2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Inhomogeneous Singular Linear Switched Systems in Discrete Time: Solvability, Reachability, and Controllability Characterizations

Sutrisno[1,2], Stephan Trenn[1]

***Abstract*— In this paper we study a novel solvability notion for discrete-time singular linear switched systems with inputs. We consider the existence and uniqueness of a solution on arbitrary finite time intervals with arbitrary inputs and arbitrary switching signals, and furthermore, we pay special attention to strict causality, i.e. the current state is only allowed to depend on past values of the state and the input. A necessary and sufficient condition for this solvability notion is then established. Furthermore, a surrogate switched system (an ordinary switched system that has equivalent input-output behavior) is derived for any solvable system. By utilizing those surrogate systems, we are able to characterize the reachability and controllability properties of the original singular systems using a geometric approach.**

I. Introduction

We consider in this study a class of switched systems where each mode is a discrete-time singular linear system of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \tag{1}$$

where $k \in \mathbb{N}$ is the time instant/step, $x(k) \in \mathbb{R}^n$ is the vector of states, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ is the vector of inputs, $\sigma : \mathbb{N} \to \{0, 1, 2, ..., \mathtt{p}\}$ is the switching signal ruling which mode $\sigma(k)$ is active at time instant k, $E_i, A_i \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times m}$. The matrices E_i are in general singular yet may be nonsingular, and thus system (1) also covers ordinary systems. The switching signal σ is triggered only by the time and not by the state vectors or the input values. Furthermore, the switching signal σ is assumed to have the following form

$$\sigma(k) = \sigma_j \text{ if } k \in [k^s_j, k^s_{j+1}),\ j = \{0, 1, 2, ...\}, \tag{2}$$

where $k^s_j \in \mathbb{N}$ denote the switching times with $k^s_0 = 0$ and $\sigma_j \in \{0, 1, ..., \mathtt{p}\}$. Assume that the switching times are strictly increasing i.e. $k^s_{j+1} > k^s_j$. This means that each mode in the switched system is active for at least one time instant every time it is active. The considered switching signal form is illustrated in Fig. 1.

The pioneering study for the non-switched case of (1) was established a few decades ago, which covers the solution theory as well as the fundamental properties including controllability, see e.g. [1], [2], [3], [4]. However, the consistency set (the set containing all consistent initial values) lacks studies until a geometric approach and a projector lemma

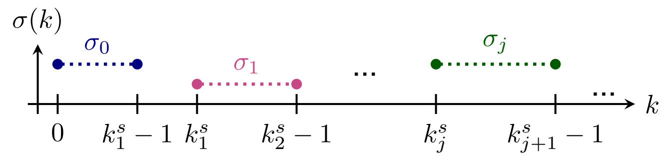

Fig. 1. The mode sequence (2)

were utilized in [5] in establishing the consistency set. Furthermore, in the same study, the one-step map was also introduced, which was then used to formulate the surrogate system, which is an ordinary system having the same input-output behavior.

Meanwhile, switched systems deserve deep research both in theory and practical applications as switching among diverse system structures is a fundamental component in many systems such as power systems [6] and electronics [7]. In particular, switched systems also arise naturally in sampled-data systems [8], [9], [10]. If all E_i are nonsingular, systems of the form (1) belong to (ordinary) linear switched systems, which have been extensively studied for the solution theory as well as its fundamental properties including observability, determinability, reachability, controllability, and stability (see e.g. [11], [12], [13], [14]). In particular, the matrices E_i in system (1) may be singular in the applications in various fields such as economic systems [15] and constrained mechanical systems [16], [17]. The solution theory for the switched system (1) is still limited in literature due to the complexity of finding a condition for the existence and uniqueness of a solution with the presence of singular E_i. A recent study in [5] provides a solvability characterization for system (1), and surrogate systems were also established; however, the corresponding one-step map formula depends on the mode at $k = -1$ which is not clear how it affects the solution, and the solution at a time instant depends not only on past states and inputs but also on the current input.

Besides the solution theory, reachability and controllability properties are essential that need to be studied for analysis and control design purposes. In the continuous time domain, controllability of system class (1) has been extensively studied, see e.g. [18], [19], however, systems in the discrete-time domain still lack studies.

In this paper, the solution theory is studied for system (1) in which the solution at any time instant depends only on past information. The corresponding necessary and sufficient conditions for this solvability notion are also established. For solvable systems, surrogate systems are then introduced

[1]Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands `s.sutrisno[@rug.nl, @live.undip.ac.id],s.trenn@rug.nl`
[2]Dept. of Mathematics, Universitas Diponegoro, Jalan Prof. Soedarto, SH. Tembalang 50275, Semarang, Indonesia

979-8-3503-0123-6/23/$31.00 ©2023 IEEE

which can be used to derive the systems' explicit solutions. Moreover, by utilizing those surrogate systems, necessary and sufficient conditions for reachability and controllability characterizations are established.

