& + Y\varpi(\sigma,\epsilon)\mathrm{d}\vartheta(\epsilon).
\end{aligned} \tag{27}
$$

By utilizing Theorem 1, the following criterion can be easily derived to ensure the FTS of system (27) in the mean square sense.

Corollary 1: System (27) is FTS in the mean square sense with respect to (δ_1,δ_2,T) if there is a matrix $P > 0$, an identity matrix $I \in \mathbb{R}^{m\times m}$, and a number $\gamma > 0$ such that

$$
\varepsilon_1\delta_1 e^{\gamma T} \leq \delta_2\lambda_{\min}(P), \tag{28}
$$

$$
\begin{bmatrix} -PD + D^TP & \frac{2\alpha}{\pi}\Gamma^T \\ * & -I \end{bmatrix} < 0, \tag{29}
$$

$$
C^TPC + \mathrm{Sym}\{PA + \Gamma\} + I - \gamma P < 0, \tag{30}
$$

where $\Gamma = PK$ and $\varepsilon_1 = \lambda_{\max}(P)$.

Remark 1: In [3], [4], [7], [11], [13], several criteria were deduced to guarantee the stabilization of stochastic parabolic PDE systems. Notably, these results were derived by exploiting the vector-valued Wirtinger's inequality, which requires that the spatial variable of stochastic parabolic PDE systems be restricted to the interval $[0,\bar{\sigma}]$, where $\bar{\sigma}$ is a positive number. This paper adopts an extended vector-valued Wirtinger's inequality to loosen the interval restriction on the spatial variable from $[0,\bar{\sigma}]$ to $[\hat{\sigma},\check{\sigma}]$.

Remark 2: The exponential stabilization issue of stochastic parabolic PDE systems was researched in [3], [4]. In contrast to [3], [4], this paper establishes the FTS results for stochastic parabolic PDE systems with uncertain parameters and discrete delay. In fact, if we let $\bar{\varphi} = (1-\check{\varphi})e^{\gamma\hat{\varphi}}$, then $\Xi(\mathscr{V}(\epsilon)) \leq e^{\gamma\epsilon}\Xi(\mathscr{V}(0))$ in formula (25) still holds when γ is negative. Apparently, the approach developed in this paper is also appropriate for the research of exponential stabilization of stochastic parabolic PDE systems. Furthermore, contrary to the spatial sampled-data control scheme in [3] and the continuous control scheme in [4], this paper further proposes a non-fragile spatial sampled-data control scheme to enhance the reliability of the controller.

Remark 3: In practice, the time delay is common. The time delay often causes system oscillation or instability, which can degrade the control performance of the system. Whereas, the impact of time delay on the FTS of stochastic parabolic PDE systems was neglected in [7], [11], [13]. This paper incorporates the influence of discrete delay in the system. On the other hand, this paper constructs a more general Lyapunov functional to obtain lower conservative results. When $P = I$ and $Q = R = 0$, $\mathscr{V}(\epsilon)$ degenerates to the Lyapunov functional established in [11], [13]. Moreover, the information on the nonlinear function is fully utilized in this paper to introduce inequalities (20)-(22) to increase the coupling relationship between system states.

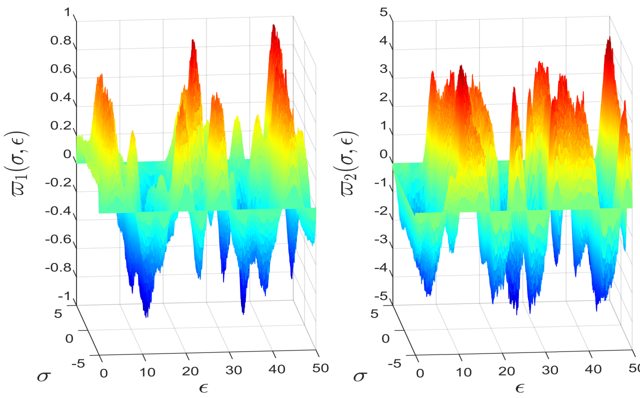

Fig. 1. Evolution of system (1) without control.

