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Performance Optimization of Angle-Based Network Localization*

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Abstract— **Recent advances in sensor network localization have enabled sensor nodes to localize themselves by using the measurements of inter-node angles. According to our earlier work, the proposed angle-based localization algorithms' performance, particularly, the convergence rate, is relatively poor, which, however, has not been adequately addressed in the existing literature. Motivated by this, this paper aims to improve the performance of angle-based localization algorithms, specifically, the stability margin, convergence rate and robustness against measurement noises. Firstly, we show that the stability margin, convergence rate and robustness of angle-based localization algorithms are commonly determined by one parameter, namely, the minimum eigenvalue of the network's localization matrix. Secondly, we formulate the performance optimization problem as an eigenvalue optimization problem, and show the non-differentiability of the eigenvalue optimization problem. By carefully choosing the decision variable, we utilize interior-point methods to obtain an optimal solution to the eigenvalue optimization problem. Finally, simulation examples validate the improvement of the algorithms' performance.**

I. INTRODUCTION

Sensor network localization is one of the fundamental problems in many multi-agent coordination tasks [1], [2]. For a static network consisting of anchor nodes and free nodes, the aim of network localization is to determine the positions of the free nodes using their sensor measurements with respect to their neighbors and the positions of the anchor nodes. The existing localization algorithms can be mainly categorized into three classes according to the sensor measurements among the nodes: distance-based [3], [4], bearing-based [5], [6], [7], [8] and angle-based [9], [10], [11].

With the development of sensor technology, such as antenna array-based Bluetooth 5.1 [12], inter-node angle measurements become more and more accessible, due to which the angle-based network localization has received intensive attention recently. The existing works on angle-based network localization mainly focus on two problems: network localizability conditions and distributed localization algorithms. Conditions for the localizability of an angle-based network have been studied in [9], [11], [10]. Sufficient localizability conditions for angle-based networks in 2-D space are obtained in [11]. For triangular sensor networks in 2-D space, a topological, necessary and sufficient localizability condition is proposed in [9], which is developed by using the theory of triangular angle rigidity. When the measurements consist of angles and displacements, localizability conditions are proposed in [10]. Only when an angle-based network is localizable, it is possible to develop a localization algorithm to estimate all the free nodes' positions. For 2-D sensor networks, distributed localization algorithms are proposed in [9], [11], in which measured angles and estimated positions are needed in the communication between neighboring nodes. The network localizability conditions and distributed localization algorithms in 3-D space or even higher dimensions can be found in [13], [10], [6].

In addition to the above mentioned two aspects, we also pay special attention to the performance of angle-based localization algorithms, since it is a fundamental concern in engineering practices. From existing works [9], [10], [14], the stability margin, convergence rate and robustness are three important performance indices of angle-based localization algorithms. Firstly, the stability margin not only verifies localization algorithms' stability, but also quantifies how much system uncertainties can be tolerated before system instability occurs. According to the stability analysis for the proposed angle-based localization algorithm in [9], the value of the stability margin equals the minimum eigenvalue of the network's localization matrix. Secondly, it is expected that the localization error could converge to zero as fast as possible, which is determined by the minimum eigenvalue of the localization matrix, according to the proposed angle-based localization algorithms [9], [14], [10]. Thirdly, when considering the existence of measurement noises, the angle-based localization algorithm proposed in [10] is globally and exponentially stable, provided that the norm of a defined error matrix is less than the minimum eigenvalue of the localization matrix. This indicates that the robustness of an angle-based localization also depends on the minimum eigenvalue of the localization matrix.

From the above introduction, we conclude that the performance of angle-based localization algorithms is mainly determined by the minimum eigenvalue of the network's localization matrix. However, according to the simulation examples in our earlier work [14], the minimum eigenvalue

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of the triangular angle-based network's localization matrix approximately equals 0.03, which indicates that the angle-based localization algorithm is of poor robustness and low convergence rate, that is, the position estimation error needs more than 300 seconds to decay to 10% of the initial error. Therefore, it is crucial to increase the minimum eigenvalue of the localization matrix. It is worth mentioning that due to the nonlinearity from decision variables, such as nodes' positions, to the minimum eigenvalue of the localization matrix, how to increase the minimum eigenvalue of the localization matrix is a challenging problem. We address the problem by formulating it as an eigenvalue optimization problem. One of the main difficulties of the eigenvalue optimization problem is that the problem is not differentiable when the eigenvalue's algebraic multiplicity is greater than one. Surprisingly, we will show that each eigenvalue's algebraic multiplicity of the angle-based network's localization matrix is at least two, which shows that the eigenvalue optimization problem is non-differentiable. In order to overcome the difficulty of non-differentiability, we reformulate the eigenvalue optimization problem as a linear semidefinite program (SDP). Furthermore, we will use the interior-point approach to obtain an optimal solution to the SDP. The main contributions of this paper are summarized as follows.

1) We show that the performance of angle-based localization algorithms is mainly determined by the minimum eigenvalue of the network's localization matrix.
2) We formulate the performance optimization problem as an eigenvalue optimization problem, which is shown to be non-differentiable. By choosing a proper decision variable, we utilize the interior-point method to obtain an optimal solution.

