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On Iterative Parameter Identification of FIR Systems With Batched Possibly Incorrect Binary-valued Observations

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Abstract— **This paper considers the problem of parameter identification for a binary output finite impulse response (FIR) system with measurement error, where the measurement error makes the binary measurement values take opposite values with a certain probability. First, the maximum likelihood estimation (MLE) of the parameters is given and an iterative algorithm with projection based on the Expectation-Maximization algorithm is presented to calculate the MLE. Furthermore, the necessary and sufficient condition for the likelihood function to have a unique maximum point is obtained. It is proved that the iterative estimation error converges to zero at an exponential rate under persistently excitation input conditions. Finally, some numerical simulation results based on a typical system show the effectiveness of the proposed algorithm.**

Index Terms— **Binary-valued observation, maximum likelihood estimate, strongly convex, system identification, exponential rate.**

I. Introduction

With the development of information technology, set-valued output systems [19] are more and more appearing in many practical scenarios, in which set-valued output refers to the output of the system can not be accurately measured, and the information that can be measured only indicate whether the output belongs to a certain set. This set-valued output system has been widely used in the fields of industrial production [1], biopharmaceutical technology [2], and information industry [12], and has important research significance, thus it has received wide-ranging attention. A representative type of set-value output system is the binary output system, whose output data are usually obtained by binary sensors. For example, in communication systems such as asynchronous transmission mode networks, the sensors that measure bit rate, queue length and other traffic information are binary [19]; in automotive systems, the exhaust gas oxygen switch sensors are also binary [10]. In these practical scenarios, due to the limited information, the set-valued sensors lead to substantial difficulties, especially in the case of binary observation, and the system identification problem is even more difficult to solve.

In recent years, research on the identification of set-valued systems has constituted a large number of literatures ([4], [6], [11], [17]). Ref. [19] proposed an empirical measure method to study the identification error of linear systems with binary-valued information. [15] proposed a non-truncated empirical measure method for finite-impulse response (FIR) systems and demonstrated its asymptotic efficiency in the sense of Cramér-Rao lower bound. [7] brought the expectation-maximization (EM) algorithm to the identification of quantized systems and provided some simulation results to justify the convergence. [20] proposed an EM-type algorithm for FIR systems and demonstrated that the algorithm has exponential convergence speed. Under a fixed-threshold quantizer, [8] presented a recursive projection algorithm for FIR models that was shown to be convergent, [14] extended the algorithm to the case of matrix input and vector output, and obtained a faster convergence rate $O(1/k)$. While, [16] introduced a unifying approach using a time-varying threshold quantizers. In addition, [18] proves that the Cramér-Rao lower bound can be reached asymptotically with appropriate weight coefficients, which shows that their algorithm is asymptotically efficient.

The above work is mainly based on accurate binary observations. However, in some real-world problems, measurements are often subject to errors caused by sensor wildcards, packet loss, communication errors, etc., which make it possible to obtain the opposite of the observed value with a certain probability. For example, in a communication system, 0-1 bits of information may be transmitted or received incorrectly. In addition, an important application of set-value identification algorithms is the problem of target classification [13]. However, there is a certain probability that the labels of the training data will be misclassified at the time of acquisition, which will affect the application of the set-value identification algorithm and lead to an incorrect classifier. Therefore, two issues need to be considered. One is whether we can accomplish the identification objective when there are incorrect observations? The other is how to design the identification algorithm and prove its consistency. Motivated by [20], we transform such set-valued system identification problems into the corresponding maximum likelihood (ML) problems.

The main contributions of this paper are as follows:

i) The ML based iterative parameter estimation algorithm is proposed for the considered FIR systems. For the considered binary output FIR system with measurement error, based

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on the ML criterion, the MLE is given. To solve MLE, an iterative solution algorithm with projection based on the EM algorithm is proposed to adapt to more general measurement error cases, which guarantees the boundedness of the iterative estimation sequence.

ii) The existence and uniqueness of the MLE are rigorously studied. For a given number of observations, by analyzing the properties of the likelihood function, the necessary and sufficient condition is given for the likelihood function to have a unique maximum point.

iii) The iterative algorithm are consistent. It is shown that the iterative estimation error converges to zero at an exponential rate under persistent excitation input conditions. Numerical simulation results based on a typical system show the effectiveness of the proposed algorithm.

The rest of the paper is organised as follows: Section II introduces the identification problem and its corresponding ML criterion, and constructs an iterative estimation algorithm. Section III analyzes the likelihood function. The necessary and sufficient condition for the existence and the uniqueness of the likelihood function is given. Section IV derives the convergence of the algorithm and obtains an exponential convergence rate. Section V illustrates the results through numerical simulations and compares them with methods that do not take into account erroneous observations. Section VI concludes the whole paper and discusses related future work.

