2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Exact differentiator with Lipschitz continuous output and optimal worst-case accuracy under bounded noise

Rodrigo Aldana-López, Richard Seeber, Hernan Haimovich, and David Gómez-Gutiérrez

***Abstract*— The online differentiation of a signal contaminated with bounded noise is addressed. A differentiator is developed that generates a Lipschitz continuous output, is exact in the absence of noise, and provides the optimal worst-case accuracy among all possible exact differentiators when noise is present. This combination of features is not shared by any previously existing differentiator. Tuning of the developed differentiator is very simple, requiring only the knowledge of a bound for the second-order derivative of the signal. The approach consists in regularizing the possibly highly noisy output of a recently introduced linear adaptive robust exact differentiator and feeding it to a first-order sliding-mode filter designed to maintain optimal accuracy. The proposed regularization and filtering of this output allows trading the speed with which exactness is obtained for the feature of a Lipschitz continuous, hence less noisy, output. An illustrative example is provided to highlight the features of the developed differentiator.**

I. Introduction

Signal differentiation encompasses the strategies and techniques by which noisy signal measurements are processed to compute an estimate of the signal's derivative. A specific strategy or technique is named a *differentiator*, and the output of a differentiator is the derivative estimate. Differentiators have attracted considerable attention in the control community given their applications in fault diagnosis, observation, and control, leading to the development, for instance, of differentiators based on algebraic [1], [2], Kalman filtering [3], [4], High-gain [5], [6], and sliding-mode [7], [8] techniques.

Assessing a differentiator's theoretical performance using qualitative properties and quantitative measures is highly desirable. These properties are *exactness* and *worst-case accuracy* [7], [9]. A differentiator is exact if its output converges in finite-time to the actual signal derivative in the absence of noise. Accuracy refers to the maximum error of the differentiator (after a certain time) for all admissible signals and noise evolutions. A quantitative measure is given by the time after which exactness is achieved, which could be asymptotically (as time tends to infinity), after a finite time, in fixed time (in a finite time uniformly bounded with respect to the initial conditions), or from the beginning (except at the initial time); each of these notions is stronger than the previous ones.

The suitability of a differentiator for a specific application depends on two aspects: (a) what is known about the noise affecting the measurements and (b) what is known about the signal whose derivatives are to be estimated. For example, if the signal's frequency content and noise are known and mostly do not overlap, then standard linear techniques may be suitable for differentiation [10], [11]. If the noise signal is stochastic and has known statistics, then differentiators based on differential algebraic operations or Kalman filters may be suitable [2], [3]. If the noise is known to be bounded and, in addition, the bound on the noise is known, then linear high-gain differentiators may be tuned in an optimal way [6], whereas if such information is not available or exact differentiation of noise-free signals is desired, then sliding-mode differentiators may be more appropriate [7].

Under the assumption that the noise is bounded by a constant N and that the second-order derivative of the signal to be differentiated has a known bound L, it has been shown that an exact differentiator cannot achieve a worst-case accuracy better than $2\sqrt{2NL}$ [7], [9]. Levant proposed in [7] what is now called a super-twisting differentiator. This differentiator employs sliding-mode techniques, is robust and exact, and features finite-time convergence and a worst-case accuracy given by $C\sqrt{NL}$, for some $C > 0$ that is a function of its parameters. Numerical methods to compute the worst-case accuracy of the super-twisting differentiator were proposed by Angulo et al. in [12]. In contrast, Seeber [13] recently presented analytical methods showing that improving the worst-case accuracy of the super-twisting differentiator reduces its convergence speed and that it cannot achieve the optimal worst-case accuracy $2\sqrt{2NL}$.

An exact differentiator for polynomial signals has been proposed by Holloway and Krstic in [14], which converges in a prescribed, fixed time. However, such a differentiator is not robust and has an unbounded worst-case differentiation error at the prescribed convergence time instant [15]. Robust exact differentiators with fixed-time convergence have been proposed in [8], [16], [17]. Such differentiators are based on differential equations with a non-Lipschitz right-hand

Corresponding author: Richard Seeber (richard.seeber@tugraz.at)

*Work supported by Agencia I+D+i, Argentina, under grants PICT 2018-1385, 2021-0730, and by Christian Doppler Research Association, Austrian Federal Ministry of Labour and Economy, and National Foundation for Research, Technology and Development, and by Consejo Nacional de Ciencia y Tecnología (CONACYT-Mexico) grant 739841.

Rodrigo Aldana-López is with the Department of Computer Science and Systems Engineering, University of Zaragoza, Zaragoza, Spain. (e-mail: rodrigo.aldana.lopez@gmail.com)

Richard Seeber is with the Christian Doppler Laboratory for Model Based Control of Complex Test Bed Systems, Institute of Automation and Control, Graz University of Technology, Graz, Austria. (e-mail: richard.seeber@tugraz.at)

Hernan Haimovich is with Centro Internacional Franco-Argentino de Ciencias de la Información y de Sistemas (CIFASIS) CONICET-UNR, 2000 Rosario, Argentina. (e-mail: haimovich@cifasis-conicet.gov.ar)

David Gómez-Gutiérrez is with Intel Tecnología de México, Intel Labs, Intelligent Systems Research Lab, Jalisco, Mexico, and with Tecnológico Nacional de México, Instituto Tecnológico José Mario Molina Pasquel y Henríquez, Unidad Académica Zapopan, Jalisco, Mexico. (e-mail: david.gomez.g@ieee.org)

side. Similarly to the super-twisting differentiator, these differentiators also feature a trade-off between worst-case accuracy and convergence speed [16]; moreover, as shown in [8], such fixed-time differentiators achieve a similar worst-case accuracy as [7] for noise signals with small bound N.

None of the previously mentioned exact differentiators achieve the optimal worst-case accuracy $2\sqrt{2NL}$, and improving their accuracy in general reduces their convergence speed. A differentiator which achieves this optimal accuracy while simultaneously being exact from the beginning is proposed by Seeber and Haimovich in [9], based on a single parameter adaptation of a finite-difference differentiator [18], [19]. This differentiator, however, features a direct feed-through from the noise to the output which causes the output to be, loosely speaking, 'highly noisy'. More formally this may be seen from the fact that the output signal inherits the discontinuous nature of the noise via the feed-through. Moreover, a lack of robustness at the initial time instant may lead to an unbounded output signal. These two features of the output signal could be detrimental in a number of practical applications. For example, the differentiator's use in controllers such as a proportional-derivative controller or the twisting sliding-mode controller could result in harmful high-frequency vibrations or chattering that may damage or reduce the life of the actuators [20], [21]. Moreover, popular chattering reduction techniques such as the boundary layer design [22] and the high-order sliding modes [23], which are efficient in the noise-free case, may be rendered ineffective in the presence of noisy estimates [24]. For instance, it was shown by Utkin in [24] that noisy estimates might make a high-order sliding-mode controller have a larger regulation error than a first-order one.