In the following we let $M^{-1}\mathcal{X}$ denote the preimage of a (possibly) singular matrix $M \in \mathbb{R}^{n\times n}$ over a set $\mathcal{X}$, i.e. $M^{-1}\mathcal{X} = \{\xi \in \mathbb{R}^n : M\xi \in \mathcal{X}\}$. A generalized inverse of $M \in \mathbb{R}^{m\times n}$ is a matrix $M^+ \in \mathbb{R}^{n\times m}$ that satisfies $MM^+M = M$ [20]; M^+ is not unique, but one possible choice is the well known Moore-Penrose inverse. Furthermore, we have that

$$M^{-1}\{x\} = \{M^+x\} + \ker M \ \forall x \in \mathbb{R}^m. \tag{3}$$

The natural numbers (including zero) are denoted by $\mathbb{N}$ and we also use the interval notation for integers, i.e. $[k_1, k_2] := \{k_1, k_1+1, \dots, k_2-1, k_2\}$. The symbol $\oplus$ denotes the direct sum of subspaces, in particular, when writing $\mathcal{V} \oplus \mathcal{W}$ we implicitly assume/require that $\mathcal{V} \cap \mathcal{W} = \{0\}$.

II. Solution Theory

A. Definition and Characterization

Recall the Inhomogeneous Singular Linear Switched System (InhSLSS) (1). For each mode i, define $\widehat{\mathcal{S}}_i := A_i^{-1}(\mathrm{im}[E_i, B_i]) = \{\xi \in \mathbb{R}^n : A_i\xi \in \mathrm{im}[E_i, B_i]\}$. In this study, we consider the following solvability notion in which we want to have, for any arbitrary switching signal, a unique solution on any finite discrete time interval $[k_0, k_1]$, $k_0, k_1 \in \mathbb{N}$, $k_0 < k_1$ with $x(k_0) = x_{k_0} \in \widehat{\mathcal{S}}_{\sigma(k_0)}$, and the state at any $k > k_0$, $x(k)$, is determined by only $x(k_0)$ and past inputs $u(k_0), u(k_0+1), \dots, u(k-1)$, i.e., the solution behavior is strictly causal.

Definition 2.1 (Solvability of InhSLSSs) We call system (1) *locally strictly causally uniquely solvable* (for short just *solvable*) if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$, all $x_{k_0} \in \widehat{\mathcal{S}}_{\sigma(k_0)}$, all input sequences $(u(k_0), u(k_0+1), ..., u(k_1-1))$ and all switching signals there exists a unique sequence $(x(k_0), x(k_0+1), \dots, x(k_1))$ with $x(k_0) = x_{k_0}$ such that (1) holds for all $k \in [k_0, k_1]$ and for some $u(k_1)$.

Strict causality is required here in which the solution at k_1, $x(k_1)$, depends only on the past states and inputs. Furthermore, a unique solution is required for the system starting from any initial time.

We present a necessary and sufficient condition for the InhSLSS (1) to become solvable in the following theorem. Furthermore, the surrogate switched system is also introduced in this theorem.

Theorem 2.2: The InhSLSS (1) is solvable in the sense of Definition 2.1 if and only if

$$E_j^+A_j\widehat{\mathcal{S}}_j + \mathrm{im}\, E_j^+B_j \subseteq \ker E_j \oplus \widehat{\mathcal{S}}_i \ \forall i, j \in \{0, 1, ..., \mathtt{p}\}. \tag{4}$$

If solvable, its solution satisfies

$$x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)}x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)}u(k), \tag{5}$$

where $\widehat{\Phi}_{i,j} = \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+A_j$, $\widehat{\Theta}_{i,j} = \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+B_j$, the matrix E_j^+ is a generalized inverse of E_j and $\Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j}$ is the canonical projector from $\ker E_j \oplus \widehat{\mathcal{S}}_i$ to $\widehat{\mathcal{S}}_i$. In particular, $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$ for all $k \in \mathbb{N}$.

Proof: Part 1: the solvability condition
Necessity: For any solution $x(k)$ at any time instant k of any mode j, the solution $x(k+1)$ of any mode i satisfies $E_jx(k+1) = A_jx(k) + B_ju(k)$ which implies, by the preimage property (3),

$$\begin{aligned} x(k+1) &\in E_j^{-1}(A_j\widehat{\mathcal{S}}_j + \mathrm{im}\, B_j) \\ &= E_j^+A_j\widehat{\mathcal{S}}_j + \mathrm{im}\, E_j^+B_j + \ker E_j. \end{aligned}$$

The solution $x(k+1)$ also satisfies $E_i\xi_1 = A_ix(k+1) + B_i\xi_2$ for some $\xi_1 \in \mathbb{R}^n$ and $\xi_2 \in \mathbb{R}^m$. Again, by the same preimage property, we have

$$x(k+1) \in A_i^{-1}(\mathrm{im}[E_i, B_i]) = \widehat{\mathcal{S}}_i. \tag{6}$$

By applying $\mathcal{U} = E_j^+A_j\widehat{\mathcal{S}}_j + \mathrm{im}\, E_j^+B_j$, $\mathcal{V} = \widehat{\mathcal{S}}_i$ and $\mathcal{W} = \ker E_j$ to the projector lemma in [21, Lem. 2.3][1], the uniqueness of $x(k+1)$ implies $E_j^+A_j\widehat{\mathcal{S}}_j + \mathrm{im}\, E_j^+B_j \subseteq \ker E_i \oplus \widehat{\mathcal{S}}_i$ for all $i, j \in \{0, 1, ..., \mathtt{p}\}$.
Sufficiency: The proof is done by induction. First, we will show that for all $x(0) = x_0 \in \widehat{\mathcal{S}}_{\sigma(0)}$, all $u(0) \in \mathbb{R}^m$, and all switching signals $\sigma(0) = j$ and $\sigma(1) = i$, there exists a unique $x(1)$ which satisfies (1) at $k = 0$ and $k = 1$ i.e.