II. Problem statement and algorithm

In this section, we will first formulate the identification problem with erroneous binary observations and then introduce the algorithms for solving the problem.

A. Problem statement

Consider the parameter identification of the following stochastic FIR systems:

$$\left\{ \begin{array}{l} y_k = \phi_k^T \theta_0 + e_k, \\ \tilde{s}_k = I_{[y_k \le C]}, \quad 1 \le k \le N, \end{array} \right. \tag{1}$$

where $\theta_0 \in \Theta \subseteq \mathbb{R}^p$ is unknown but time invariant p-dimensional parameter vector with p being known, $\phi_k \in \mathbb{R}^p$ is the regressor vector consisting current and past inputs, $y_k \in \mathbb{R}$ and $e_k \in \mathbb{R}$ are the system output and noise, respectively; $\tilde{s}_k \in \{0,1\}$ is the binary-valued observation generated by the comparison between the system output and a given sensor threshold $C \in \mathbb{R}; I$ is the indicator function and N is the data length.

In practice, due to sensor wild values, data packet loss, etc., the system's binary output $\tilde{s}_k$ cannot be obtained precisely and its opposite result may be observed with a certain probability. Mathematically, the observed binary-valued output can be represented by

$$s_k = \begin{cases} \tilde{s}_k, & \text{with probability } p_k, \\ 1 - \tilde{s}_k, & \text{with probability } 1 - p_k, \end{cases} \tag{2}$$

where $p_k \in (0,1)$ may be a function of θ_0, C and ϕ_k to characterize the probability that the binary observation is correct.

The goal of this paper is to construct an identification algorithm to estimate the unknown parameter vector θ_0 based on the input data $\mathscr{I}_N = \{\phi_1, \phi_2, \ldots, \phi_N\}$ and the observation $\mathscr{O}_N = \{s_1, s_2, \ldots, s_N\}$. First, we give the following assumptions.

Assumption 1: Matrix $A = \sum_{k=1}^N \phi_k \phi_k^T$ is positive definite.

Assumption 2: For all $k \le N$, system noise $\mathscr{E}_N = \{e_1, e_2, \ldots, e_N\}$ is independent with a zero-mean and variance 1 Gaussian distribution.

Remark 1: Assumption 1 is the mathematical description of persistent excitation condition, which is a common assumption in the research of system identification. Assumption 2 is about the noise. For the general case where the error $\{e_k\}$ obeys normal distribution with mean 0 and variance σ^2, by transformation $y_k^\star = y_k/\sigma,\ \theta^\star = \theta/\sigma,\ e_k^\star = e_k/\sigma,\ C^\star = C/\sigma$, system (1) is converted to satisfy Assumption 2.

B. Maximum likelihood criterion

In statistics, MLE is a method of estimating the parameters of a hypothetical probability distribution given some observations. By maximizing the likelihood function, the observed data are most probable under the assumed statistical model. This maximum point is called the maximum likelihood estimate. Now we consider systems (1) and (2). For any $1 \le k \le N$, given the input data ϕ_k and the parameter θ, by the law of total probability, the corresponding probabilities of observation $s_k = 1$ and $s_k = 0$ are as follows.

$$\begin{aligned} & P\{s_k = 1 \mid \phi_k, \theta\} \\ =& P\{s_k = \tilde{s}_k\} P\{s_k = 1 \mid s_k = \tilde{s}_k, \phi_k, \theta\} \\ & + P\{s_k = 1 - \tilde{s}_k\} P\{s_k = 1 \mid s_k = 1 - \tilde{s}_k, \phi_k, \theta\} \\ =& [2p_k - 1] F(C - \phi_k^T \theta) + 1 - p_k, \\ & P\{s_k = 0 \mid \phi_k, \theta\} = 1 - P\{s_k = 1 \mid \phi_k, \theta\} \\ =& [1 - 2p_k] F(C - \phi_k^T \theta) + p_k, \end{aligned}$$

where $F(x)$ is the cumulative distribution function (CDF) of the standard normal distribution. Denote

$$G(C - \phi_k^T \theta) = [2p_k - 1] F(C - \phi_k^T \theta) + 1 - p_k. \tag{3}$$

Using the conditional probabilities above, we can construct the following likelihood function given the input data $\mathscr{I}_N$,

$$\begin{aligned} & L_N(\theta) = P\{\mathscr{O}_N \mid \mathscr{I}_N, \theta\} \\ =& \prod_{\{k: s_k = 1\}} G\left(C - \phi_k^{\mathrm{T}} \theta\right) \cdot \prod_{\{k: s_k = 0\}} \left(1 - G\left(C - \phi_k^{\mathrm{T}} \theta\right)\right). \end{aligned}$$

Since $\log x$ is a strictly increasing function and $L_N(\theta)$ can be assumed to be positive without loss of generality, maximizing $L_N(\theta)$ is equivalent to maximizing the log-likelihood function:

$$\begin{aligned} l_N(\theta) = \log L_N(\theta) = \sum_{k=1}^N & \left[\log[G(C - \phi_k^{\mathrm{T}} \theta)] I_{[s_k = 1]} \right. \\ & \left. + \log[1 - G(C - \phi_k^{\mathrm{T}} \theta)] I_{[s_k = 0]}\right]. \end{aligned} \tag{4}$$

It is often more convenient to work with $l_N(\theta)$ and the corresponding MLE is the point that maximizes the log-likelihood function:

$$\theta_N = \arg\max_{\theta} l_N(\theta). \tag{5}$$

Intuitively, this MLE will select the parameter values that make the observations most probable.