The present paper develops an exact differentiator that achieves optimal worst-case accuracy and has a Lipschitz continuous output. To this end, the optimal differentiator proposed in [9] is combined with a first-order sliding-mode filter. The resulting differentiator is easy to tune, featuring only a single parameter that determines the convergence speed and the Lipschitz constant of the output signal, but does not otherwise impact the optimal differentiation accuracy. The main features of the developed differentiator are theoretically established and illustrated by means of an example.

Notation: $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}$ denote the nonnegative, the positive and the whole real numbers, respectively; $\mathbb{N}$ denotes the natural numbers. $\lceil a \rceil$ denotes the least integer not less than $a \in \mathbb{R}$. One-sided limits of a function f at time instant T from above are written as $\lim_{t\to T^+} f(t)$, $\limsup_{t\to T^+} f(t)$, and $\liminf_{t\to T^+} f(t)$. If $\alpha \in \mathbb{R}$, then $|\alpha|$ denotes its absolute value. 'Almost everywhere' is abbreviated as 'a.e.'.

II. Preliminaries

We recall here the performance-related properties named *worst-case error*, *exactness*, and *accuracy* as introduced and precisely quantified in [9]. Afterward, we recall the differentiator in [9], which was shown to be exact from the beginning and to achieve optimal accuracy in the form of the lowest possible worst-case differentiation error.

A. Performance Measures for Differentiators

Let $\mathcal{F}$ denote the set of functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that f is differentiable and $\dot{f}$ is Lipschitz continuous on $\mathbb{R}_{\geq 0}$. Consider the differentiation of such a signal f from a measurement $u = f + \eta$ which is corrupted by a uniformly bounded noise η. Denoting by $\mathcal{E}$ the set of all functions $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which are uniformly bounded on $\mathbb{R}_{\geq 0}$, the corresponding classes of signals to consider, from which the measurements are generated, are given by

$$\mathcal{F}_L = \{f \in \mathcal{F} : \left|\ddot{f}(t)\right| \leq L \text{ a.e. on } \mathbb{R}_{\geq 0}\} \tag{1a}$$

$$\mathcal{E}_N = \{\eta \in \mathcal{E} : |\eta(t)| \leq N \text{ for all } t \geq 0\}. \tag{1b}$$

Write $\mathcal{F}_L + \mathcal{E}_N = \{f + \eta : f \in \mathcal{F}_L, \eta \in \mathcal{E}_N\}$ for the set of inputs u with fixed L and N. The possible inputs to the differentiator then belong to the set

$$\mathcal{U} = \bigcup_{\substack{L\geq 0\\ N\geq 0}} (\mathcal{F}_L + \mathcal{E}_N). \tag{2}$$

A differentiator is a causal operator $\mathcal{D} : \mathcal{U} \to (\mathbb{R}_{\geq 0} \to \mathbb{R})$ that maps the measured signal u to an estimate $\mathcal{D}u$ for the derivative of f. For future reference, for every $R \geq 0$, define the class of signals with a bounded second derivative that, in addition, have a bounded initial value and initial derivative

$$\mathcal{F}_L^R := \{f \in \mathcal{F}_L : |f(0)| \leq R, |\dot{f}(0)| \leq R\}. \tag{3}$$

The next definitions recall concepts that are useful to describe the features required for a differentiator in this work.

Definition 1 (Worst-case error [9])**.** *Let $L, N, R \in \mathbb{R}_{\geq 0}$. A differentiator $\mathcal{D}$ is said to have worst-case error $M_N^{L,R}(t)$ from time $t \geq 0$ over the signal class $\mathcal{F}_L^R$ with noise bound N if*

$$M_N^{L,R}(t) = \sup_{\substack{f\in\mathcal{F}_L^R\\ \eta\in\mathcal{E}_N}} \sup_{\tau\geq t} \left|\dot{f}(\tau) - [\mathcal{D}(f+\eta)](\tau)\right|. \tag{4}$$

Definition 2 (Exactness [9])**.** *A differentiator $\mathcal{D}$ is said to be exact in finite time over $\mathcal{F}_L$, if for each $R \in \mathbb{R}_{\geq 0}$ there exists $t_R \in \mathbb{R}_{>0}$ such that $M_0^{L,R}(t_R) = 0$.*

The time t_R in Definition 2 is called a convergence time bound of the differentiator and relates to the case without measurement noise. In the following, the notion of convergence time functions *in presence of noise* is introduced, based on bounds for the asymptotic accuracy C_L as defined in [9, Definition 3.6]. Loosely speaking, such a function bounds from above the time after which the differentiator with noisy input achieves the corresponding accuracy.

Definition 3 (Accuracy)**.** *A differentiator $\mathcal{D}$ is said to have accuracy bound $\hat{C}_{L,\overline{N}} \in \mathbb{R}_{\geq 0}$ for signals in $\mathcal{F}_L$ with noise bounds less than $\overline{N} \in \mathbb{R}_{>0}$, if there exists a function $\hat{\mathcal{T}} : \mathbb{R}_{\geq 0} \times [0, \overline{N}) \to \mathbb{R}_{\geq 0}$ that is continuous in its second argument such that*

$$M_N^{L,R}[\hat{\mathcal{T}}(R, N)] \leq \hat{C}_{L,\overline{N}}\sqrt{NL} \tag{5}$$

holds for all $N \in [0, \overline{N})$ and $R \geq 0$. In this case, $\hat{\mathcal{T}}$ is called a convergence time function in presence of noise for $\hat{C}_{L,\overline{N}}$.

Remark 1. *Note that $\hat{C}_{L,\overline{N}}$ defined in Definition 3 differs from the asymptotic accuracy C_L defined in [9, Definition 3.6] mainly in the fact that noise amplitudes N up to some given maximum noise amplitude $\overline{N}$ are considered. Any accuracy bound $\hat{C}_{L,\overline{N}}$ as defined above is an upper bound for the asymptotic accuracy C_L, i.e., $C_L \leq \hat{C}_{L,\overline{N}}$. Hence, according to [9, Proposition 3.10], the lowest possible (i.e., optimal) value of $\hat{C}_{L,\overline{N}}$ is given by $2\sqrt{2}$.*

It is easy to see that, from any convergence time function in presence of noise $\hat{\mathcal{T}}$, a convergence time bound t_R in absence of noise is obtained according to $t_R = \hat{\mathcal{T}}(R, 0)$.