$$\begin{aligned} E_jx(1) &= A_jx(0) + B_ju(0) \\ E_i\xi &= A_ix(1) + B_i\nu \end{aligned}$$

for some $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^m$. Again, by the preimage property (3), the latter is equivalent to

$$\begin{aligned} x(1) &\in E_j^{-1}(A_jx_0 + B_ju(0)) \\ &= \{E_j^+A_jx_0 + E_j^+B_ju(0)\} + \ker E_j \\ x(1) &\in A_i^{-1}(\mathrm{im}[E_i, B_i) = \widehat{\mathcal{S}}_i. \end{aligned}$$

The condition $E_j^+(A_jx(0) + B_ju(0)) \subseteq \widehat{\mathcal{S}}_i \oplus \ker E_j$ implies that

$$\{E_j^+A_jx_0 + E_j^+B_ju(0)\} + \ker E_j \cap \widehat{\mathcal{S}}_i$$

is a singleton (be the projector lemma) for all $x_0 \in \widehat{\mathcal{S}}_j$ and all $u(0) \in \mathbb{R}^m$. Thus, a vector $x(1) \in \mathbb{R}^n$ satisfying (1) exists and is unique. Repeating the same argument, we can show that for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$, all $x(k_0) \in \widehat{\mathcal{S}}_{\sigma(k_0)}$, all $(u(0), u(1), ...)$ and all switching signals, a unique solution $(x(k_0), x(k_0+1), ..., x(k_1))$ exists and is determined only by past states and inputs.
Part 2: the surrogate system (5)
For every time instant k, switching signal σ, solution $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$, and input $u(k) \in \mathbb{R}^m$, the intersection $E_{\sigma(k)}^+(A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)) + \ker E_{\sigma(k)} \cap \widehat{\mathcal{S}}_{\sigma(k+1)}$ provides $x(k+1)$. Putting $\mathcal{U} = E_{\sigma(k)}^+(A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k))$, $\mathcal{V} = \widehat{\mathcal{S}}_{\sigma(k+1)}$ and $\mathcal{W} = \ker E_{\sigma(k)}$ into formula (7) in the

[1]For subspaces $\mathcal{U}, \mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is a singleton for all $u \in \mathcal{U}$ if and only if $\mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$. In that case

$$\mathcal{V} \cap (\{u\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}}u\}, \tag{7}$$

where $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} \oplus \mathcal{W} \to \mathcal{V}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$.

projector lemma proves that $x(k+1)$ satisfies (5). Finally, the inclusion $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$ is a direct consequence of $x(k)$ solving (1); this can also be seen from (6). ∎

By utilizing the surrogate ordinary switched system (5), the explicit solution of (1) can be written as

$$\begin{aligned} x(k) =& \Pi_{j=1}^{k} \widehat{\Phi}_{\sigma(k+1-j),\sigma(k-j)} x(0) \\ &+ \Pi_{j=1}^{k-1} (\widehat{\Phi}_{\sigma(j+1),\sigma(j)}) \widehat{\Theta}_{\sigma(1),\sigma(0)} u(0) + \cdots \\ &+ \widehat{\Phi}_{\sigma(k),\sigma(k-1)} \widehat{\Theta}_{\sigma(k-1),\sigma(k-2)} u(k-2) \\ &+ \widehat{\Theta}_{\sigma(k),\sigma(k-1)} u(k-1). \end{aligned} \tag{8}$$

B. Discussion on Unswitched Systems

The results are also valid for unswitched systems of the form

$$Ex(k+1) = Ax(k) + Bu(k), \; k \in \mathbb{N}, \tag{9}$$

with $\widehat{\mathcal{S}} := A^{-1}(\mathrm{im}[E, B]) = \{\xi \in \mathbb{R}^n : A\xi \in \mathrm{im}[E, B]\}$. This is presented in the following corollary. For unswitched systems, the solvability notion in Definition 2.1 is considered with a constant switching signal where its mode corresponds to (9).

Corollary 2.3 (Solvability of unswitched systems) System (9) is solvable (in the sense of Definition 2.1) if, and only if,

$$E^+ A\widehat{\mathcal{S}} + \mathrm{im}\, E^+ B \subseteq \ker E \oplus \widehat{\mathcal{S}}. \tag{10}$$

If solvable, its solution satisfies

$$x(k+1) = \widehat{\Phi} x(k) + \widehat{\Theta} u(k), \; x(0) \in \widehat{\mathcal{S}}, \; k = 0, 1, \dots, \tag{11}$$

where $\widehat{\Phi} = \Pi_{\widehat{\mathcal{S}}}^{\ker E} E^+ A$, $\widehat{\Theta} = \Pi_{\widehat{\mathcal{S}}}^{\ker E} E^+ B$, E^+ is a generalized inverse of E and $\Pi_{\widehat{\mathcal{S}}}^{\ker E}$ is the canonical projector from $\ker E \oplus \widehat{\mathcal{S}}$ to $\widehat{\mathcal{S}}$. In particular, $x(k) \in \widehat{\mathcal{S}}$ for all $k \geq 0$.

One crucial observation for the solvability of switched systems related to unswitched systems is that solvability for individual modes (as unswitched systems) is in general not sufficient for switched systems composed of those modes to become also solvable. This is already confirmed by the system in Example 4.1 where a switched system composed of solvable individual modes may be not solvable, we thus have that the condition (10) satisfied by each mode is not sufficient for switched systems to become solvable.