Remark 2: MLE does not have optimal properties for finite samples. However, as with other estimation methods, MLE has many attractive limiting properties such as consistency and asymptotic normality as the sample size increases to infinity, i.e., under smoothness assumptions [9], the MLE satisfies: $\lim_{N\to\infty} \theta_N = \theta$. Here, we focus on the estimation problem from another perspective. With a fixed sample size, how to solve (5) as accurate as possible.

C. Iterative estimate algorithm

We apply the idea of the EM algorithm to construct the iterative estimate algorithm for solving (5). The EM algorithm is an iterative method for finding the (local) maximum likelihood of parameters in a model. Since we are concerned with solving (5) for a given data $\mathscr{I}_N$ and $\mathscr{O}_N$, for ease of description, in this paper, the symbols $l(\theta)$ are used in the absence of conflicts. Given the estimate $\hat{\theta}_{N,t}$ (abbreviated as $\hat{\theta}_t$) at the t-th iteration, the EM algorithm constructs a function $l(\theta|\hat{\theta}_t)$ that satisfies the following two properties:

(i) $l(\theta|\hat{\theta}_t) \le l(\theta)$ holds for all θ, (ii) $l(\hat{\theta}_t|\hat{\theta}_t) = l(\hat{\theta}_t)$, and then calculate $\hat{\theta}_{t+1} = \arg\max_\theta l(\theta \mid \hat{\theta}_t)$. Thus, we have

$$l(\hat{\theta}_{t+1}) \ge \max l(\theta \mid \hat{\theta}_t) \ge l(\hat{\theta}_t \mid \hat{\theta}_t) = l(\hat{\theta}_t).$$

This ensures that the log-likelihood function is not reduced. The construction process for the function $l(\theta|\hat{\theta}_t)$ is the E-step and the maximization process is the M-step. Although the EM iterations do increase the likelihood function of the observed data, there is usually no guarantee that the series converge to the MLE. However, for the possibly incorrect binary-valued model (1) and (2), under some effective condition, we can construct a function $l(\theta|\hat{\theta}_t)$ and the EM series converges to the MLE. The function $l(\theta|\hat{\theta}_t)$ is defined as

$$\begin{aligned} l(\theta|\hat{\theta}_t) &= -\frac{1}{2}\theta^T\left(\sum_{k=1}^N \phi_k\phi_k^T\right)\theta + \left[\left(\sum_{k=1}^N \phi_k\phi_k^T\right)\hat{\theta}_t \right.\\ &\left. - \left(\sum_{k=1}^N \phi_k\left[\frac{g(C-\phi_k^T\hat{\theta}_t)I_{[s_k=1]}}{G(C-\phi_k^T\hat{\theta}_t)} - \frac{g(C-\phi_k^T\hat{\theta}_t)I_{[s_k=0]}}{1-G(C-\phi_k^T\hat{\theta}_t)}\right]\right)\right]^T\theta \\ &+ l_1(\hat{\theta}_t), \end{aligned}$$

where $l_1(\hat{\theta}_t)$ is the part independent of θ, $G(\cdot)$ is defined in (3) and $g(\cdot) = G'(\cdot)$. To guarantee the boundedness of $\hat{\theta}_t$, we introduce the following projection operator:

Definition 1. For a given convex compact set $\Theta \subseteq \mathbb{R}^p$ and a positive definite matrix A, the projection operator $\Pi_{\Theta,A}(\cdot)$ is defined as $\Pi_{\Theta,A}(x) = \arg\min_{\theta\in\Theta} (x-\theta)^T A(x-\theta),\ \forall x \in \mathbb{R}^p$.

Under Assumption 2, the iterative estimate for the batched possibly incorrect binary-valued observations is provided in Algorithm 1.

Algorithm 1 Iterative estimate for the batched possibly incorrect binary-valued observations:

Input and Initialization: The input data $\mathscr{I}_N = \{\phi_1, \phi_2, \ldots, \phi_N\}$, the binary-valued observation $\mathscr{O}_N = \{s_1, s_2, \ldots, s_N\}$, the probability of correct observation $P_N = \{p_1, p_2 \ldots, p_N\}$, the threshold value C, the initialization value $\hat{\theta}_1$ and the positive definite matrix $A = \sum_{k=1}^N \phi_k\phi_k^{\mathrm{T}}$.