B. Problem statement

The problem addressed in this paper is the following. Let $L > 0$ be known. Design a differentiator $\mathcal{D}$ with the following features:

i) $\mathcal{D}$ has *Lipschitz continuous output*;
ii) $\mathcal{D}$ has optimal accuracy bound $\hat{C}_{L,\overline{N}} = 2\sqrt{2}$ with an $\overline{N}$ that can be made arbitrarily large by appropriate tuning;
iii) $\mathcal{D}$ is exact in finite time over $\mathcal{F}_L$.

Existing exact sliding-mode differentiators, such as the super-twisting differentiator [7] and its variants, do not fulfill item ii) as shown in [13, Proposition 3.1], whereas the optimal exact differentiator from [9] does not fulfill item i). In addition to designing a differentiator having all of the above features, the present paper also derives a closed-form expression for a corresponding convergence time function in presence of noise $\hat{\mathcal{T}}(R, N)$.

C. Optimal exact differentiation

Recall the optimal exact differentiator $\mathcal{D}_{\mathrm{w}}$ from [9] with output $y_{\mathrm{w}} = \mathcal{D}_{\mathrm{w}} u$ given by

$$y_{\mathrm{w}}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \lim_{T \to 0^+} \dfrac{u(t) - u(t-T)}{T} & \text{if } t > 0, \hat{T}(t) = 0 \\ \dfrac{u(t) - u(t - \hat{T}(t))}{\hat{T}(t)} & \text{if } t > 0, \hat{T}(t) > 0 \end{cases} \tag{6a}$$

where the time difference $\hat{T}(t)$ is adapted according to

$$\hat{T}(t) = \min\left\{ t, \overline{T}, 2\gamma(t)\sqrt{\frac{\hat{N}(t)}{L}} \right\} \tag{6b}$$

with an arbitrary function $\gamma : \mathbb{R}_{\geq 0} \to [1, \overline{\gamma}]$ and an estimate $\hat{N}(t)$ for the noise amplitude that is determined from the measurement u according to

$$\hat{N}(t) = \frac{1}{2} \sup_{\substack{T \in (0, \overline{T}] \\ T \leq t \\ \sigma \in [0, T]}} \left(|Q(t, T, \sigma)| - \frac{L\sigma(T - \sigma)}{2} \right) \tag{6c}$$

wherein $Q(t, T, \sigma)$ is defined as

$$Q(t, T, \sigma) = u(t - \sigma) - u(t) + \frac{u(t) - u(t-T)}{T}\sigma. \tag{6d}$$

The differentiator features two parameters: a window-length parameter $\overline{T} \in \mathbb{R}_{\geq 0}$ which determines how much of the past evolution of u is considered for computing the output y_{w}, and an upper bound $\overline{\gamma} \geq 1$ for the function γ. The latter may in practice be chosen as $\overline{\gamma} = 1$ to keep $\hat{T}(t)$ and hence the estimation delay of the differentiator as small as possible.

In [9], it was proved that $y_{\mathrm{w}}(t)$ is well-defined for all $t \geq 0$ and any $u \in \mathcal{U}$, which also implies that the limit in (6a) exists. Furthermore, it was shown that this differentiator is[1] exact from the beginning and achieves optimal asymptotic accuracy bound $\hat{C}_L = 2\sqrt{2}$ with convergence time function $\hat{\mathcal{T}}(R, N) = \sqrt{2N/L}$. However, depending on the features of the noise, the output of this differentiator is not guaranteed to be continuous. This happens because the (possibly discontinuous) noise η enters directly into the expression for $y_{\mathrm{w}}(t)$ in (6a) through the input $u = f + \eta$.

Next, we prove an additional auxiliary bound on the differentiator output $\mathcal{D}_{\mathrm{w}} u$, which is important to establish its local boundedness on $\mathbb{R}_{>0}$ later on.

Lemma 1. *Let $L \in \mathbb{R}_{>0}$ and consider the differentiator $\mathcal{D}_{\mathrm{w}}$ defined in (6) with parameters $\bar{\gamma} \geq 1$ and $\bar{T} \in \mathbb{R}_{>0} \cup \{\infty\}$. Then, for all $u \in \mathcal{U}$ and all $t > 0$, the differentiator output $y_{\mathrm{w}} = \mathcal{D}_{\mathrm{w}} u$ satisfies*

$$|y_{\mathrm{w}}(t)| \leq \begin{cases} \frac{Lt}{2} + \frac{|u(t) - u(0)|}{t} & \text{if } t \leq \overline{T} \\ \frac{L\overline{T}}{2} + \frac{|u(t) - u(t - \overline{T})|}{\overline{T}} & \text{otherwise.} \end{cases} \tag{7}$$

The proof is given in the appendix.

III. Differentiator with Lipschitz continuous output

Comparing the features of the differentiator $\mathcal{D}_{\mathrm{w}}$ in (6) with those required in our problem statement, it is clear that the aim is to trade a reduction in the speed with which exactness is attained (in a finite time instead of from the beginning) for the feature of a Lipschitz continuous output. The key idea is to filter the output $y_{\mathrm{w}} = \mathcal{D}_{\mathrm{w}} u$ by means of a first-order sliding-mode system.

For this purpose, a regularization $\mathcal{D}_{\mathrm{m}}$ of the differentiator $\mathcal{D}_{\mathrm{w}}$ is first introduced in Section III-A. The proposed differentiator $\mathcal{D}$ with Lipschitz continuous output is then stated in Section III-B.

A. Differentiator output regularization

The output of the differentiator $\mathcal{D}_{\mathrm{w}}$ of (6), namely y_{w} in (6a), may lack not only continuity but also (Lebesgue) measurability. To ensure that a filter that takes y_{w} as input has a well-defined solution, y_{w} must be (at least) a measurable function. One of the main reasons for this lack of measurability is the fact that the supremum in (6c) is taken over an uncountable set, apart from the fact that neither the noise η

[1] For the formal definition of exactness from the beginning, refer to [9, Definition 2.4].

nor the function γ are assumed to be Lebesgue measurable in the present paper. To ensure measurability we introduce the following regularization. For any function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$, its regularization $v^{\ddagger} : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty, \infty\}$ is defined as

$$v^{\ddagger}(t) = \begin{cases} v(0) & \text{if } t = 0 \\ \frac{\limsup_{\varepsilon \to 0^+} v(t-\varepsilon) + \liminf_{\varepsilon \to 0^+} v(t-\varepsilon)}{2} & \text{if } t > 0 \end{cases} \tag{8}$$

with $\infty + (-\infty) := 0$ in case both limits are infinite.