III. Reachability and Controllability: Single Switch Case

The basic intuition for reachability is to find the set of all final states reachable within finite time steps starting from a given initial state. Meanwhile, controllability (to zero) deals with finding initial values that can be brought to zero within some finite time steps. Those two notions are in fact equivalent when considering continuous-time non-switched systems, see e.g. [22, Lem. 2.3]. However, they are not equivalent in discrete time; this is already well-known in ordinary systems, see e.g. [23]. In singular systems, this is also true, see the forthcoming Remark 3.8.

We restrict our attention in this section to only single switch switching signals considered on the finite time domain $[0, K],\; K \in \mathbb{N}$ of the form (see also Fig. 2 for illustration)

$$\sigma(k) = \begin{cases} 0, & 0 \leq k < k^s, \\ 1, & k^s \leq k \leq K. \end{cases} \tag{12}$$

Thus, in this section, we consider switched systems composed of two modes; it starts from mode (E_0, A_0, B_0) with the corresponding consistency space $\widehat{\mathcal{S}}_0$ and switches at the switching time k^s to mode (E_1, A_1, B_1) with the corresponding consistency space $\widehat{\mathcal{S}}_1$.

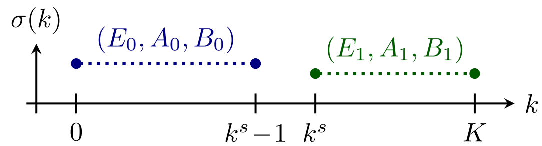

Fig. 2. Single switch switching signal for (1)

A. Definitions

The reachability and controllability notions considered in this study are mathematically defined for the switched system (1) as follows:

Definition 3.1 (Reachability from zero) A state $x_f \in \widehat{\mathcal{S}}_1$ of the InhSLSS (1) is called *reachable from zero* on $[0, K], K > k^s$ w.r.t. the single switch switching signal given by (12) if with $x(0) = 0$, there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K) = x_f$.

Definition 3.2 (Reachable set and reachability) The reachable set from zero of system (1) on $[0, K], K > k^s$ w.r.t. σ of the form (12) is the set of all final states $x_f \in \widehat{\mathcal{S}}_1$ which are reachable from zero on $[0, K]$ and denoted by $\mathcal{R}^{\sigma}_{[0,K]}$. In particular, the InhSLSS (1) is called *reachable from zero* on $[0, K]$ if $\mathcal{R}^{\sigma}_{[0,K]} = \widehat{\mathcal{S}}_1$.

Definition 3.3 (Controllability to zero) A consistent initial state $x_0 \in \widehat{\mathcal{S}}_0$ of (1) is called *controllable to zero* on $[0, K], K > k^s$ w.r.t. the single switch switching signal of the form (12) if with $x(0) = x_0$, there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K) = 0$.

Definition 3.4 (Controllable set and controllability) The controllable set to zero of system (1) on $[0, K], K > k^s$ is the set of all consistent initial states $x_0 \in \widehat{\mathcal{S}}_0$ which are controllable to zero on $[0, K]$ and denoted by $\mathcal{C}^{\sigma}_{[0,K]}$. In particular, the InhSLSS (1) is called *controllable to zero* on $[0, K]$ if $\mathcal{C}^{\sigma}_{[0,K]} = \widehat{\mathcal{S}}_0$.

B. Characterizations

Let $\mathcal{R}_i(k) = \mathrm{im}\, R_i(k) = \mathrm{im}\left[\widehat{\Theta}_i, \widehat{\Phi}_i\widehat{\Theta}_i, \cdots, \widehat{\Phi}_i^{k-1}\widehat{\Theta}_i\right]$ for mode $i = 0, 1$, and define the following subspaces

$$\begin{aligned} \mathcal{P}_0 =& \widehat{\mathcal{S}}_0 \cap \mathcal{R}_0(k^s - 1), \\ \mathcal{P}_1 =& \widehat{\mathcal{S}}_1 \cap \Big(\widehat{\Phi}_1^{K-k^s} \widehat{\Phi}_{1,0} \mathcal{P}_0 + \mathrm{im}\, \widehat{\Phi}_{1,0}^{K-k^s} \widehat{\Theta}_{1,0} \\ &+ \mathcal{R}_1(K - k^s)\Big). \end{aligned} \tag{13}$$

We present the main result for the reachability characterization in the following theorem.

Theorem 3.5 (Reachability) Consider the solvable InhSLSS (1). Let $\mathcal{R}^{\sigma}_{[0,K]}$ be its reachable set on $[0, K]$ w.r.t. the single switch switching signal (12). Then

$$\mathcal{P}_1 = \mathcal{R}^{\sigma}_{[0,K]}, \tag{14}$$

where $\mathcal{P}_1$ is given by (13). In particular, the InhSLSS (1) is reachable if, and only if, $\mathcal{P}_1 = \widehat{\mathcal{S}}_1$.