Iterative: Under assumption 2, the estimate $\hat{\theta}_{t+1}$ is iterated by the following equation

$$\begin{aligned} \hat{\theta}_{t+1} = \hat{\theta}_t - \left(\sum_{k=1}^N \phi_k\phi_k^{\mathrm{T}}\right)^{-1}\left(\sum_{k=1}^N \phi_k \cdot g\left(C-\phi_k^{\mathrm{T}}\hat{\theta}_t\right)\right. \\ \left.\left[\frac{I_{[s_k=1]}}{G(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)} - \frac{I_{[s_k=0]}}{1-G(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)}\right]\right), \end{aligned} \tag{6}$$

or the projection version

$$\hat{\theta}_{t+1} = \Pi_{\Theta,A}\{\arg\max_{\theta} l(\theta|\hat{\theta}_t)\}. \tag{7}$$

Under some conditions on p_k, Algorithm 1 can be proved to converge to MLE (5) with an exponential rate. Before that, we discuss about the existence and uniqueness of (5).

III. Existence and Uniqueness of the MLE

In this section, we investigated the properties of MLE by analyzing the log-likelihood function (4). The concavity of the likelihood functions with incorrect binary observations is mainly investigated. In practice, both the communication process and the FIR process cause measurement errors. In the following, we give a functional form of the probability of correct observation, p_k:

$$p_k = h(C, \theta_0, \phi_k), \tag{8}$$

where the function $h(C, \theta_0, \phi_k)$ has a value range of $(0, 1)$.

Remark 3: When $h(C, \theta_0, \phi_k) = p^*$, $p^* \in (0, 1)$, it can characterize the situation where the communication process produces the error, which is independent of the system. In addition, due to the sensitivity of the sensor, the FIR process may also result in erroneous observations. Note that in practice, the probability of a correct observation occurring should be related to the error between the true value and the threshold value. The function $h(\cdot)$ can be constrained to be $h(C, \theta_0, \phi_k) = K(C - \theta_0^T\phi_k)$ with $K(\cdot)$ being some function, which describes the situation in which the FIR process leads to errors.

We first consider the case $p_k = p^*$ with $0 < p^* < 1$ being known. Denote $f(x) = F'(x)$ as the probability dense function of the standard normal distribution and we have $G(x) = 1 - F(x) - p^* + 2p^*F(x)$ and $g(x) = G'(x) = (2p^* - 1)f(x)$. However, in this case, the existence and uniqueness of the log-likelihood function (4) is only satisfied on a finite set in the parameter space. The following lemma illustrate this result.

Lemma 1: Let $p_k = p^*$ with $0 < p^* < 1$. Then, for all $k \leq N$, the log-likelihood function given in (4) is not a concave function on $\mathbb{R}^p$. While, given a finite set Θ in parameter space, if $p^* \geq \inf_{|x|\leq C+LM} \frac{xF(x)+f(x)-x}{2xF(x)+2f(x)-x}$, the log-likelihood function given in (4) is concave on Θ, where $L = \sup_{\theta\in\Theta} \|\theta\|$ and $M = \sup_k \|\phi_k\|$.

Proof: For any $k \leq N$, calculate the gradient vector and Hessian matrix of log-likelihood function:

$$\nabla l(\theta) = \left[\sum_{\{s_k=1\}} \frac{-g(C-\phi_k^T\theta)}{G(C-\phi_k^T\theta)} + \sum_{\{s_k=0\}} \frac{g(\phi_k^T\theta - C)}{G(\phi_k^T\theta - C)} \right] \phi_k, \tag{9}$$

$$\nabla^2 l(\theta) = -\sum_{k=1}^N \Bigg[\Bigg(\frac{g(x)xG(x)+g^2(x)}{G^2(x)}|_{x=C-\phi_k^T\theta} \cdot I_{[s_k=1]} + \frac{g(x)xG(x)+g^2(x)}{G^2(x)}|_{x=\phi_k^T\theta - C} \cdot I_{[s_k=0]} \Bigg) \phi_k\phi_k^T \Bigg].$$

Defining $p(x) = \frac{g(x)xG(x)+g^2(x)}{G^2(x)}$, and noting that $G(x) = 1 - G(-x)$ and $g(x) = g(-x)$, we can rewrite the Hessian matrix as

$$\nabla^2 l(\theta) = -\sum_{k=1}^N \left[p\left(C-\phi_k^T\theta\right) I_{[s_k=1]} + p\left(\phi_k^T\theta - C\right) I_{[s_k=0]} \right] \phi_k\phi_k^T. \tag{10}$$