The remainder of this paper is organized as follows. We provide preliminaries in Section II. The main results are shown in Section III. Section IV provides simulation examples to validate our method.

II. PRELIMINARIES

A. Notations

Consider a 2-D static sensor network consisting of n_a anchor nodes and n_f free nodes. Let $\mathcal{V}_a = \{1, \cdots, n_a\}$, $\mathcal{V}_f = \{n_a+1, \cdots, n_a+n_f\}$ and $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_f$ denote the sets of anchor nodes, free nodes, and all nodes, respectively. The number of all nodes is $|\mathcal{V}| = n_a + n_f = n$. The positions of the anchor nodes are denoted by $p_a = \left[p_1^{\mathrm{T}}, \cdots, p_{n_a}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{2n_a}$. The positions of the free nodes are denoted by $p_f = \left[p_{n_a+1}^{\mathrm{T}}, \cdots, p_{n_a+n_f}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{2n_f}$. Let $\sum_g$ be the fixed global coordinate frame. All the positions of the nodes are respect to $\sum_g$. We assume that no overlapping points exist in $p = \left[p_a^{\mathrm{T}}, p_f^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{2n}$. Let I_2, $\mathbf{1}_n$, $\otimes$, $\lambda_{\max}$ and $\lambda_{\min}$ be the 2-by-2 identity matrix, $n \times 1$ column vector of all ones, the Kronecker product, the maximum eigenvalue and the minimum eigenvalue of a symmetric real matrix, respectively. Let $\|\cdot\|$ be the Euclidean norm of a vector or the spectral norm of a matrix. Denote the 2-D rotation matrix with rotation angle θ by $\bar{R}(\theta) \in \mathbb{R}^{2\times 2}$.

B. Description of angle-based localization algorithms' performance

For $\forall i, j \in \mathcal{V}_a \cup \mathcal{V}_f$, define the bearing from node i to node j in $\sum_g$ by $b_{ij} := (p_j - p_i)/\|p_j - p_i\|$, where p_i and p_j represent the coordinates of i and j in $\sum_g$, respectively. Assume that each node i measures the angle $\alpha_{kij} \in [0, 2\pi)$ with respect to its neighboring nodes $k, j \in \mathcal{V}_a \cup \mathcal{V}_f$ under the counterclockwise direction, which can be calculated by [15]

$$\alpha_{kij} = \begin{cases} \arccos\left(b_{ij}^{\mathrm{T}} b_{ik}\right), \text{ if } b_{ij}^{\mathrm{T}} b_{ik}^{\perp} \geq 0, \\ 2\pi - \arccos\left(b_{ij}^{\mathrm{T}} b_{ik}\right), \text{ otherwise}, \end{cases} \tag{1}$$

where $b_{ik}^{\perp} := \bar{R}\left(\frac{\pi}{2}\right) b_{ik} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} b_{ik}$, $j, k \in \mathcal{N}_i$, and $\mathcal{N}_i$ denotes the neighbor set of node i. Note that α_{kij} can be obtained using local bearing measurements b_{ij}^i and b_{ik}^i, which are measured in the node i's local coordinate frame $\sum_i$.

We recall the definition of angularity from [15] to describe a network with triple-vertex angle constraints. Let an angularity be denoted by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, where $\mathcal{A} = \{(i,j,k) | i,j,k \in \mathcal{V}, i \neq j \neq k\}$ denotes an angle set. We say that $\mathcal{A}$ is a triangular angle set if for every $(i_1, j_1, k_1) \in \mathcal{A}$, there also exists $\{(j_1, k_1, i_1), (k_1, i_1, j_1)\} \subseteq \mathcal{A}$. Denote the trigraph [9, Subsection 2.3] and the total number of triangles in the trigraph by $\mathcal{T}(\mathcal{V}, \mathcal{A})$ and m, respectively.

For the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, let $\alpha^* := \left[\cdots, \alpha_{ijk}^*, \alpha_{jki}^*, \alpha_{kij}^*, \cdots\right]^{\mathrm{T}} \in \mathbb{R}^{|\mathcal{A}|}$, $(i,j,k) \in \mathcal{A}$, where these constant angle constraints defined by $\mathcal{A}$ are calculated by the position p. An angle-induced linear equation is established as [9]

$$A_i^{\triangle ijk}(\alpha^*) p_i + A_j^{\triangle ijk}(\alpha^*) p_j + A_k^{\triangle ijk}(\alpha^*) p_k = 0, \tag{2}$$

where $A_i^{\triangle ijk}(\alpha^*) := \left(\sin\alpha_{jki}^* I_2 - \sin\alpha_{ijk}^* \bar{R}^{\mathrm{T}}(\alpha_{kij}^*)\right) \in \mathbb{R}^{2\times 2}$, $A_j^{\triangle ijk}(\alpha^*) := \left(\sin\alpha_{ijk}^* \bar{R}^{\mathrm{T}}(\alpha_{kij}^*)\right) \in \mathbb{R}^{2\times 2}$, and $A_k^{\triangle ijk}(\alpha^*) := \left(-\sin\alpha_{jki}^* I_2\right) \in \mathbb{R}^{2\times 2}$. According to (2), the triangular angle rigidity matrix $R_{\mathcal{A}}(\alpha^*)$ can be written as [9]

$$\begin{array}{c} \\ 1\text{st } \triangle \\ \cdots \\ \triangle ijk \\ \cdots \\ m\text{th } \triangle \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots & \text{Vertex } k & \cdots \end{array} \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_i^{\triangle ijk} & \cdots & A_j^{\triangle ijk} & \cdots & A_k^{\triangle ijk} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}.$$