Proof: From the explicit solution formula (8), the solution of (1) with $x(0) = 0$ at $k = K > k^s$ can be written as

$$x(K) = R_1(K-k^s)\begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(k^s) \end{bmatrix} + \widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0}u(k^s-1)$$
$$+\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}R_0(k^s-1)\begin{bmatrix} u(k^s-2) \\ u(k^s-3) \\ \vdots \\ u(0) \end{bmatrix}. \tag{15}$$

Step 1: Reachable space
Step 1.a: Proof of $\mathcal{P}_1 \supseteq \mathcal{R}^{\sigma}_{[0,K]}$. Pick any reachable state $x(K) \in \mathcal{R}^{\sigma}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that (15) is satisfied i.e. $x(K) \in \widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{P}_0 + \operatorname{im}\widehat{\Phi}_{1,0}^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K-k^s)$. On the other hand, from the proof of Theorem 2.2, note that $x(k) \in \widehat{\mathcal{S}}_0$ for all $k \in [0, k^s)$ and $x(k) \in \widehat{\mathcal{S}}_1$ for all $k \in [k^s, K]$. Thus, $x(K) \in \widehat{\mathcal{S}}_1 \cap \Big(\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{P}_0 + \operatorname{im}\widehat{\Phi}_{1,0}^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K-k^s)\Big) = \mathcal{P}_1$, and hence $\mathcal{R}^{\sigma}_{[0,K]} \subseteq \mathcal{P}_1$.
Step 1.b: Proof of $\mathcal{P}_1 \subseteq \mathcal{R}^{\sigma}_{[0,K]}$. Pick any $x_f \in \mathcal{P}_1$. Then, there exists a vector $\bar{u} \in \mathbb{R}^{(K\times m)\times 1}$ with the structure $\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}$ with $\bar{u}_1 \in \mathbb{R}^{(k^s-1\times m)\times 1}$, $\bar{u}_2 \in \mathbb{R}^{m\times 1}$, and $\bar{u}_3 \in \mathbb{R}^{(K-k^s\times m)\times 1}$ such that

$$R_1(K-k^s)\bar{u}_1 + \widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0}\bar{u}_2 +$$
$$\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}R_0(k^s-1)\bar{u}_3 = x_f,$$

i.e. x_f is reachable (from zero) by considering $\bar{u}$ as the input. Thus, $x_f \in \mathcal{R}^{\sigma}_{[0,K]}$, and hence $\mathcal{P}_1 \subseteq \mathcal{R}^{\sigma}_{[0,K]}$. Altogether, we get $\mathcal{P}_1 = \mathcal{R}^{\sigma}_{[0,K]}$.
Step 2: reachability
This is the direct consequence of its definition and the first part of this theorem. ∎

Remark 3.6: Note that reachable from zero on $[0, K]$ is equivalent to *reachable* on $[0, K]$ i.e. every $x_f \in \mathcal{R}^{\sigma}_{[0,K]}$ is reachable from any consistent initial value $x_0 \in \widehat{\mathcal{S}}_0$. This can be seen from the fact that putting the term of the solution that contains the nonzero initial value, $\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}x_0$, into (15) yields the same reachable set.

We now present the main result for the controllability characterization. First, define the subspaces

$$\mathcal{Q}_1 = \widehat{\mathcal{S}}_1 \cap \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1}\mathcal{R}_1(K-k^s),$$
$$\mathcal{Q}_0 = \widehat{\mathcal{S}}_0 \cap \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1}\Big[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\Big]. \tag{16}$$

Theorem 3.7: Consider the solvable InhSLSS (1). Let $\mathcal{C}^{\sigma}_{[0,K]}$ be its *controllable set to zero* on $[0, K]$ w.r.t. the single switch switching signal given by (12). Then

$$\mathcal{C}^{\sigma}_{[0,K]} = \mathcal{Q}_0, \tag{17}$$

where $\mathcal{Q}_0$ is defined in (16). In particular, the InhSLSS (1) is controllable if, and only if, $\mathcal{Q}_0 = \widehat{\mathcal{S}}_0$.

Proof: Setting the solution at $k = K > k^s$ of (1) under the single switch switching signal (12) with $x(0) = x_0 \in \widehat{\mathcal{S}}_0$ as zero gives us

$$0 = x(K) = \widehat{\Phi}_1^{K-k^s}x(k^s)$$
$$[\widehat{\Theta}_1, \widehat{\Phi}_1\widehat{\Theta}_1, \cdots, \widehat{\Phi}_1^{K-k^s-1}\widehat{\Theta}_1]\begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(k^s) \end{bmatrix}, \tag{18}$$

i.e. $x(k^s) \in \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1}\mathcal{R}_1(K-k^s)$. The solution at $k = k^s$ can be written as

$$x(k^s) = \widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}x_0 + \widehat{\Phi}_{1,0}R_0(k^s-1)\begin{bmatrix} u(k^s-2) \\ u(k^s-3) \\ \vdots \\ u(0) \end{bmatrix} \tag{19}$$
$$+\widehat{\Theta}_{1,0}u(k^s-1),$$

i.e. $x_0 \in \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1}\Big[\{x(k^s)\} + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\Big]$. Pick any controllable to zero state $x_0 \in \mathcal{C}^{\sigma}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that (18) holds. Together with the knowledge of $x(k^s) \in \widehat{\mathcal{S}}_1$, it implies that

$$x_0 \in \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1}\Big[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\Big].$$