Thus, the concaveness of the log-likelihood function is equivalent to the negative definite property of the Hessian matrix $\nabla^2 l(\theta)$. Since $A = \sum_{k=1}^N \phi_k\phi_k^T \geq 0$, if $\nabla^2 l(\theta) \leq 0$ for any $\theta \in \mathbb{R}^p$, then we have $p(x) \geq 0$ for any $x \in \mathbb{R}$. It follows that $g(x)xG(x) + g^2(x) \geq 0$ for ant $x \in \mathbb{R}$, that is

$$\begin{aligned} &x(1-F(x)-p^*+2p^*F(x))+(2p^*-1)f(x) \geq 0,\ \forall x > 0, \\ \Leftrightarrow\ &p^*\left[2xF(x)+2f(x)-x\right] \geq xF(x)+f(x)-x,\ \forall x > 0. \end{aligned} \tag{11}$$

Since $2F(x) - 1 \geq 0$ for any $x \geq 0$ and $2F(x) - 1 \leq 0$ for any $x \leq 0$, it follows that $[2F(x) - 1]x \geq 0$. By noting that $f(x) > 0$ for any $x \in \mathbb{R}$, we have $2xF(x) + 2f(x) - x = [2F(x) - 1]x + 2f(x) > 0$ for any $x \in \mathbb{R}$. Therefore, if (11) holds for any $x \in \mathbb{R}$, then p^* satisfies

$$p^* \geq \inf_x \frac{xF(x)+f(x)-x}{2xF(x)+2f(x)-x}. \tag{12}$$

While, by use of L'Hôpital's rule [3], we can obtain $\lim\limits_{x\to-\infty} \frac{xF(x)+f(x)-x}{2xF(x)+2f(x)-x} = 1$, which implies that $p^* \geq 1$. This contradiction shows that the log-likelihood function cannot be concave on $\mathbb{R}^p$ as long as p^* is not equal to 1. The remainder of the Lemma is directly justified by (12). This completes the proof. ■

Remark 4: Lemma 1 shows that the log-likelihood function cannot be concave on $\mathbb{R}^p$ for the fixed probability case. However if restricted to a finite set in the parameter space, the Hessian matrix $\nabla^2 l(\theta)$ may still be negative definite and Algorithm I will work. Simulations in Section V also verify the validity of Algorithm 1 in this case. Furthermore, in practice, the probability of the channel error can be estimated through a large number of prior experiments.

When $h(C, \theta_0, \phi_k) = K(C - \theta_0^T\phi_k)$, the function $G(\cdot)$ defined by the formula (3) and its derivative $g(\cdot)$ can be computed as $G(x) = (2K(x) - 1)F(x) + 1 - K(x)$ and $g(x) = G'(x) = 2K'(x)F(x) + (2K(x)-1)f(x) - K'(x)$. In this case, the following theorem gives the condition that the log-likelihood function has at most one maximum point.

Theorem 1: Consider the system (1)-(2) and the correct observation probability function form (8) with $h(C, \theta, \phi_k) = K(C - \theta^T\phi_k)$, and assume that Assumptions 1-2 hold. If $K(x)$ satisfies $\inf_{|x|\leq C+LM} \frac{g(x)xG(x)+g^2(x)}{G^2(x)} > 0$, where $L = \sup_{\theta\in\Theta} \|\theta\|$ and $M = \sup_k \|\phi_k\|$, then the log likelihood function $l(\theta)$ given in (4) has at most one maximum point.

Proof: Noting that $\inf_{|x|\leq C+LM} \frac{g(x)xG(x)+g^2(x)}{G^2(x)} > 0$, if limited on the set Θ, then for all $k \geq N$, there exists $\epsilon > 0$ such that $\min_{k\leq N} \left(p\left(C-\phi_k^T\theta\right) I_{[s_k=1]} + p\left(\phi_k^{\mathrm{T}}\theta - C\right) I_{[s_k=0]}\right) \geqslant \epsilon$. It follows that $\nabla^2 l(\theta) \leqslant -\epsilon \sum_{k=1}^N \phi_k\phi_k^{\mathrm{T}} = -\epsilon A$. By Assumption 2, it infers the strongly concave property of the function $l(\theta)$ on the set Θ. From the strongly concave property on Θ, log-likelihood function $l(\theta)$ given in (4) has at most one maximum point on Θ. ■

To reveal the condition that MLE exists, following the results in [20], we use the following condition.

Definition 2. Denote $\Psi = [\phi_1(I_{[s_1=0]} - I_{[s_1=1]}), \ldots, \phi_N(I_{[s_N=0]} - I_{[s_N=1]})]$. Given input $\mathscr{I}_N$ and binary-valued observations $\mathscr{O}_N$, if there exists a non-zero vector $\gamma \in \mathbb{R}^n$ such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$, then the data $(\mathscr{I}_N, \mathscr{O}_N)$ is called ineffective, otherwise it is called effective.