Let $D(\alpha^*) := R_{\mathcal{A}}^{\mathrm{T}}(\alpha^*) R_{\mathcal{A}}(\alpha^*) \in \mathbb{R}^{2n\times 2n}$ and partition the matrix $R_{\mathcal{A}} = \begin{bmatrix} R_{\mathcal{A}}^a & R_{\mathcal{A}}^f \end{bmatrix}$ into anchor nodes' part $R_{\mathcal{A}}^a \in \mathbb{R}^{2m\times 2n_a}$ and free nodes' part $R_{\mathcal{A}}^f \in \mathbb{R}^{2m\times 2n_f}$, under which the matrix $D(\alpha^*)$ can be rewritten in the form of

$$D(\alpha^*) = R_{\mathcal{A}}^{\mathrm{T}}(\alpha^*) R_{\mathcal{A}}(\alpha^*) = \begin{bmatrix} D_{aa} & D_{af} \\ D_{fa} & D_{ff} \end{bmatrix}, \tag{3}$$

where $D_{aa} = (R_{\mathcal{A}}^a)^{\mathrm{T}} R_{\mathcal{A}}^a \in \mathbb{R}^{2n_a\times 2n_a}$, $D_{af} = (R_{\mathcal{A}}^a)^{\mathrm{T}} R_{\mathcal{A}}^f \in \mathbb{R}^{2n_a\times 2n_f}$, $D_{fa} = \left(R_{\mathcal{A}}^f\right)^{\mathrm{T}} R_{\mathcal{A}}^a \in \mathbb{R}^{2n_f\times 2n_a}$, and $D_{ff} =$

$\left(R_{\mathcal{A}}^{f}\right)^{\mathrm{T}} R_{\mathcal{A}}^{f} \in \mathbb{R}^{2n_f \times 2n_f}$. We define D_{ff} as the network's localization matrix since it plays a key role in network localization.

According to [9, Section 5], a distributed localization algorithm can be designed as

$$\dot{\hat{p}}_f(t) = -D_{ff}\hat{p}_f(t) - D_{fa}p_a = -D_{ff}\left(\hat{p}_f(t) - p_f\right), \quad (4)$$

where $\hat{p}_f$ denotes the estimated positions of the free nodes, and $p_f = -D_{ff}^{-1}D_{fa}p_a$ if the network is localizable and D_{ff} is nonsingular.

Now we give the stability condition for the distributed localization algorithm (4).

Lemma 1: [9, Theorem 8] If $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is localizable and p is generic [15, Definition 4], $\hat{p}_f(t)$ converges to p_f globally and exponentially under the distributed localization algorithm (4). The estimation error $\left\|p_f - \hat{p}_f(t)\right\|$ satisfies

$$\left\|p_f - \hat{p}_f(t)\right\| \leq \left\|p_f - \hat{p}_f(0)\right\| e^{-\lambda_{\min}(D_{ff})t}.$$

When considering the existence of measurement noises, we denote the error matrices due to measurement noises by ΔD_{ff} and ΔD_{fa}, and define $\hat{D}_{ff} := D_{ff} + \Delta D_{ff}$ and $\hat{D}_{fa} := D_{fa} + \Delta D_{fa}$. Then, the distributed localization algorithm (4) becomes

$$\dot{\hat{p}}_f(t) = -\hat{D}_{ff}\hat{p}_f(t) - \hat{D}_{fa}p_a = -\hat{D}_{ff}\left(\hat{p}_f(t) + \hat{D}_{ff}^{-1}\hat{D}_{fa}p_a\right), \quad (5)$$

which holds if $\hat{D}_{ff}$ is nonsingular.

Now, we give the condition for the nonsingularity of $\hat{D}_{ff}$.

Lemma 2: [10, Theorem 10] Given a localizable network $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with nonsingular D_{ff}, then $\hat{D}_{ff}$ is nonsingular if the error matrix ΔD_{ff} satisfies

$$\left\|\Delta D_{ff}\right\| < \lambda_{\min}\left(D_{ff}\right). \quad (6)$$

Combining Lemma 1 and Lemma 2, one has the following proposition.

Proposition 1: For a triangular sensor network $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with the localization algorithm (4), the following statements hold.