The following lemma, which is proven in the appendix, shows that applying the regularization to a locally bounded function yields a Lebesgue measurable function.

Lemma 2. *Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be locally bounded on $\mathbb{R}_{>0}$. Then, the function $v^{\ddagger}$ defined in* (8) *takes only finite values, i.e., $v^{\ddagger}(t) \in \mathbb{R}$ for all $t \in \mathbb{R}_{\geq 0}$, is locally bounded on $\mathbb{R}_{>0}$, and is Lebesgue measurable.*

Remark 2. *A function v is locally bounded on $\mathbb{R}_{>0}$ if for every $t > 0$ there exists a (sufficiently small) neighborhood of t where v is bounded. Note that this does not preclude v from being unbounded in a right-neighborhood of zero. For example, v defined as $v(0) = 0$, $v(t) = \frac{1}{t}$ for $t > 0$ is locally bounded on $\mathbb{R}_{>0}$; in this case, actually, $v^{\ddagger} = v$.*

Define a new, intermediate differentiator $\mathcal{D}_{\mathrm{m}}$ whose output $y_{\mathrm{m}} = \mathcal{D}_{\mathrm{m}} u$ is a regularized version of y_{w}, namely

$$y_{\mathrm{m}}(t) = y_{\mathrm{w}}^{\ddagger}(t). \tag{9}$$

In the following result, we show that $\mathcal{D}_{\mathrm{m}}$ achieves the same accuracy as $\mathcal{D}_{\mathrm{w}}$, while guaranteeing a measurable output.

Proposition 1. *Let $L \in \mathbb{R}_{>0}$ and consider the differentiator $\mathcal{D}_{\mathrm{m}}$ with output $y_{\mathrm{m}} = \mathcal{D}_{\mathrm{m}} u$ defined by* (6) *and* (9)*, with parameters $\overline{\gamma} \in [1, 1 + \sqrt{2}]$ and $\overline{T} \in \mathbb{R}_{>0} \cup \{\infty\}$. Then, the following statements are true:*

a) the output $\mathcal{D}_{\mathrm{m}} u$ is Lebesgue measurable for all $u \in \mathcal{U}$;
b) the worst-case differentiation error of $\mathcal{D}_{\mathrm{m}}$ fulfills $M_N^{L,R}(t) \leq 2\sqrt{2NL}$ for all $N \in [0, L\overline{T}^2/2)$ and all $t > \sqrt{2N/L}$. △

Remark 3. *Note that in contrast to [9, Theorem 5.1], the error bound does not necessarily hold for $t = \sqrt{2N/L}$ due to the regularization. Nevertheless, $\mathcal{D}_{\mathrm{m}}$ can be seen to have optimal accuracy bound $\hat{C}_{L,\overline{N}} = 2\sqrt{2}$ for signals with noise bounds less than $\overline{N} = \frac{L\overline{T}^2}{2}$ from Definition 3.*

Proof. For item a), note that every $u \in \mathcal{U}$ is locally bounded on $\mathbb{R}_{\geq 0}$. Lemma 1 then implies that $y_{\mathrm{w}} = \mathcal{D}_{\mathrm{w}} u$ is locally bounded on $\mathbb{R}_{>0}$, which allows to conclude Lebesgue measurability of $\mathcal{D}_{\mathrm{m}} u = y_{\mathrm{w}}^{\ddagger}$ using Lemma 2.

Regarding item b), from [9, Theorem 5.1] we have that the output $y_{\mathrm{w}} = \mathcal{D}_{\mathrm{w}}(f + \eta)$ of $\mathcal{D}_{\mathrm{w}}$ satisfies

$$\left|\dot{f}(\tau) - y_{\mathrm{w}}(\tau)\right| \leq 2\sqrt{2NL} \tag{10}$$

for all $f \in \mathcal{F}_L$, $\eta \in \mathcal{E}_N$, and all $\tau \geq \sqrt{2N/L}$. Note that $\limsup_{\varepsilon \to 0^+} \dot{f}(\tau - \varepsilon) = \dot{f}(\tau)$ since $\dot{f}(t)$ is Lipschitz continuous. Therefore, using (10), it follows that

$$\begin{aligned} &\left|\dot{f}(\tau) - \limsup_{\varepsilon \to 0^+} y_{\mathrm{w}}(\tau - \varepsilon)\right| \\ &= \left|\limsup_{\varepsilon \to 0^+} \left(\dot{f}(\tau - \varepsilon) - y_{\mathrm{w}}(\tau - \varepsilon)\right)\right| \leq 2\sqrt{2NL} \end{aligned} \tag{11}$$

for all $t > \sqrt{2N/L}$. The same conclusion applies to $\liminf_{\varepsilon \to 0^+} y_{\mathrm{w}}(t - \varepsilon)$ and in turn to $y_{\mathrm{m}}(t)$, completing the proof. □

B. Exact differentiator with Lipschitz continuous output

Define the output of the proposed differentiator as the Filippov [25] solution to

$$\dot{y}(t) = -\kappa \operatorname{sign}(y(t) - y_{\mathrm{m}}(t)), \quad y(0) = 0, \tag{12}$$

with κ a positive design parameter, i.e., by applying a first-order sliding-mode filter to the output y_{m} of the regularization. Note that this output may be unbounded in a right-neighborhood of $t = 0$; nevertheless, the right-hand side of (12) is uniformly bounded by virtue of the sign function and Lebesgue measurable as a consequence of y_{m} being Lebesgue measurable according to Proposition 1-a). The proposed differentiator is then defined by (6), (9), and (12). The design parameter κ should be selected greater than the upper bound L for the Lipschitz constant of $\dot{f}$, as will be shown in the following.