Remark 5: Definition 1 is given for observed and input data. The existence of the nonzero vector γ causes the maximum likelihood function to increase in the direction along γ, so that there is no maximum point. See the proof of Theorem 2 for details.

The following theorem gives the sufficient and necessary condition for the existence of a unique maximum point of the log-likelihood function on the set of parameter Θ.

Theorem 2: Consider the system (1)-(2) and the form of the correct observation probability function (8) with $h(C, \theta_0, \phi_k) = K(C - \theta^T\phi_k)$, and assume that Assumptions 1-2 hold . If $K(x)$ satisfies the assumptions in Theorem 1 and $\forall |x| \leq C + LM$, we have $G'(x) = 2K'(x)F(x) + (2K(x)-1)f(x) - K'(x) > 0$ where $L = \sup_{\theta\in\Theta} \|\theta\|$ and $M = \sup_k \|\phi_k\|$, then there exists a unique maximum point of the log-likelihood function (4) on Θ if and only if the data set $(\mathscr{I}_N, \mathscr{O}_N)$ is effective.

Proof: We prove the theorem from two directions.

Sufficiency. The proof is similar to that of Theorem 2 in [20], and so, omitted here.

Necessity. If $(\mathscr{I}_N, \mathscr{O}_N)$ is ineffective, then there exists a non-zero vector $\gamma \in \mathbb{R}^n$ such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$. Since $\Psi\Psi^{\mathrm{T}} = \sum_{k=1}^N \phi_k\phi_k^{\mathrm{T}}$, by Assumption 1 we have $\Psi^{\mathrm{T}}\gamma > 0$, which implies that there exists at least one positive component for vector $\Psi^{\mathrm{T}}\gamma$. Given any parameter θ, define a scalar function $h_{\theta,\gamma}(r)$ as follows: $h_{\theta,\gamma}(r) = l(\theta + r\gamma) = \sum_{k=1}^N \left[\log\left[G\left(-\phi_k^T\gamma r + C - \phi_k^T\theta\right)\right] I_{[s_k=1]}\right.$

$+\log[G(\phi_k^T\gamma r+\phi_k^T\theta-C)]I_{[s_k=0]}\big]$. Note that the k-th component of $\Psi^{\mathrm{T}}\gamma$ is $\left(\Psi^{\mathrm{T}}\gamma\right)_k=-\phi_k^{\mathrm{T}}\gamma I_{[s_k=1]}+\phi_k^{\mathrm{T}}\gamma I_{[s_k=0]}$ which is consistent with the coefficient of r in $h_{\theta,\gamma}(r)$. This together with the assumption $G'(x)=2K'(x)F(x)+(2K(x)-1)f(x)-K'(x)>0$ and that $\log(\cdot)$ is increasing concludes that $h_{\theta,\gamma}(r)$ is a strictly increasing function. Assume that there exists a maximum value point θ^* of $l(\theta)$, which contradicts with that $h_{\theta^*,\gamma}(r)=l(\theta+r\gamma)$ is increasing. Then, $l(\theta)$ does not have any maximum points. This completes the proof. ■

IV. Convergence of the Estimation Error

In this section, we will use the Lyapunov method to prove the convergence of Algorithm 1 in the case of communication process error and FIR process error.

As Theorem 2 shows, if data $(\mathscr{I}_N,\mathscr{O}_N)$ is ineffective, then the log-likelihood function does not have any finite maximum point. Hence, the following assumption is necessary.

Assumption 3: Data $(\mathscr{I}_N,\mathscr{O}_N)$ is effective.

The following theorem proves the convergence of the Algorithm 1.

Theorem 3: Consider the system (1)-(2), the correct observation probability function form (8) where $h(C,\theta_0,\phi_k)=K(C-\theta_0^T\phi_k)$ or $h(C,\theta_0,\phi_k)=p^*$, and the iterative solution Algorithm 1, if it is assumed that Assumptions 1-3 holds, then there exists $\epsilon\in(0,1)$ such that the sequence $\{\hat{\theta}_t\}$ iteratively generated by (6) or (7) satisfies

$$\left\|\hat{\theta}_t-\hat{\theta}\right\|\leqslant\sqrt{\frac{Q_1}{\lambda_{\min}(A)}}\cdot\frac{\sqrt{(1-\epsilon)}^t}{1-\sqrt{(1-\epsilon)}},$$

where $\hat{\theta}$ is the MLE given in (5); $A=\sum_{k=1}^N\phi_k\phi_k^{\mathrm{T}}$, $\lambda_{\min}(A)$ is the minimal eigen value of A; $Q_1=(1-\epsilon)^{-1}(\hat{\theta}_2-\hat{\theta}_1)^{\mathrm{T}}A(\hat{\theta}_2-\hat{\theta}_1)$; and $\|\cdot\|$ is the Euclidean norm.