1) The localization algorithm (4) is asymptotically stable if and only if the network is localizable which holds if and only if D_{ff} is nonsingular, i.e., $\lambda_{\min}(D_{ff}) > 0$;
2) The convergence rate of the localization error $\tilde{p}_f(t) = \hat{p}_f(t) - p_f$ to zero is higher if $\lambda_{\min}(D_{ff})$ is larger;
3) The robustness of (4) against measurement noises is higher if $\lambda_{\min}(D_{ff})$ is larger.

Proposition 1 indicates that a larger $\lambda_{\min}(D_{ff})$ will make positive effects on the performance of the angle-based localization algorithm (4).

III. MAIN RESULTS

In this section, we design an algorithm to optimize the performance of the angle-based localization algorithm.

A. Selection of the decision variable

It is expected to maximize $\lambda_{\min}(D_{ff})$ since a larger $\lambda_{\min}(D_{ff})$ implies better performance of angle-based localization algorithms according to Proposition 1. Note that if $A_i^{\triangle s}(\alpha^*)p_i + A_j^{\triangle s}(\alpha^*)p_j + A_k^{\triangle s}(\alpha^*)p_k = 0$ given in (2) is an angle-induced linear equation, where $s \in \{1,\cdots,m\}$, denotes the sth triangle in $\mathcal{T}(\mathcal{V},\mathcal{A})$, and i,j,k denote the vertices of the sth triangle, $\sqrt{\beta_s}\left(A_i^{\triangle s}(\alpha^*)p_i + A_j^{\triangle s}(\alpha^*)p_j + A_k^{\triangle s}(\alpha^*)p_k\right) = 0, \beta_s \in \mathbb{R}^+$ is also an angle-induced linear equation, which can be written in the compact form of

$$\left(\mathrm{diag}\left[\sqrt{\beta_1},\cdots,\sqrt{\beta_m}\right] \otimes I_2\right) R_{\mathcal{A}}(\alpha^*)p = \mathbf{0}. \quad (7)$$

From (7), the new D_{ff} is modified to

$$\begin{aligned} D_{ff}(\beta) = &\left(\left(\mathrm{diag}\left[\sqrt{\beta_1},\cdots,\sqrt{\beta_m}\right] \otimes I_2\right) R_{\mathcal{A}}^{f}(\alpha^*)\right)^{\mathrm{T}} \\ &\cdot \left(\left(\mathrm{diag}\left[\sqrt{\beta_1},\cdots,\sqrt{\beta_m}\right] \otimes I_2\right) R_{\mathcal{A}}^{f}(\alpha^*)\right) \\ =&\beta_1 e_1^{\mathrm{T}} e_1 + \cdots + \beta_m e_m^{\mathrm{T}} e_m \\ =&\beta_1 E_1 + \cdots + \beta_m E_m, \end{aligned} \quad (8)$$

where $\beta = [\beta_1,\cdots,\beta_m]^{\mathrm{T}} \in \mathbb{R}^m$, $R_{\mathcal{A}}(\alpha^*) = [e_1^{\mathrm{T}},\cdots,e_m^{\mathrm{T}}]^{\mathrm{T}}$, $e_i \in \mathbb{R}^{2\times 2n_f}$, and $E_i := e_i^{\mathrm{T}} e_i \in \mathbb{R}^{2n_f\times 2n_f}, i \in \{1,\cdots,m\}$. The matrix D_{ff} in (8) is a linear combination of symmetric matrices E_i with coefficients β_i, $i = 1,\cdots,m$, which indicates that the function $D_{ff}(\beta)$ is convex. Taking β as the decision variable for the eigenvalue optimization problem provides us freedoms to adjust the structure of D_{ff} so that $\lambda_{\min}(D_{ff})$ can be optimized. The parameter vector β can also be considered as a weighted vector of triangles in the network, as shown in Fig. 1. Note that a normalization is needed for β to guarantee the feasibility of the problem, namely $\beta_i > 0$ and $\sum_{i=1}^{m}\beta_i = 1$, since if there is no constraint on β, $\lambda_{\min}(D_{ff}(\beta))$ could be changed arbitrarily. Under the definition of $D_{ff}(\beta)$ in (8) and constraints on β, the eigenvalue optimization problem can be formulated as

$$\begin{aligned} \max_{\beta} \quad & \lambda_{\min}(D_{ff}(\beta)) = \lambda_{\min}(\beta_1 E_1 + \cdots + \beta_m E_m) \\ \text{s.t} \quad & \beta > 0, \quad \beta^{\mathrm{T}}\mathbf{1}_m = 1. \end{aligned} \quad (9)$$

Indeed, the parameter vector β is not the only type of decision variables that can be chosen for the eigenvalue optimization problem. For example, if we take the positions of the anchor nodes, namely p_a, as the decision variable, the eigenvalue optimization problem could be formulated as

$$\max_{p_a} \quad \lambda_{\min}(D_{ff}(p_a)).$$

Firstly, the optimal solution p_a^* for the above eigenvalue optimization problem may be a local solution since the convexity of $\lambda_{\min}(D_{ff}(p_a))$ cannot be guaranteed, which is caused by the nonlinearity of the function $D_{ff}(p_a)$ [16]. Secondly, the number of saddle points will increase as the network scale increases [17], which makes it much difficult to obtain a global and optimal solution.