Theorem 1. *Let $L > 0, N \geq 0$ and consider the differentiator $\mathcal{D}$ with output $y = \mathcal{D}u$ defined by* (6)*,* (9)*, and* (12) *with parameters $\overline{\gamma} \in [1, 1 + \sqrt{2}]$, $\overline{T} \in \mathbb{R}_{>0} \cup \{\infty\}$, and $\kappa > L$. Then, the following statements are true:*

a) $\mathcal{D}$ has a Lipschitz continuous output for any $u \in \mathcal{U}$.
b) $\mathcal{D}$ has accuracy bound $\hat{C}_{L,\overline{N}} = 2\sqrt{2}$ for signals in $\mathcal{F}_L$ with noise bounds less than $\overline{N} = \frac{L\overline{T}^2}{2}$, with corresponding convergence time function in presence of noise given by

$$\hat{\mathcal{T}}(R, N) = 2\sqrt{\frac{2N}{L}} + \frac{R}{\kappa - L}. \tag{13}$$

Proof. Item a) follows by noting that $\dot{y}(t)$ exists almost everywhere and is bounded by κ, therefore $y(t)$ is Lipschitz.

For item b), let $f \in \mathcal{F}_L^R$, $\eta \in \mathcal{E}_N$ with $N < \overline{N}$, and define the differentiation error $e(t) = y(t) - \dot{f}(t)$ with $y = \mathcal{D}(f + \eta)$. From (12), e satisfies

$$\dot{e}(t) = -\kappa \operatorname{sign}(e(t) - \eta_{\mathrm{m}}(t)) - \ddot{f}(t) \tag{14}$$

and $|e(0)| \leq R$, with $\eta_{\mathrm{m}} = y_{\mathrm{m}} - \dot{f} = \mathcal{D}_{\mathrm{m}} f - \dot{f}$ being Lebesgue measurable according to Proposition 1-a) and bounded by

$$|\eta_{\mathrm{m}}(t)| \leq 2\sqrt{2NL} \tag{15}$$

for all $t > \tilde{\mathcal{T}}(N) := \sqrt{2N/L}$ according to Proposition 1-b). For $t \in [0, \tilde{\mathcal{T}}(N)]$ it follows from $|\dot{e}| \leq \kappa + L$ that

$$|e(t)| \leq (\kappa + L)t + |e(0)| \leq (\kappa + L)\tilde{\mathcal{T}}(N) + R \tag{16}$$

holds. For $t > \tilde{\mathcal{T}}(N)$, consider $V(e) = |e|$ as a Lyapunov function. Then, $V = |e| > 2\sqrt{2NL}$ implies that its time derivative $\dot{V}$ along (14) satisfies

$$\dot{V} = -\kappa \operatorname{sign}(e) \operatorname{sign}(e - \eta_\mathrm{m}(t)) - \ddot{f}(t) \operatorname{sign}(e) \le -\kappa + L \tag{17}$$

since $\operatorname{sign}(e - \eta_\mathrm{m}) = \operatorname{sign}(e)$ due to (15). Noting that $V(\tilde{\mathcal{T}}(N)) \le (\kappa + L)\tilde{\mathcal{T}}(N) + R$, it will now be shown using the comparison principle that $V(e(t)) \le 2\sqrt{2NL}$ holds for all $t \ge \hat{\mathcal{T}}(R, N)$, proving the claim that the worst-case error satisfies $M_L^{R,N}(\hat{\mathcal{T}}(R, N)) \le 2\sqrt{2NL} = \hat{C}_{L,\overline{N}}\sqrt{NL}$ with $\hat{C}_{L,\overline{N}} = 2\sqrt{2}$. To see this, suppose to the contrary that $V(e(t)) > 2\sqrt{2NL}$ holds for some $t \ge \hat{\mathcal{T}}(R, N)$. Then, the differential inequality (17) may be integrated backward in time to obtain $V(e(t)) > 2\sqrt{2NL}$ for all $t \in [\tilde{\mathcal{T}}(N), \hat{\mathcal{T}}(R, N)]$ along with the contradiction

$$\begin{aligned} V(\tilde{\mathcal{T}}(N)) &\ge V(\hat{\mathcal{T}}(R, N)) + (\kappa - L)(\hat{\mathcal{T}}(R, N) - \tilde{\mathcal{T}}(N)) \\ &> 2\sqrt{2NL} + (\kappa - L)\tilde{\mathcal{T}}(N) + R \\ &= (\kappa + L)\tilde{\mathcal{T}}(N) + R, \end{aligned} \tag{18}$$

because $2\sqrt{2NL} = 2L\tilde{\mathcal{T}}(N)$. □

IV. Example

In the following, an illustrative example is shown in order to demonstrate the features of the introduced differentiators and to compare them to results from literature. To demonstrate the effect of the regularization (8), the output (9) of the regularized differentiator $\mathcal{D}_\mathrm{m}$ is first computed analytically, from which the output (12) of the proposed differentiator $\mathcal{D}$ is then obtained by means of a numerical simulation.

Let $\mathcal{V} \subset [6, 10]$ be a dense Vitali set, i.e., a non-measurable set whose closure is $[6, 10]$ and which has the property that for each $x \in \mathbb{R}$ there exists one and only one $v \in \mathcal{V}$ such that $v - x$ is a rational number. Consider differentiation of the signal $f(t) = t^2/2 + 5t$ from a measurement $u = f + \eta$ subject to the noise with $N = 1$ given by

$$\eta(t) = \begin{cases} 1 - \sqrt{t} & t < 4 \\ 1 & t \in [4, 5) \\ 1 - \frac{7}{4}(t-5)^2 & t \in [5, 6) \\ -\frac{1}{2} + \frac{1}{2}\mu(t) & t \in \mathcal{V} \\ \frac{1}{2} + \frac{1}{2}\mu(t) & \text{otherwise,} \end{cases} \tag{19}$$

with $\mu(t) = \cos \omega t$ and positive parameter $\omega = 10^3$. This noise consists of a square-root arc on $[0, 4)$, of a constant on $[4, 5)$, and of a parabola arc on $[4, 6)$; it is not Lebesgue measurable on the interval $[6, 10]$; and it is a high-frequency deterministic signal for $t > 10$.