Before giving the proof of the theorem, the following properties of the projection operator are given.

Lemma 2: [5] The projection operator given in Definition 1 satisfies $\forall x,y\in\mathbb{R}^p$, $\|\Pi_{\Theta,A}(x)-\Pi_{\Theta,A}(y)\|_A\leq\|x-y\|_A$.

Proof: First, from equation (9), we convert equation (6) into $\hat{\theta}_{t+1}=\Pi_{\Theta,A}(\hat{\theta}_t+(\sum_{k=1}^N\phi_k\phi_k^{\mathrm{T}})^{-1}\nabla l(\hat{\theta}_t))=\Pi_{\Theta,A}(\hat{\theta}_t+A^{-1}(\nabla l(\hat{\theta}_t)))$. Iterating these equations and by Lemma 2, we can get

$$\|\hat{\theta}_{t+1}-\hat{\theta}_t\|_A\leq\|\hat{\theta}_t-\hat{\theta}_{t-1}+A^{-1}(\nabla l(\hat{\theta}_t)-\nabla l(\hat{\theta}_{t-1}))\|_A. \quad (13)$$

By the mean value theorem, we have

$$\nabla l(\hat{\theta}_t)-\nabla l(\hat{\theta}_{t-1})=\nabla^2 l(\check{\theta}_{t,t-1})(\hat{\theta}_t-\hat{\theta}_{t-1}), \quad (14)$$

where $\check{\theta}_{t,t-1}$ between $\hat{\theta}_t$ and $\hat{\theta}_{t-1}$, i.e., there exists a constant $0<c<1$ such that $\check{\theta}_{t,t-1}=(1-c)\hat{\theta}_t+c\hat{\theta}_{t-1}$. From (13) and (14), we have

$$\|\hat{\theta}_{t+1}-\hat{\theta}_t\|_A\leq\|(I+A^{-1}(\nabla^2 l(\check{\theta}_{t,t-1})))(\hat{\theta}_t-\hat{\theta}_{t-1})\|_A. \quad (15)$$

By Theorem 1, there exists $\epsilon>0$ such that for any $t\geq1$, $-A<\nabla^2 l\left(\check{\theta}_{t,t-1}\right)\leqslant-\epsilon\sum_{k=1}^N\phi_k\phi_k^T=-\epsilon A$. Let $B_t=-\nabla^2 l\left(\check{\theta}_{t,t-1}\right)$, $x_t=\hat{\theta}_{t+1}-\hat{\theta}_t$. Then, from (15) we get $\|x_t\|_A\leq\|(I-A^{-1}B_t)(x_{t-1})\|_A$. Let the Lyapunov function $Q_t=(1-\epsilon)^{-t}x_t^TAx_t$, then we have $Q_t\leq(1-\epsilon)^{-t}x_{t-1}^{\mathrm{T}}\left(I_n-A^{-1}B_t\right)^{\mathrm{T}}\cdot A\left(I_n-A^{-1}B_t\right)x_{t-1}$. The rest of the proof is similar to Theorem 3 in [20], which we omit here. ■

This theorem proves the convergence of the iterative algorithm, while obtaining the convergence rate. This is also with the same convergence rate as that of the iterative algorithm obtained for the model without errors.

V. Simulation Study

In this section, we illustrate the main results with a number of simulations.

A. Log-likelihood function curve

To show the log-likelihood function intuitively, we restrict the model dimension to $p=1$. Consider the system $y_k=\theta_0\phi_k+e_k$ with true parameter $\theta_0=1$.

Data Generate Process. Fix sample size $N=20$. Then, we generate the input ϕ_k and the noise e_k with the following Matlab codes: $\mathrm{Phi}=\mathrm{randn}(1,\mathrm{N})$; $\mathrm{E}=\mathrm{randn}(1,\mathrm{N})$.

Log-likelihood function curve. For the fixed probability case, we set $p_k=0.99$ and $p_k=1$, where the latter is indeed the case without error. For the time-varying probability case with $p_k=K(C-\theta\phi)$ with $K(x)=1-\frac{1}{10}e^{-1/2x^2}$. Log-likelihood functions $l(\theta)$ of these three cases, where $\theta\in(-4,4)$, are all shown in Fig. 1. The figure shows that for the fixed probability case, the log-likelihood function is not concave as long as there is a slight probability of the error, while it is still concave in some finite set of the parameter space. For some special time-varying probability case, the log-likelihood function is indeed concave.

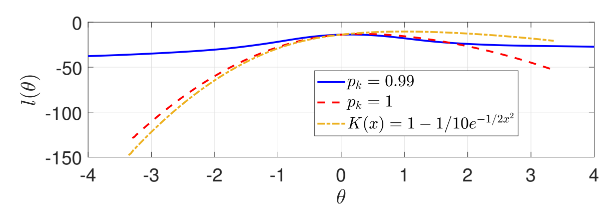

Fig. 1: Log-likelihood function for the fixed probability case.