Now, since $x_0 \in \widehat{\mathcal{S}}_0$ we have $x_0 \in \widehat{\mathcal{S}}_1 \cap \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1}\Big[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\Big]$. Hence $\mathcal{C}^{\sigma}_{[0,K]} \subseteq \mathcal{Q}_0$.
Now, pick any $\xi \in \mathcal{Q}_0$. Then, $\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\xi = \Big[\varsigma + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1)\bar{u}_1 + \widehat{\Theta}_{1,0}\bar{u}_2\Big]$, for some $\varsigma \in \mathcal{Q}_1$, $\bar{u}_1 \in \mathbb{R}^{(k^s\times m)\times 1}$ and $\bar{u}_2 \in \mathbb{R}^{m\times 1}$. Vector $\varsigma \in \mathcal{Q}_1$ implies that there exists a vector $\bar{u}_3 \in \mathbb{R}^{(K-k^s)\times m\times 1}$ such that $\widehat{\Phi}_1^{K-k^s}\varsigma = \mathcal{R}_1(K-k^s)\bar{u}_3$. Now, take $\bar{u} \in \mathbb{R}^{(K\times m)\times n}$ of the form $\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}$. Then with this input, $x(0) = \xi$ is brought to zero on $[0, K]$ i.e. $x(0) = \xi$ is controllable to zero. Thus, $\xi \in \mathcal{C}^{\sigma}_{[0,K]}$, and hence $\mathcal{Q}_0 \subseteq \mathcal{C}_{[0,K]}$.
Finally, the controllability part is the direct consequence of its definition and the result from the first part of this proof. ∎

Remark 3.8 (Reachability vs Controllability) In ordinary systems, there are three important observations regarding the relationship between reachability and controllability i.e. (1) reachability implies controllability to zero, (2) controllability to zero does not always imply reachability, and (3) they are equivalent when the state's coefficient matrix is nonsingular [24]. For solvable singular systems, with singular matrix E, the first two statements are still true, however, in contrast, the equivalency between reachability and controllability to zero never happens as the matrix $\widehat{\Phi}$ in (11) is always singular. The proof for the first statement is obvious since, in reachability, the zero (final) state is also reachable from any consistent initial value i.e. it is controllable to zero. The second statement is illustrated by the forthcoming Example 4.3 as a counter-example.

C. Discussion on Unswitched Systems

As the solvability results, the reachability and controllability characterizations derived above are also valid for the unswitched system (9). This is stated in the following corollary where the reachability and controllability notions for unswitched systems are the same as in Definitions 3.1-3.4.

Corollary 3.9: Consider the solvable InhSLS (9), and let $\mathcal{R}_{[0,K]}$ be its reachable set from zero on $[0,K]$ and $\mathcal{C}_{[0,K]}$ be its controllable set to zero on $[0,K]$. Then.

$$\mathcal{R}_{[0,K]} = \widehat{\mathcal{S}} \cap \operatorname{im} R(K) \tag{20}$$

and

$$\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K)), \tag{21}$$

where $R(k) = [\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{k-1}\widehat{\Theta}]$, and the matrices $\widehat{\Phi}$ and $\widehat{\Theta}$ are as in (11). In particular, the system is reachable from zero if, and only if, $\widehat{\mathcal{S}} \cap \operatorname{im} R(K) = \widehat{\mathcal{S}}$, or equivalently, $\widehat{\mathcal{S}} \subseteq \operatorname{im} R(K)$, and the system is controllable to zero if, and only if, $\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}}$, or equivalently, $\widehat{\mathcal{S}} \subseteq \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$.

IV. Illustrative Examples

The following examples illustrate nonsolvable and solvable InhSLSSs.

Example 4.1: Consider system (1) composed of two modes represented by the matrix triplets

$$(E_0, A_0, B_0) = \left(\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right),$$
$$(E_1, A_1, B_1) = \left(\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right).$$

Geometric computations provide that

$$\ker E_0 = \operatorname{span}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \quad \ker E_1 = \operatorname{span}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),$$
$$\widehat{\mathcal{S}}_0 = \operatorname{span}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \quad \widehat{\mathcal{S}}_1 = \operatorname{span}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right).$$

The condition $E_i^+ A_i \widehat{\mathcal{S}}_i + \operatorname{im}[E_i^+ B_i] \subseteq \ker E_i \oplus \widehat{\mathcal{S}}_i, \ \forall i = 0,1$ is satisfied, however, $\widehat{\mathcal{S}}_1 \cap \ker E_0 \neq \{0\}$ and also $\widehat{\mathcal{S}}_0 \cap \ker E_1 \neq \{0\}$, thus, switched systems composed of those two modes are not solvable.

Example 4.2: Consider system (1) composed of

$$(E_0, A_0, B_0) = \left(\begin{bmatrix}-1&1&0\\1&-1&-1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&-1&1\\0&1&1\\0&-1&0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\end{bmatrix}\right),$$

$$(E_1, A_1, B_1) = \left(\begin{bmatrix}-1&0&1\\1&-1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&-1\\0&-1&0\\1&1&-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right).$$

with

$$\begin{aligned}\ker E_0 &= \operatorname{span}\{(1,1,0)^\top\},\\ \ker E_1 &= \operatorname{span}\{(1,1,1)^\top\},\\ \widehat{\mathcal{S}}_0 &= \operatorname{span}\{(1,0,0)^\top, (0,0,1)^\top\},\\ \widehat{\mathcal{S}}_1 &= \operatorname{span}\{(1,0,1)^\top, (0,1,1)^\top\}.\end{aligned}$$