Another alternative is that we could take the feedback gain $k_c \in \mathbb{R}^+$ as the decision variable, which can be added in (4) [9], [13], i.e.,

$$\dot{\hat{p}}_f(t) = -k_c D_{ff}\left(\hat{p}_f(t) - p_f\right).$$

Then, the eigenvalue optimization problem can be formulated as a linear matrix inequality (LMI) form [18], i.e.,

$$\begin{aligned} &\text{Find} \quad k_c > 0, \\ &\text{s.t} \quad D_{ff}(k_c) := k_c D_{ff} \succeq \bar{\lambda} I_{2n_f}, \end{aligned}$$

where $\bar{\lambda} \in \mathbb{R}^+$ is a given desired minimum eigenvalue for D_{ff}. Note that this eigenvalue optimization problem can be considered as a special case of (9) when taking $\beta_i = k_c, i = 1, \cdots, m$ and freeing the equality constraint $\sum_{i=1}^m \beta_i = 1$. However, this solution $\beta_i = k_c, i = 1, \cdots, m$ may not be optimal.

According to the above discussion, taking β as the decision variable has several advantages. Firstly, the solution to the problem (9) is globally optimal since the cost function and constraints are all convex. Secondly, the vector β has a physical meaning which can be interpreted as the importance of the triangles in $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. Thirdly, due to the smoothness of the problem (9), powerful methods from smooth analysis [19] can be applied.

B. Properties of the localization matrix's eigenvalues

In this subsection, we analyze the algebraic multiplicity of $\lambda_{\min}(D_{ff}(\beta))$. We first present the following lemma.

Lemma 3: For $\beta_i > 0, i = 1, \cdots, m$, $D_{ff}(\beta)$ can be described by

$$D_{ff}(\beta) = \begin{bmatrix} a_1 I_2 & b_{12}\bar{R}(\theta_{12}) & \cdots & b_{1n_f}\bar{R}(\theta_{1n_f}) \\ b_{12}\bar{R}^{\mathrm{T}}(\theta_{12}) & a_2 I_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n_f}\bar{R}^{\mathrm{T}}(\theta_{1n_f}) & \cdots & \cdots & a_{n_f} I_2 \end{bmatrix}, \quad (10)$$

where $a_i = \sum_{s=1}^m \beta_s \varepsilon_i^{\triangle s}$, $\varepsilon_i^{\triangle s} \in \mathbb{R}^+$, $b_{ij} \in \mathbb{R}^+$ and $\theta_{ij} \in \mathbb{R}$ denote the corresponding coefficient and rotation angle after normalizing $\sum_{s=1}^m \beta_s \varepsilon_{ij}^{\triangle s} \bar{R}\left(\theta_{ij}^{\triangle s}\right)$, $\varepsilon_{ij}^{\triangle s} \in \mathbb{R}$, $\theta_{ij}^{\triangle s} \in \mathbb{R}$, $i \neq j$, respectively.

The proof of Lemma 3 can be found in APPENDIX V-A.

Theorem 2: Suppose that $\beta_i > 0, \forall i = 1, \cdots, n_f$. The algebraic multiplicity of $\lambda_j(D_{ff}(\beta))$ is equal to the geometric multiplicity of $\lambda_j(D_{ff}(\beta)), \forall j = 1, \cdots, 2n_f$, which is always an even number.

The proof of Theorem 2 can be found in APPENDIX V-B.

C. Eigenvalue optimization approach

Combining Theorem 2, we can derive that $\lambda_{\min}(D_{ff}(\beta))$ is non-differentiable according to [20], [21]. To overcome the difficulty due to the non-differentiability of (9), we transform the problem into a SDP [22], [20], i.e.,

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t} \quad & E(x) := \lambda I + \sum_{i=1}^m x_i E_i \succeq \mathbf{0}, \\ & x_i \geq 0, i = 1, \cdots, m, \\ & b^{\mathrm{T}} x = 1, \end{aligned} \quad (11)$$

where $f(x) := a^{\mathrm{T}} x$, $x := \left[\beta^{\mathrm{T}}, \lambda\right]^{\mathrm{T}} \in \mathbb{R}^{m+1}$, $a = [0, \cdots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^{m+1}$, and $b = [1, \cdots, 1, 0]^{\mathrm{T}} \in \mathbb{R}^{m+1}$. In (11), we transform the max-min eigenvalue optimization problem into a typical min-max eigenvalue optimization problem using the min-max principle, namely $\max \lambda_{\min}(D_{ff}(\beta)) = \min \lambda_{\max}(-D_{ff}(\beta))$, since $D_{ff}(\beta)$ is positive definite when the localization algorithm (4) is asymptotically stable according to Proposition 1. The optimization objective for (11) is the scalar λ under the LMI constraint $\lambda I + \sum_{i=1}^m x_i E_i \succeq \mathbf{0}$, which is equivalent to minimizing the maximum eigenvalue of $-D_{ff}(\beta)$ [22]. Importantly, the problem (11) is still convex since the cost function $f(x)$ and all constraints are linear.