Consider the differentiators $\mathcal{D}_\mathrm{w}$, $\mathcal{D}_\mathrm{m}$, and $\mathcal{D}$ whose outputs are defined in (6), (9), and (12), respectively, with parameters $L = 1$, $\overline{\gamma} = 1$, $\overline{T} = \infty$, and $\kappa = 2.5$. In the following, the outputs $y_\mathrm{w} = \mathcal{D}_\mathrm{w} u$ of the original optimal exact differentiator $\mathcal{D}_\mathrm{w}$ and $y_\mathrm{m} = \mathcal{D}_\mathrm{m} u$ of the regularized optimal exact differentiator $\mathcal{D}_\mathrm{m}$ will be calculcated analytically. Since the noise is discontinuous at $t = 4$, $\hat{N}(t) = 1$ follows for $t \ge 4$ using the same arguments as in [9, Proposition 5.6]. For $t \in [0, 4)$, $T \in (0, t)$, and $\sigma \in [0, T]$, we have

$$Q(t, T, \sigma) = -\frac{\sigma(T - \sigma)}{2} - q(t, T, \sigma) \tag{20}$$

according to (6d) with

$$q(t, T, \sigma) = \sqrt{t - \sigma} - \sqrt{t} + (\sqrt{t} - \sqrt{t - T})\frac{\sigma}{T}, \tag{21}$$

and since this expression is positive due to the concavity of the square root, then

$$|Q(t, T, \sigma)| - \frac{\sigma(T - \sigma)}{2} = q(t, T, \sigma) \tag{22}$$

is obtained for the expression in (6c). One may verify that this expression is maximal for $T = t$ and $\sigma = \frac{3t}{4}$, yielding $\hat{N}(t) = q(t, t, 3t/4)$ for $t < 4$, resulting together with (6b) in

$$\hat{N}(t) = \begin{cases} \frac{\sqrt{t}}{4} & t < 4 \\ 1 & t \ge 4, \end{cases} \quad \hat{T}(t) = \begin{cases} t & t \in [0, 1] \\ \sqrt[4]{t} & t \in (1, 4) \\ 2 & t \ge 4. \end{cases} \tag{23}$$

Noting that $t \in \mathcal{V}$ implies $t + 2, t - 2 \notin \mathcal{V}$ by virtue of $\mathcal{V}$ being a Vitali set, the output $y_\mathrm{w} = \mathcal{D}_\mathrm{w} u$ of (6) is then given by

$$y_\mathrm{w}(t) = \begin{cases} 0 & t = 0 \\ \dot{f}(t) - \frac{t}{2} - \frac{1}{\sqrt{t}} & t \in (0, 1] \\ \dot{f}(t) - \frac{\sqrt[4]{t}}{2} - \frac{\sqrt{t} - \sqrt{t - \sqrt[4]{t}}}{\sqrt[4]{t}} & t \in (1, 4) \\ \dot{f}(t) - 1 + \frac{\sqrt{t-2}}{2} & t \in [4, 5) \\ \dot{f}(t) - 1 + \frac{\sqrt{t-2}}{2} - \frac{7(t-5)^2}{8} & t \in [5, 6) \\ \dot{f}(t) - \frac{7}{4} + \frac{1}{4}\mu(t) & t \in [6, 7), t \in \mathcal{V} \\ \dot{f}(t) - \frac{5}{4} + \frac{1}{4}\mu(t) & t \in [6, 7), t \notin \mathcal{V} \\ \dot{f}(t) - \frac{7}{4} + \frac{7(t-7)^2}{8} + \frac{\mu(t)}{4} & t \in [7, 8), t \in \mathcal{V} \\ \dot{f}(t) - \frac{5}{4} + \frac{7(t-7)^2}{8} + \frac{\mu(t)}{4} & t \in [7, 8), t \notin \mathcal{V} \\ \dot{f}(t) - \frac{3}{2} + \frac{\mu(t) - \mu(t-2)}{4} & t \in [8, \infty), t \in \mathcal{V} \\ \dot{f}(t) - \frac{1}{2} + \frac{\mu(t) - \mu(t-2)}{4} & t \in [8, \infty), t - 2 \in \mathcal{V} \\ \dot{f}(t) - 1 + \frac{\mu(t) - \mu(t-2)}{4} & \text{otherwise.} \end{cases} \tag{24}$$

By applying the regularization (8) to this function according to (9), the output $y_\mathrm{m} = \mathcal{D}_\mathrm{m} u$ of the regularized differentiator is obtained as

$$y_\mathrm{m}(t) = \begin{cases} y_\mathrm{w}(t) & t \in [0, 4) \\ \dot{f}(t) - \frac{\sqrt[4]{t}}{2} - \frac{\sqrt{t} - \sqrt{t - \sqrt[4]{t}}}{\sqrt[4]{t}} & t = 4 \\ \dot{f}(t) - 1 + \frac{\sqrt{t-2}}{2} & t \in (4, 5] \\ \dot{f}(t) - 1 + \frac{\sqrt{t-2}}{2} - \frac{7(t-5)^2}{8} & t \in (5, 6] \\ \dot{f}(t) - \frac{3}{2} + \frac{1}{4}\mu(t) & t \in (6, 7] \\ \dot{f}(t) - \frac{3}{2} + \frac{7(t-7)^2}{8} + \frac{1}{4}\mu(t) & t \in (7, 8] \\ \dot{f}(t) - 1 + \frac{\mu(t) - \mu(t-2)}{4} & t \in (8, 10] \\ \dot{f}(t) - \frac{3}{4} + \frac{\mu(t) - \mu(t-2)}{4} & t \in (10, 12] \\ \dot{f}(t) - 1 + \frac{\mu(t) - \mu(t-2)}{4} & t \in (12, \infty). \end{cases} \tag{25}$$

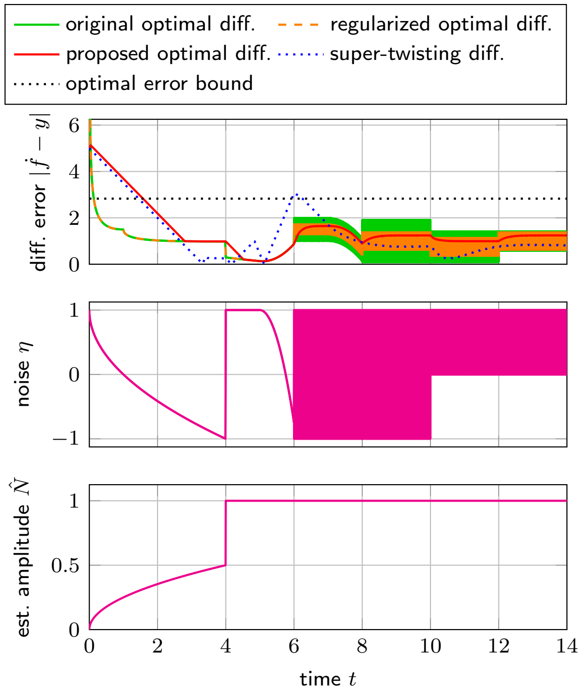

Fig. 1: Differentiation error obtained from an analytical computation for the original optimal exact differentiator from [9] and the regularized optimal exact differentiator (6), (9), and obtained from a numerical forward Euler simulation with step size $\Delta = 5 \cdot 10^{-4}$ for the proposed optimal exact differentiator with Lipschitz continuous output (6), (9), (12) and the super-twisting differentiator (26) with parameters $L = 1$, $\bar{\gamma} = 1$, $\overline{T} = \infty$, and $\kappa = 2.5$, as well as $\lambda_1 = 2.3$, $\lambda_2 = \frac{\kappa}{L} = 2.5$, differentiating the signal $f(t) = \frac{t^2}{2} + 5t$ corrupted by discontinuous, not Lebesgue measurable, and high-frequency noise $\eta(t)$ as in (19) with $\omega = 10^3$. Also shown is the differentiators' noise amplitude estimate $\hat{N}(t)$.