The solvability condition (4) is satisfied, and thus switched systems composed of those modes are solvable. With

$$E_0^+ = \begin{bmatrix}-1/2&0&0\\1/2&0&0\\-1&-1&0\end{bmatrix}, \qquad E_1^+ = \begin{bmatrix}-1/3&1/3&0\\-1/3&-2/3&0\\2/3&1/3&0\end{bmatrix},$$

$$\Pi_{\widehat{\mathcal{S}}_0}^{\ker E_0} = \begin{bmatrix}1&-1&0\\0&0&0\\0&0&1\end{bmatrix}, \qquad \Pi_{\widehat{\mathcal{S}}_1}^{\ker E_1} = \begin{bmatrix}0&-1&1\\-1&0&1\\-1&-1&2\end{bmatrix},$$

$$\Pi_{\widehat{\mathcal{S}}_1}^{\ker E_0} = \begin{bmatrix}1/2&-1/2&1/2\\-1/2&1/2&1/2\\0&0&1\end{bmatrix}, \quad \Pi_{\widehat{\mathcal{S}}_0}^{\ker E_1} = \begin{bmatrix}1&-1&0\\0&0&0\\0&-1&1\end{bmatrix},$$

we have the surrogate system (5) with

$$\widehat{\Phi}_{0,0} = \begin{bmatrix}-1&1&-1\\0&0&0\\-1&0&-2\end{bmatrix}, \qquad \widehat{\Phi}_{1,0} = \begin{bmatrix}-1&1/2&-3/2\\0&-1/2&-1/2\\-1&0&-2\end{bmatrix},$$

$$\widehat{\Phi}_{0,1} = \begin{bmatrix}0&-1&0\\0&0&0\\0&-1&-1\end{bmatrix}, \qquad \widehat{\Phi}_{1,1} = \begin{bmatrix}0&-1&-1\\0&0&-1\\0&-1&-2\end{bmatrix},$$

$$\widehat{\Theta}_{0,0} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \widehat{\Theta}_{1,0} = \begin{bmatrix}1\\0\\1\end{bmatrix},$$

$$\widehat{\Theta}_{0,1} = \begin{bmatrix}-1\\0\\0\end{bmatrix}, \qquad \widehat{\Theta}_{1,1} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

With the switching signal $\sigma(k) = 0$ for $k < 5$ and $\sigma(k) = 1$ for $k \geq 5$, the solution of the switched system is given in Fig. 3

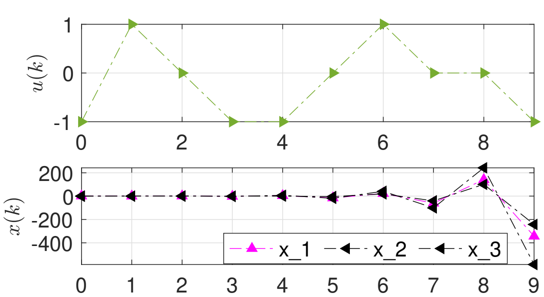

Fig. 3. Solution of the switched system in Example 4.2

The following example illustrates a controllable (unswitched) system that is not reachable. The system in this example is also a counter-example for the observation in Remark 3.8.

Example 4.3: Consider system (9) with

$$(E, A, B) = \left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&1\\1&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right).$$

Its consistency space is $\widehat{\mathcal{S}} = \operatorname{span}\begin{pmatrix}1\\-1\\0\end{pmatrix}$. It is solvable as (10) is satisfied, e.g. with $E^+ = E = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $\operatorname{im}[E^+A, E^+B] = \operatorname{im}\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} = \operatorname{span}\begin{pmatrix}1\\0\\0\end{pmatrix} \subset \ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^3$. With $\Pi_{\widehat{\mathcal{S}}}^{\ker E} = \begin{bmatrix}1&0&0\\-1&0&0\\0&0&0\end{bmatrix}$, we have its surrogate system

(11) with $(\widehat{\Phi},\widehat{\Theta}) = \left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$. Since for all $K > 0$, $\operatorname{im} R(K) = \operatorname{span}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathcal{R}_{[0,K]} = \widehat{\mathcal{S}} \cap \operatorname{im} R(K) = \{0\}$ i.e. it is unreachable on $[0,K]$ for any $K \geq 0$. However, it is controllable to zero on $[0,K]$ for any $K > 0$; this can be seen from the fact that $\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1}(\operatorname{im} R(K)) = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \widehat{\mathcal{S}}$. //

This part is closed by the reachability and controllability analysis of the system in Example 4.2.

Example 4.4: Recall the system in Example 4.2. Each mode as an individual system is not reachable on $[0,K]$ for all $K > 0$ since

$$\forall K,\ \mathcal{R}^0_{[0,K]} = \{0\} \neq \mathcal{S}_0 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\},$$

$$\mathcal{R}^1_{[0,1]} = \{0\} \neq \mathcal{S}_1 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}, \text{ and}$$

$$\text{for } K > 1,\ \mathcal{R}^1_{[0,K]} = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \neq \mathcal{S}_1,$$

where $\mathcal{R}^i_{[0,K]}$ and $\mathcal{C}^i_{[0,K]}$ for $i = 0, 1$ are reachable set and controllable set on $[0,K]$ for mode 0 and 1 respectively.