We use the interior-point method [22], [20], [23] to solve the problem (11). By utilizing the logarithmic barrier functions $\log(\cdot)$ and $\log\det(\cdot)$, we approximately reformulate the constrained optimization problem (11) as

$$\begin{aligned} \min_x \quad & t f(x) + \phi(x) \\ \text{s.t} \quad & b^{\mathrm{T}} x = 1, \end{aligned} \quad (12)$$

where $t \in \mathbb{R}^+$, and $\phi(x) := -\sum_{i=1}^m \log x_i - \log\det E(x)$. Denote the domain of $\phi(x)$ by $\operatorname{dom} \phi = \left\{x \in \mathbb{R}^{m+1} \middle| x_i > 0, i = 1, \cdots, m, E(x) \succeq \mathbf{0}\right\}$. Note that if x is a strictly feasible solution to (12), namely, $x \in \operatorname{dom} \phi \cap \left\{x \middle| x \in \mathbb{R}^{m+1}, b^{\mathrm{T}} x = 1\right\}$, it also satisfies all the constraints in (11). The cost function $t f(x) + \phi(x)$ is convex since the sum of convex functions $t f(x)$ and $\phi(x)$ is also convex, which indicates that the problem (12) has a global optimal solution. Besides, since the cost function in (12) is second-order differentiable due to the second-order differentiability of $t f(x)$ and $\phi(x)$, we can use Newton's methods [23, Section 9.5] to solve (12) when t is fixed. The scalar t denotes a weighted parameter, which is updated according to

$$t_{k+1} = \mu t_k, \quad (13)$$

where $\mu > 1$ and $k = 1, 2, \cdots$ denotes the iteration sequence number. The choice of the parameter μ involves a trade-off in the numbers of inner and outer steps required in Algorithm 1. According to [23, Section 11.3], for μ in a range from around 3 to 100 or so, the total number of steps, namely the result of the number of the inner steps multiplies the number of the outer steps, remains approximately constant, which indicates that choice of μ is not particularly critical.

The complete process of the interior-point method is shown in Algorithm 1. At each iteration k, we compute the central point $x^*(t_k)$ starting from the previously computed central point by utilizing the Newton's method to solve (12) when $t = t_k$. Then we calculate t_{k+1} following (13). The scalar $\varepsilon \in \mathbb{R}^+$ denotes the accurate threshold and the stop criterion $\frac{m_c}{t_k} < \varepsilon$, where m_c denotes the number of inequality constraints in (11), is developed in Proposition 3.

Denote the optimal value of $f(x)$ by f^*. Note that the problem (12) is an approximation of (11), and $x^*(t)$ is the optimal solution to (12). The gap between f^* and $f(x^*(t))$ is described in the following proposition.

Proposition 3: [23, Section 11.2] Consider the optimization problem (12). Assume $t > 0$, $\mu > 1$ and $\varepsilon > 0$. For every strictly feasible initial $x(t_0)$, under Algorithm 1, the

Algorithm 1 Interior-point method

Require: Strictly feasible x, initial $t_0 > 0$, factor μ, and tolerance ε.

for $k = 1, 2, \cdots$ **do**

Compute the central point $x^*(t_k)$ by minimizing (12), starting at x.

Update x by $x = x^*(t_k)$.

if $\frac{m_c}{t_k} < \varepsilon$ **then**

Quit the loop.

end if

Calculate t_{k+1} following (13).

end for

return The optimization solution x to the problem (11).

gap associated with $x^*(t)$ is described as

$$f(x^*(t)) - f^* \leq \frac{m_c}{t}, \tag{14}$$

i.e., $x^*(t)$ is no more than $\frac{m_c}{t}$-suboptimal.

Proposition 3 shows that the gap $\frac{m_c}{t}$ converges to zero when t goes to infinity, which indicates that the optimal solution $x^*(t)$ of (12) is close enough to that of (11) when the gap $\frac{m_c}{t}$ is less than the specified accurate threshold ε.

Remark 1: We give a complexity analysis for Algorithm 1. According to Proposition 3, the accurate threshold ε is achieved after $\left\lceil \frac{\log(m_c/\varepsilon t_0)}{\log\mu} \right\rceil$ centering steps. At every centering step, the required number of Newton steps is $\frac{m_c(\mu-1-\log\mu)}{\gamma} + c$, where $\gamma, c \in \mathbb{R}$ are constants [23, Section 11.5]. Using $m_c = m+1$, the total number of Newton steps in relation to the scale of the network is $\left\lceil \frac{\log((m+1)/\varepsilon t_0)}{\log\mu} \right\rceil \left(\frac{(m+1)(\mu-1-\log\mu)}{\gamma} + c \right)$, which indicates that the total number of Newton steps increases as m, i.e., the number of triangles in the network, increases.

Remark 2: The initial value $x(t_0)$ may only affect the dynamic characteristics of the convergent process when executing Algorithm 1 according to [23, Section 11.3]. The proposed eigenvalue optimization method in this paper is centralized. When applying the proposed performance optimization method to engineering practices, it requires to calculate the optimal β offline and modify D_{ff} in the distributed angle-based localization algorithm (4) according to (8). Developing a distributed form for the proposed eigenvalue optimization method is one of our future works.