From the function y_{m}, the output $y = \mathcal{D}u$ of the proposed Lipschitz optimal exact differentiator is computed by means of a numerical simulation using a forward Euler integration of (12) with the step size $\Delta = 5 \cdot 10^{-4}$. For comparison purposes, a numerical forward Euler simulation, with identical step size, of a super-twisting differentiator [7]

$$\dot{z}_1(t) = z_2(t) + \lambda_1 \sqrt{L} \sqrt{u^\ddagger(t) - z_1(t)}\, \mathrm{sign}(u^\ddagger(t) - z_1(t))$$
$$\dot{z}_2(t) = \lambda_2 L \,\mathrm{sign}(u^\ddagger(t) - z_1(t)) \tag{26}$$

with output $y_{\mathrm{s}}(t) = z_2(t)$ is performed using the measurement $u^\ddagger$ regularized according to (8) as an input. To achieve similar convergence speed as the proposed differentiator, $\lambda_2 = \frac{\kappa}{L} = 2.5$ is chosen, and $\lambda_1 = 2.3$ is selected.

Fig 1 depicts the differentiation error of the three optimal differentiators and the super-twisting differentiator along with the noise signal $\eta(t)$ and the differentiators' estimate $\hat{N}(t)$ for its amplitude. One can see that the discontinuous, high-frequency noise, which is additionally not Lebesgue measurable on the interval $[6, 10]$, causes the output y_{w} of the original differentiator $\mathcal{D}_{\mathrm{w}}$ to be discontinuous (in addition to also being not Lebesgue measurable). The regularized differentiator $\mathcal{D}_{\mathrm{m}}$ removes the non-measurable components, resulting in a Lebesgue measurable output y_{m} with smaller, but still discontinuous and high-frequency variation of the differentiation error. Additionally, the lack of Lipschitz continuity of the noise can be seen to cause an unbounded differentiation error near $t = 0$ for both differentiators. The proposed exact differentiator $\mathcal{D}$, in contrast, features an output that is Lipschitz continuous with Lipschitz constant $\kappa = 2.5$ and hence significantly attenuates the high-frequency oscillations. Moreover, its differentiation error can be seen to converge to within the optimal error bound $2\sqrt{2NL} \approx 2.83$, according to Theorem 1, in a time bounded from above by $\hat{\mathcal{T}}(5, 1) = 2\sqrt{2} + \frac{10}{3} \approx 6.17$. The super-twisting differentiator, in comparison, also features a Lipschitz continuous output y_{s} with Lipschitz constant κ and converges in a similar time as the proposed differentiator, but can be seen to exceed the optimal error bound near $t = 6$ in accordance with the results obtained in [13].

V. Conclusions

This paper introduced what is, to the best of our knowledge, the first exact differentiator with a Lipschitz continuous output that attains optimal worst-case accuracy. To achieve this result, we extend an optimal base differentiator by Seeber and Haimovich [9] using a sliding-mode filter designed to maintain the same worst-case accuracy as [9] while providing a Lipschitz continuous estimate for the derivative. Doing so requires Lebesgue measurability of the filter's input, which is ensured by regularizing the base differentiator's output. Compared to the base differentiator in [9], our approach was shown to exhibit superior attentuation of high frequency noise at the cost of reducing the speed of convergence, which in [9] occurs instantaneously in the noise-free case. Moreover, we give an upper bound on its convergence time in presence of noise, at which it attains optimal accuracy. The proposed differentiator thus has similar features as the well-known super-twisting differentiator with the additional advantage of having optimal worst-case differentiation accuracy and a known convergence time bound in presence of measurement noise. In the future, the robustness and implementation of the differentiator in presence of sampled measurements may be studied.

Appendix
Proofs of all Lemmata

Proof of Lemma 1. Let $t > 0$, define $\tau = \min\{t, \overline{T}\}$, and distinguish the cases $\hat{T}(t) = 0$ and $\hat{T}(t) > 0$. In the first case, the inequality

$$|u(t - \tau) + y_{\mathrm{w}}(t)\tau - u(t)| \le \frac{L\tau^2}{2} \tag{27}$$

is obtained from [9, Lemma 5.8] by setting $\mu = \sigma = \tau$ in that lemma. In the second case, (27) is obtained from [9, Lemma 5.9] with $\hat{\sigma} = \tau$. In either case, (27) implies

$$|y_{\mathrm{w}}(t)|\tau \le \frac{L\tau^2}{2} + |u(t) - u(t - \tau)|, \tag{28}$$

which after division by τ yields the claimed inequality. $\square$

Proof of Lemma 2. Since v is locally bounded, both limits always yield finite values in $\mathbb{R}$ and are themselves locally bounded as functions of t on $\mathbb{R}_{>0}$. It suffices to show that $\limsup_{\epsilon\to 0^+} v(t-\epsilon)$ is measurable because measurability of the limit inferior follows analogously and the definition of $v^\ddagger$ at the single point $t=0$ does not affect measurability. First, note that with the restrictions $\epsilon \in (0,t)$ and $n \in \mathbb{N}$,

$$\begin{aligned}\limsup_{\epsilon\to 0^+} v(t-\epsilon) &= \lim_{\epsilon\to 0^+} \sup_{\tau\in[t-\epsilon,t)} v(\tau) \\ &= \lim_{n\to\infty} \sup_{\tau\in\left[t-\frac{1}{n},t\right)} v(\tau) \end{aligned} \tag{29}$$

holds for all $t > 0$, where the first equality follows by definition and the change of variables $\tau = t-\epsilon$, and the second one is true with n restricted to the integers, because the argument of the limit is monotonous. Define a sequence of functions $g_n : \mathbb{R}_{>0} \to \mathbb{R}$, $n \in \mathbb{N}$ as