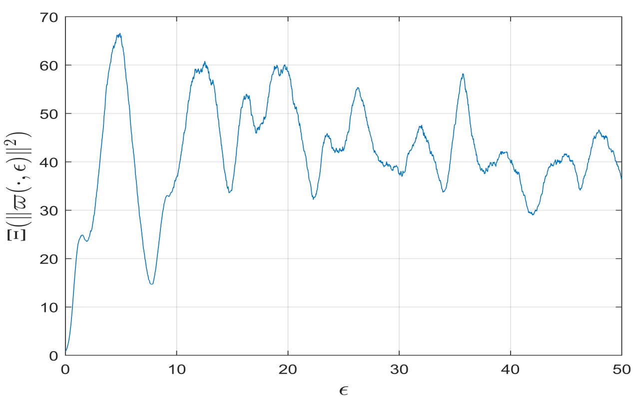

Fig. 2. Trajectory of $\Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right)$ without control.

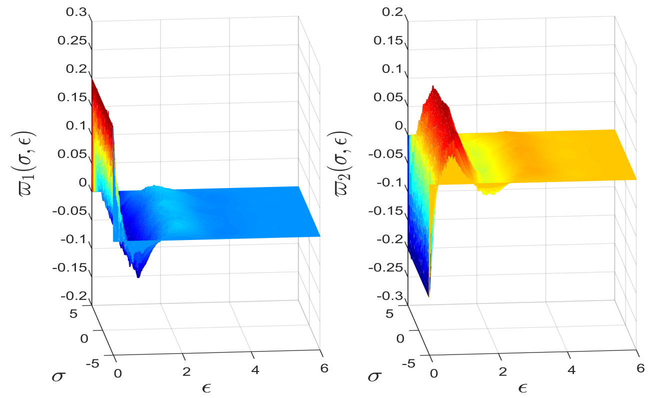

Fig. 3. Evolution of system (1) with control.

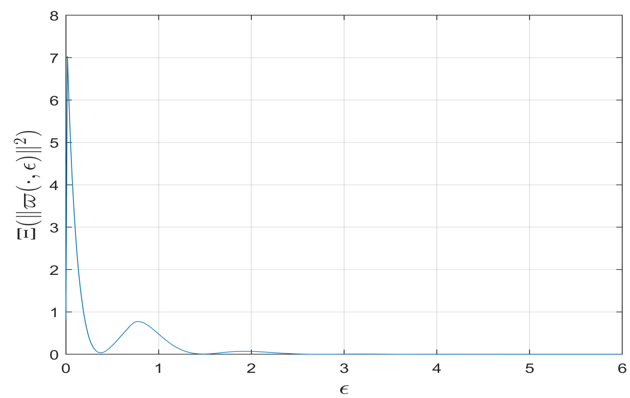

Fig. 4. Trajectory of $\Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right)$ with control.

IV. Numerical Simulations

We give an example to show the feasibility of the obtained results in this section.