IV. SIMULATION

The sensor network is shown in Fig. 1, which consists of $m = 6$ triangles: $\triangle 234, \triangle 346, \triangle 467, \triangle 457, \triangle 578$ and $\triangle 167$. The positions of the two anchor nodes are $p_1 = [0.3, -1.0]^{\mathrm{T}}$ and $p_2 = [-2.7, 2.7]^{\mathrm{T}}$, and the positions of the free nodes are $p_3 = [-6.0, 2.1]^{\mathrm{T}}, p_4 = [-0.6, 2.2]^{\mathrm{T}}, p_5 = [0.3, 1.2]^{\mathrm{T}}, p_6 = [-3.0, 0.3]^{\mathrm{T}}, p_7 = [0.1, -0.1]^{\mathrm{T}}$ and $p_8 = [0.7, 0.0]^{\mathrm{T}}$. The initial decision variable $\beta = \frac{1}{m} \cdot \mathbf{1}_m$ and $\lambda_{\min}\left(D_{ff}(\beta(0))\right) = 0.0075$. Let the initial estimation $\hat{p}_f(0) = 1.5 p_f$.

Through Algorithm 1, we obtain the optimal solution $\beta^* = [0.4686, 0.0978, 0.1011, 0.0907, 0.0507, 0.1911]^{\mathrm{T}}$ and $\lambda_{\min}\left(D_{ff}(\beta^*)\right) = 0.0183 \approx 2.46 \lambda_{\min}\left(D_{ff}(\beta(0))\right)$, which has been improved effectively by our method. The weights β_1 and β_6 corresponding to $\triangle 234$ and $\triangle 167$ have been obviously increased a lot with respect to $\beta_1(0)$ and $\beta_6(0)$. One physical interpretation for this is that the eigenvalues of E_1 and E_6 defined in (8) are relatively small, which is one of the reasons that makes the minimum eigenvalue of $D_{ff}(\beta)$ small. If large coefficients β_1 and β_6 are given in the front of E_1 and E_6, the minimum eigenvalue of $D_{ff}(\beta)$ can be increased.

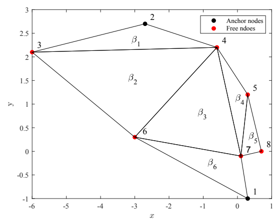

Fig. 1. Network topology with 8 nodes and 6 triangles.

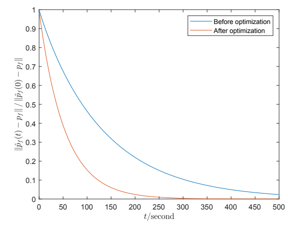

Fig. 2. Position estimation errors without measurement noises.

Fig. 2 shows the evolution of the position estimation error $\left\|\hat{p}_f(t) - p_f\right\|$ without the existence of measurement noises under the angle-based localization algorithm (4). The convergence rate of the position estimation error to zero is faster after optimization.

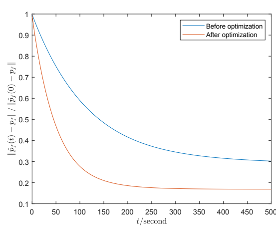

Fig. 3. Position estimation errors when $\left\|\Delta D_{ff}\right\| = 0.0037$.

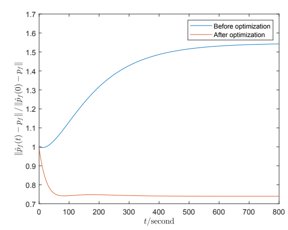

Fig. 4. Position estimation errors when $\left\|\Delta D_{ff}\right\| = 0.0129$.

To simulate noisy sensor environments, each element of the error matrices ΔD_{ff} and ΔD_{fa} is generated by using white noises. When $\left\|\Delta D_{ff}\right\| = 0.0037 < \lambda_{\min}\left(D_{ff}(\beta(0))\right) < \lambda_{\min}\left(D_{ff}(\beta^*)\right)$, Fig. 3 shows that the position estimation error is less after optimization. When $\lambda_{\min}\left(D_{ff}(\beta(0))\right) < \left\|\Delta D_{ff}\right\| = 0.0129 < \lambda_{\min}\left(D_{ff}(\beta^*)\right)$, Fig. 4 shows that the angle-based localization algorithm (5) has better robustness against larger measurement noises after performance optimization. These two simulation examples validate that our performance optimization method can improve the stability margin and robustness of the angle-based localization algorithm (5).

V. CONCLUSIONS

This paper has developed an optimization method to improve the performance of the angle-based localization algorithm. Firstly, we have shown that the performance of angle-based localization algorithms is mainly determined by the minimum eigenvalue of the network's localization matrix.

Secondly, we have formulated the performance optimization problem as an eigenvalue optimization problem, and shown the non-differentiability of the eigenvalue optimization problem. We have utilized the interior-point method to obtain an optimal solution to the eigenvalue optimization problem.