On the time domain $[0,K]$ with $K = 1$, mode 0 is controllable ($\mathcal{C}^0_{[0,1]} = \mathcal{S}_0$), however, mode 1 is uncontrollable since $\mathcal{C}^1_{[0,1]} = \operatorname{span}\begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix} \neq \mathcal{S}_1$. For longer time observations, both modes are always controllable since for all $K > 1$,

$$\mathcal{C}^0_{[0,K]} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \mathcal{S}_0 \text{ and}$$

$$\mathcal{C}^0_{[0,K]} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} = \mathcal{S}_1.$$

Consider now switched systems with the mode sequence $(0,1)$ on the time domain $[0,K]$ with $K = 10$ and with the switching time $k^s = 1, 2, \ldots, 9$. The switched system is unreachable but controllable for all k^s since for all $k^s = 1, 2, \ldots, 9$,

$$\mathcal{R}^\sigma_{[0,10]} = \{0\} \neq \mathcal{S}_1 \text{ and } \mathcal{C}^\sigma_{[0,10]} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \mathcal{S}_0.$$

With the mode sequence $(1,0)$, the characterization results are the same, i.e., the switched system is unreachable but controllable on $[0,10]$ for all switching times $k^s = 1, 2, \ldots, 9$.

V. Summary

Solution theory for discrete-time singular linear switched systems has been investigated for which strict causality is required for the solutions. Moreover, surrogate ordinary systems have been introduced for solvable systems, and are then utilized for reachability and controllability characterizations for the original singular systems. Geometric criteria have been derived with single switch switching signals.

Future work will focus on the case of multiple switches and also on studying how reachability and controllability depend on the switching times. Furthermore, an extension of the solution theory to the nonlinear case will be investigated.

References

[1] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Autom. Control*, vol. 22, pp. 312–321, 1977.

[2] ——, "Time-invariant descriptor systems," *Automatica*, vol. 14, pp. 473–480, 1978.

[3] L. Dai, *Singular Control Systems*, ser. Lecture Notes in Control and Information Sciences. Berlin: Springer-Verlag, 1989, no. 118.

[4] R. Bru, C. Coll, and E. Anchez, "About positively discrete-time singular systems," *System and control:theory and applications*, pp. 44–48, 01 2000.

[5] P. K. Anh, P. T. Linh, D. D. Thuan, and S. Trenn, "The one-step-map for switched singular systems in discrete-time," in *Proc. 58th IEEE Conf. Decision Control (CDC) 2019*, 2019, pp. 605–610.

[6] W. Hu, Z. Ding, X. Chen, and X. Long, "Controllability analysis of flyback converter with constant power load based on switched linear systems," *Energy Reports*, vol. 8, pp. 747–754, 2022.

[7] F. Liao, Y. Zhu, and D. Zhou, "Observer-based fault estimation for a class of discrete-time switched affine systems: An application to the dc-dc converter," *Journal of the Franklin Institute*, vol. 358, no. 15, pp. 7992–8011, 2021.

[8] S. Wang, Y. Bi, J. Bi, X. Wang, W. Qi, S. Cong, and Z. Sheng, "Robust control of sampled-data system subject to input time-delay," in *2016 35th Chinese Control Conference (CCC)*, 2016, pp. 2957–2960.

[9] T.-T. Tang, T. Shi, and C. Zhang, "Reliable model predictive control for linear sampled-data system with input constraints," in *2018 5th International Conference on Information, Cybernetics, and Computational Social Systems (ICCSS)*, 2018, pp. 399–403.

[10] L. Etienne, A. Girard, and L. Greco, "Stability and stabilizability of discrete-time dual switching systems with application to sampled-data systems," *Automatica*, vol. 100, pp. 388–395, 2019.

[11] S. S. Ge, Z. Sun, and T. H. Lee, "Reachability and Controllability of Switched Linear Discrete-Time Systems," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1–12, 2001.

[12] M. Babaali and M. Egerstedt, "Observability of switched linear systems," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science. Springer, 2004, vol. 2993, pp. 48–63.

[13] M. Egerstedt and M. Babaali, "On observability and reachability in a class of discrete-time switched linear systems," in *Proc. of the American Control Conference*, 2005, pp. 1179–1180.

[14] Z. Sun and S. S. Ge, *Switched linear systems*, ser. Communications and Control Engineering. London: Springer-Verlag, 2005.

[15] A. A. Belov, O. G. Andrianova, and A. P. Kurdyukov, "Control of discrete-time descriptor systems," *Cham: Springer International Publishing*, vol. 39, 2018.

[16] T. Schmidt, *Parametrschaetzung bei mehrkoerpersystemen mit zwangsbedingungen*. VDIVerlag, Dusseldorf, 1994.

[17] G.-R. Duan, *Analysis and Design of Descriptor Linear Systems*. Springer New York, NY, 2010.

[18] F. Küsters, M. G.-M. Ruppert, and S. Trenn, "Controllability of switched differential-algebraic equations," *Syst. Control Lett.*, vol. 78, no. 0, pp. 32 – 39, 2015.

[19] F. Küsters and S. Trenn, "Duality of switched ODEs with jumps," in *Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan*, 2015, to appear.

[20] A. Ben-Israel and T. Greville, *Generalized Inverse*. New York: Springer-Verlag, 2003.

[21] Sutrisno and S. Trenn, "Nonlinear switched singular systems in discrete-time: The one-step map and stability under arbitrary switching signals," *European Journal of Control*, 2023, in press.

[22] T. Berger and T. Reis, "Controllability of linear differential-algebraic systems - a survey," in *Surveys in Differential-Algebraic Equations I*, ser. Differential-Algebraic Equations Forum, A. Ilchmann and T. Reis, Eds. Berlin-Heidelberg: Springer-Verlag, 2013, pp. 1–61.

[23] K. Ogata, *Modern Control Engineering*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1990.

[24] B. Porter, "Reachability and controllability of discrete-time linear dynamical systems," *International Journal of General Systems*, vol. 2, no. 2, pp. 115–115, 1975.