Example 1: Consider system (1) with parameters as below: $D = \mathrm{diag}\{0.3, 0.3\}$, $A = \mathrm{diag}\{-1, -1\}$, $B = (b_{\imath\jmath})_{2\times 2}$, $C = (c_{\imath\jmath})_{2\times 2}$, where $b_{11} = 2$, $b_{12} = -0.1$, $b_{21} = -5$, $b_{22} = 3$, $c_{11} = -1.5$, $c_{12} = -0.1$, $c_{21} = -0.2$, and $c_{22} = -2.5$. Besides, $M = \mathrm{diag}\{0.1, 0.5\}$, $N_1 = \mathrm{diag}\{0.3, 0.1\}$, $N_2 = \mathrm{diag}\{0.3, 0.2\}$, $N_3 = \mathrm{diag}\{0.3, 0.3\}$, $N_4 = \mathrm{diag}\{0.3, 0.4\}$, $E = \mathrm{diag}\{0.2, 0.1\}$, $H = \mathrm{diag}\{0.1, 0.3\}$, and $F(\sigma, \epsilon) = \mathrm{diag}\{e^{-|\sigma|}\sin(\epsilon), e^{-|\sigma|}\sin(\epsilon)\}$, $\sigma \in [-5, 5]$. Let $\varphi(\epsilon) = \frac{e^{\epsilon}}{1+e^{\epsilon}}$, $\zeta(\varpi(\sigma,\epsilon)) = \tanh(\varpi(\sigma,\epsilon))$, $\Lambda(\epsilon, \varpi(\sigma,\epsilon), \varpi(\sigma, \epsilon - \varphi(\epsilon))) = 0.15\varpi(\sigma,\epsilon) + 0.2\varpi(\sigma, \epsilon-\varphi(\epsilon))$. Then, we can get $\hat{\varphi} = 1$, $\check{\varphi} = 0.25$, $\varrho_1 = 0.045$, $\varrho_2 = 0.08$, $\Pi_1 = \mathrm{diag}\{0, 0\}$, and $\Pi_2 = \mathrm{diag}\{1, 1\}$. The spatiotemporal evolution of system (1) without control and the trajectory of $\Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right)$ are depicted in Figs. 1 and 2 respectively, where the initial value is $\psi(\sigma, s) = (0.2, -0.2)^T$. Let $\delta_1 = 0.9$, $\delta_2 = 35$, and $T = 5$. From Fig. 2, it is easy to get that system (1) without control is not robust finite-time stable in the mean square sense with respect to $(0.9, 35, 5)$.

To achieve the robust FTS of system (1) with the above parameters, we design the non-fragile spatial sampled-data control scheme as $u(\sigma,\epsilon) = (K + \Delta K(\sigma_\iota, \epsilon))\varpi(\sigma_\iota, \epsilon)$, where $E = \mathrm{diag}\{0.2, 0.1\}$, $H = \mathrm{diag}\{0.1, 0.3\}$, $G(\sigma_\iota, \epsilon) = \mathrm{diag}\{e^{-|\sigma_\iota|}\cos(\epsilon), e^{-|\sigma_\iota|}\cos(\epsilon)\}$, and $\sigma_{\iota+1} - \sigma_\iota = 0.2$ with $\iota = 0, 1, \ldots, 49$. Selecting $\gamma = 0.01$ and solving with the LMI toolbox yields

$$P = \begin{bmatrix} 2.7234 & 0.1903 \\ 0.1903 & 0.3069 \end{bmatrix}, \quad Q = \begin{bmatrix} 5.3059 & 0.1680 \\ 0.1680 & 1.5637 \end{bmatrix},$$
$$R = \begin{bmatrix} 2.3998 & 0.5460 \\ 0.5460 & 0.5168 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -9.1294 & 0.4227 \\ -0.7112 & -2.8551 \end{bmatrix}.$$

Easily, we have

$$K = P^{-1}\Gamma = \begin{bmatrix} -3.3348 & 0.8420 \\ -0.2489 & -9.8256 \end{bmatrix}.$$

Moreover, we can get $\lambda_{\min}(P) = 0.2920$, $\lambda_{\max}(P) = 2.7383$, $\lambda_{\max}(Q) = 5.3134$, and $\lambda_{\max}(R) = 2.5466$. The conditions of Theorem 1 can be easily checked to hold by calculation. To visualize the correctness of the theoretical results, we depict the spatiotemporal evolution of system (1) with control and the trajectory of $\Xi\left(\|\varpi(\cdot,\epsilon)\|^2\right)$ in Figs. 3 and 4, respectively. As can be seen in Fig. 4, system (1) can achieve robust FTS in the mean square sense with respect to $(0.9, 35, 5)$.