$$g_n(t) = \sup_{\tau\in\left[\frac{\lceil nt\rceil-1}{n},t\right)} v(\tau), \tag{30}$$

where $\lceil nt\rceil$ denotes the least integer not less than nt. Note that g_n is piecewise monotone and is therefore measurable, cf. [26, Page 17]. In the following, we show that

$$\lim_{n\to\infty} \sup_{\tau\in[t-\frac{1}{n},t)} v(\tau) = \lim_{n\to\infty} g_n(\tau), \tag{31}$$

which implies measurability of $\limsup_{\epsilon\to 0^+} v(t-\epsilon)$ by virtue of being the pointwise limit of measurable functions according to [26, Theorem 2.6]. Since $t - \frac{1}{n} \le \frac{\lceil nt\rceil-1}{n}$ for all $n, t > 0$, we have

$$\lim_{n\to\infty} \sup_{\tau\in[t-\frac{1}{n},t)} v(\tau) \ge \lim_{n\to\infty} \sup_{\tau\in\left[\frac{\lceil nt\rceil-1}{n},t\right)} v(\tau). \tag{32}$$

Moreover, for all $n, t > 0$ there exists an integer $m > n$ such that $t - \frac{1}{m} \ge \frac{\lceil nt\rceil-1}{n}$ holds, implying the inequality $\sup_{\tau\in[t-\frac{1}{m},t)} v(\tau) \le \sup_{\tau\in\left[\frac{\lceil nt\rceil-1}{n},t\right)} v(\tau)$, and thus

$$\lim_{m\to\infty} \sup_{\tau\in[t-\frac{1}{m},t)} v(\tau) \le \lim_{n\to\infty} \sup_{\tau\in\left[\frac{\lceil nt\rceil-1}{n},t\right)} v(\tau), \tag{33}$$

which together with (32) implies (31). $\square$

REFERENCES

[1] S. Ibrir, "Online exact differentiation and notion of asymptotic algebraic observers," *IEEE transactions on Automatic control*, vol. 48, no. 11, pp. 2055–2060, 2003.

[2] M. Mboup, C. Join, and M. Fliess, "Numerical differentiation with annihilators in noisy environment," *Numerical algorithms*, vol. 50, pp. 439–467, 2009.

[3] J. Moore, "Optimum differentiation using Kalman filter theory," *Proceedings of the IEEE*, vol. 56, no. 5, pp. 871–871, 1968.

[4] S. Ibrir, "New differentiators for control and observation applications," in *Proceedings of the 2001 American Control Conference.(Cat. No. 01CH37148)*, vol. 3. IEEE, 2001, pp. 2522–2527.

[5] Y. Chitour, "Time-varying high-gain observers for numerical differentiation," *IEEE Transactions on Automatic Control*, vol. 47, no. 9, pp. 1565–1569, 2002.

[6] L. K. Vasiljevic and H. K. Khalil, "Differentiation with high-gain observers the presence of measurement noise," in *Conference on Decision and Control*. IEEE, 2006, pp. 4717–4722.

[7] A. Levant, "Robust Exact Differentiation via Sliding Mode Technique," *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.

[8] E. Cruz-Zavala, J. A. Moreno, and L. M. Fridman, "Uniform robust exact differentiator," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2727–2733, 2011.

[9] R. Seeber and H. Haimovich, "Optimal robust exact differentiation via linear adaptive techniques," *Automatica*, vol. 148, p. 110725, 2023.

[10] B. Carlsson, "Maximum flat digital differentiator," *Electronics letters*, vol. 27, no. 8, pp. 675–677, 1991.

[11] I. W. Selesnick, "Maximally flat low-pass digital differentiator," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 49, no. 3, pp. 219–223, 2002.

[12] M. T. Angulo, J. A. Moreno, and L. Fridman, "Optimal gain for the super-twisting differentiator in the presence of measurement noise," in *2012 American Control Conference (ACC)*. IEEE, 2012, pp. 6154–6159.

[13] R. Seeber, "Worst-case error bounds for the super-twisting differentiator in presence of measurement noise," *Automatica*, vol. 152, p. 110983, 2023.

[14] J. Holloway and M. Krstic, "Prescribed-time observers for linear systems in observer canonical form," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3905–3912, 2019.

[15] R. Aldana-Lopez, R. Seeber, H. Haimovich, and D. Gomez-Gutierrez, "On inherent robustness and performance limitations of a class of prescribed-time algorithms," *ArXiv:2205.02528*, 5 2022. [Online]. Available: https://arxiv.org/abs/2205.02528v1

[16] L. Fraguela, M. T. Angulo, J. A. Moreno, and L. Fridman, "Design of a prescribed convergence time uniform Robust Exact Observer in the presence of measurement noise," in *Proc. IEEE Conf. on Decis. and Control*, 2012, pp. 6615–6620.

[17] R. Seeber, H. Haimovich, M. Horn, L. M. Fridman, and H. De Battista, "Robust exact differentiators with predefined convergence time," *Automatica*, vol. 134, p. 109858, 2021.

[18] S. Diop, J. Grizzle, and F. Chaplais, "On numerical differentiation algorithms for nonlinear estimation," in *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)*, vol. 2. IEEE, 2000, pp. 1133–1138.

[19] A. Levant, "Finite differences in homogeneous discontinuous control," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1208–1217, 2007.

[20] ——, "Chattering analysis," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1380–1389, 2010.

[21] U. Pérez-Ventura and L. Fridman, "When is it reasonable to implement the discontinuous sliding-mode controllers instead of the continuous ones? frequency domain criteria," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 810–828, 2019.

[22] J.-J. E. Slotine, "Sliding controller design for non-linear systems," *International Journal of control*, vol. 40, no. 2, pp. 421–434, 1984.

[23] Y. Shtessel, C. Edwards, L. Fridman, A. Levant, Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, "Higher-order sliding mode controllers and differentiators," *Sliding mode control and observation*, pp. 213–249, 2014.

[24] V. Utkin, "Discussion aspects of high-order sliding mode control," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 829–833, 2015.

[25] A. F. Filippov, *Differential equations with discontinuous righthand sides*, ser. Mathematics and its Applications (Soviet Series). Dordrecht: Kluwer Academic Publishers Group, 1988, vol. 18, translated from the Russian.

[26] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*. Providence, RI, USA: American Mathematical Society, 1994.