B. Convergence of the proposed iterative algorithm

In this section, the convergence of the proposed iterative algorithm (6) or (7) is demonstrated by numerical simulations. The brief simulation procedure is as follows:

Step 1. Data generation. Fix the data length $N=1000$, the model dimension $p=3$, the sensor threshold $C=1$, and the model parameter $\theta=(-1,0,1)^{\mathrm{T}}$. Error $\mathscr{E}_N$ and input data $\mathscr{I}_N$ are generated based on standard normal distribution. Matlab codes are: error $=[\mathrm{randn}(N,1)]$; phi $=[\mathrm{randn}(N,3)]$. The binary-valued observations $\mathscr{O}_N$ are generated according to the model (1) and (2).

Step 2. Initial vector selection. To demonstrate that the EM algorithm can converge to a unique MLE, with the same effective data $\{\mathscr{I}_N,\mathscr{O}_N\}$, we adopt a random vector as the iterative initial vector $\hat{\theta}_1$. $\hat{\theta}_1$ are generated by the Matlab code: initial $=\mathrm{randn}(3,1)$.

Step 3. Parameter estimate. Based on the initial value $\hat{\theta}_1$ and iteration process (6), we can generate the iteration estimates $\{\hat{\theta}_t, t \geqslant 1\}$.

The simulation results of the iterative estimation are shown in Fig. 2 with $K(x) = 1 - 1/8e^{-x^2}$ and $p^* = 0.8$. With different initial vectors, all the estimated components $\{\hat{\theta}_t\}$ converge to a unique ML estimate, which is fairly close to the true parameters. Finally, we have also estimated this case using the algorithm in [20] without considering the possible incorrectness and the results are shown in Fig. 3 with $K(x) = 1 - 1/10e^{-1/2x^2}$. The figure shows that the algorithm in [20] will not converge to the true parameters when there exists possible incorrectness.

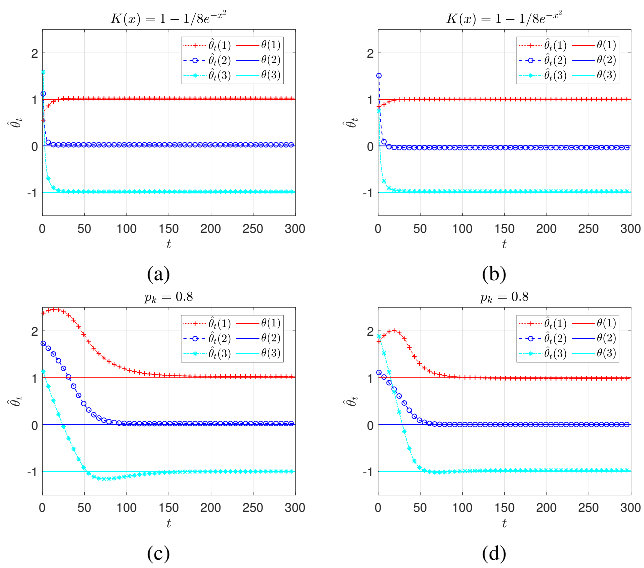

Fig. 2: The estimated components of $\{\hat{\theta}_t\}$ with different initial vectors for the case $K(x) = 1 - 1/8e^{-x^2}$ and $p^* = 0.8$.

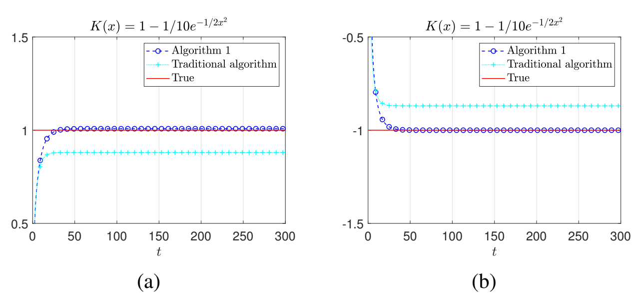

Fig. 3: The estimated components $\{\hat{\theta}_t\}$ using the algorithm in [20] and Algorithm 1.

VI. Conclusion

In this paper, we consider the parameter identification problem of a binary output FIR system with measurement error, in which the measurement error makes the binary measurement values have a certain probability of getting the opposite values. Firstly, the likelihood function are calculated and the MLE is given. Secondly, based on the EM algorithm, the iterative solution algorithm of MLE is given. To ensure the boundedness of the iterative estimation sequence, the iterative algorithm with projection is proposed. In addition, the necessary and sufficient conditions for the uniqueness of the MLE are given and the iterative estimation error is proved to converge to zero at an exponential rate. Finally, numerical simulation results demonstrate the effectiveness of the proposed method. Further consideration can be given to applying the algorithm to the general case and designing more robust algorithms.

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