V. CONCLUSIONS

Stochastic disturbance, structural uncertainty, and time delay typically impact the normal operation of the system and lead to performance degradation. The perturbation of these three practical factors have been considered simultaneously in this paper to construct the stochastic parabolic PDE systems with parameter uncertainty and discrete delay. In fact, in addition to the parameter uncertainty of the controlled system, the non-fragile problem in the controller cannot be ignored. To improve the reliability of the controller and further lower control costs, this paper has designed a non-fragile spatial sampled-data control scheme. Then, the robust FTS of stochastic parabolic PDE systems has been realized via the suggested control scheme, and the corresponding stabilization conditions have been obtained. At last, the correctness of the established results has been clarified by utilizing a numerical example.

REFERENCES

[1] K. Ding and Q. Zhu, "Fuzzy intermittent extended dissipative control for delayed distributed parameter systems with stochastic disturbance: A spatial point sampling approach," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 6, pp. 1734–1749, Jun. 2022.

[2] J. Deutscher, "Robust cooperative output regulation for a network of parabolic PDE systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 1, pp. 451–459, Jan. 2022.

[3] W. Kang, X. N. Wang, and B. Z. Guo, "Observer-based fuzzy quantized control for a stochastic third-order parabolic PDE system," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 53, no. 1, pp. 485–494, Jan. 2023.

[4] H. N. Wu and X. M. Zhang, "Boundary static output feedback control for nonlinear stochastic parabolic partial differential systems via fuzzy-model-based approach," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 10, pp. 2581–2591, Oct. 2020.

[5] H. N. Wu and X. M. Zhang, "Exponential stabilization for 1-D linear Itô-type state-dependent stochastic parabolic PDE systems via static output feedbac," *Automatica*, vol. 121, p. 109173, Nov. 2020.

[6] Z. Yan, W. Zhang, and G. Zhang, "Finite-time stability and stabilization of Itô stochastic systems with Markovian switching: Mode-dependent parameter approach," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2428–2433, Sep. 2015.

[7] J. Man, Z. Zeng, and Y. Sheng, "Finite-time fuzzy boundary control for 2-D spatial nonlinear parabolic PDE systems," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 9, pp. 3278–3289, Sep. 2023.

[8] X. Song, M. Wang, J. H. Park, and S. Song, "Spatial-$\mathcal{L}_\infty$-norm-based finite-time bounded control for semilinear parabolic PDE systems with applications to chemical-reaction processes," *IEEE Transactions on Cybernetics*, vol. 52, no. 1, pp. 178–191, Jan. 2022.

[9] G. Zong, H. Ren, and H. R. Karimi, "Event-triggered communication and annular finite-time H_∞ filtering for networked switched systems," *IEEE Transactions on Cybernetics*, vol. 51, no. 1, pp. 309–317, Jan. 2021.

[10] J. Li and Y. Liu, "Adaptive control of uncertain coupled reaction-diffusion dynamics with equidiffusivity in the actuation path of an ODE system," *IEEE Transactions on Automatic Control*, vol. 66, no. 2, pp. 802–809, Feb. 2021.

[11] X. Z. Liu, K. N. Wu, and W. H. Zhang, "Mean square finite-time boundary stabilisation and H_∞ boundary control for stochastic reaction-diffusion systems," *International Journal of Systems Science*, vol. 50, no. 7, pp. 1388–1398, May 2019.

[12] Z. P. Wang and H. N. Wu, "Sampled-data fuzzy control with guaranteed cost for nonlinear parabolic PDE systems via static output feedback," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 10, pp. 2452–2465, Oct. 2020.

[13] K. N. Wu, Y. Z. Wang, and Z. Wang, "Spatial sampled-data control for stochastic reaction-diffusion systems," *Journal of the Franklin Institute*, vol. 357, no. 17, pp. 12538–12554, Nov. 2020.

[14] Y. Qiu, C. Hua, and Y. Wang, "Nonfragile sampled-data control of T-S fuzzy systems with time delay," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 8, pp. 3202–3210, Aug. 2022.

[15] J. W. Wang and H. N. Wu, "Some extended Wirtinger's inequalities and distributed proportional-spatial integral control of distributed parameter systems with multi-time delays," *Journal of the Franklin Institute*, vol. 352, no. 10, pp. 4423–4445, Oct. 2015.