APPENDIX

A. Proof of Lemma 3

Proof: Conducting basic trigonometric calculations for the angle-induced coefficient matrix $A_i^{\triangle s}$ defined in (2), where s denotes the sth triangle in $\mathcal{T}(\mathcal{V},\mathcal{A})$, one has

$$\begin{aligned} &\left(A_i^{\triangle s}\right)^{\mathrm{T}} A_i^{\triangle s} = \varepsilon_i^{\triangle s} I_2, \\ &\left(A_i^{\triangle s}\right)^{\mathrm{T}} A_j^{\triangle s} = \begin{bmatrix} \varepsilon_{ij,1}^{\triangle s} & \varepsilon_{ij,2}^{\triangle s} \\ -\varepsilon_{ij,2}^{\triangle s} & \varepsilon_{ij,1}^{\triangle s} \end{bmatrix} = \varepsilon_{ij}^{\triangle s} \bar{R}\left(\theta_{ij}^{\triangle s}\right), \end{aligned} \tag{15}$$

where $i,j \in \{1,\cdots,n_f\}$, $i \neq j$, $\varepsilon_i^{\triangle s} \in \mathbb{R}^+$, $\varepsilon_{ij,1}^{\triangle s} \in \mathbb{R}$, $\varepsilon_{ij,2}^{\triangle s} \in \mathbb{R}$, and $\varepsilon_{ij}^{\triangle s} \in \mathbb{R}$ are constants related to those interior angles within $\triangle s$, and $\theta_{ij}^{\triangle s} \in \mathbb{R}$ denotes the corresponding rotation angle. Then according to the definition of $D_{ff}(\beta)$ in (8), the 2-by-2 diagonal blocks and off-diagonal blocks of $D_{ff}(\beta)$ can be described by

$$D_{ff}[i,i] = \sum_{s=1}^{m} \beta_s \varepsilon_i^{\triangle s} I_2, D_{ff}[i,j] = \sum_{s=1}^{m} \beta_s \varepsilon_{ij}^{\triangle s} \bar{R}\left(\theta_{ij}^{\triangle s}\right), \tag{16}$$

respectively. Consequently, we can conclude the result. ■

B. Proof of Theorem 2

Proof: Since $D_{ff}(\beta) = D_{ff}^{\mathrm{T}}(\beta)$, $D_{ff}(\beta)$ is diagonalizable. Then the algebraic multiplicity of $\lambda_j\left(D_{ff}(\beta)\right)$ is equal to its geometric multiplicity.

We then prove that the multiplicity of each $\lambda_j\left(D_{ff}(\beta)\right), j = 1,\cdots,2n_f$ is an even number. According to [9, Theorem 6], if the network $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is localizable and $\beta_i > 0$, $D_{ff}(\beta)$ is positive definite. Therefore, $\lambda_{\min}(D_{ff}(\beta)) > 0$ and all the eigenvectors of $D_{ff}(\beta)$ are nonzero. According to Lemma 3, suppose that $x = \left[x_1,\cdots,x_{2n_f}\right]^{\mathrm{T}} \in \mathbb{R}^{2n_f}$ is the eigenvector corresponding to an eigenvalue $\lambda_j(D_{ff}(\beta)), j = 1,\cdots,2n_f$, i.e., $\left(\lambda_j(D_{ff}(\beta))I_{2n_f} - D_{ff}(\beta)\right)x = 0$, whose component form is

$$\begin{bmatrix} (\lambda_j - a_1)I_2 & -b_{12}\bar{R}(\theta_{12}) & \cdots & -b_{1n_f}\bar{R}(\theta_{1n_f}) \\ -b_{12}\bar{R}^{\mathrm{T}}(\theta_{12}) & (\lambda_j - a_2)I_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -b_{1n_f}\bar{R}^{\mathrm{T}}(\theta_{1n_f}) & \cdots & \cdots & (\lambda_j - a_{n_f})I_2 \end{bmatrix} x = \mathbf{0}. \tag{17}$$

Now we construct a new vector $\bar{x} = \left[x_2, -x_1, x_4, -x_3, \cdots, x_{2n_f}, -x_{2n_f-1}\right]^{\mathrm{T}} \in \mathbb{R}^{2n_f}$. It can be easily verified that $\left(\lambda_j(D_{ff}(\beta))I_{2n_f} - D_{ff}(\beta)\right)\bar{x} = 0$. This implies that $\bar{x}$ is also an eigenvector of $D_{ff}(\beta)$ corresponding to the eigenvalue $\lambda_j(D_{ff}(\beta))$. Since $x \neq 0$ and $x \neq \bar{x}$, the multiplicity of $\lambda_j(D_{ff}(\beta))$ is at least two. Since this holds for an arbitrary $j = 1,\cdots,2n_f$, the multiplicity of each eigenvalue of $D_{ff}(\beta)$ is an even number. ■

REFERENCES

[1] S. Zhao, "Affine Formation Maneuver Control of Multiagent Systems," *IEEE Trans. Automat. Contr.*, vol. 63, no. 12, pp. 4140–4155, 2018.

[2] A. Kleiner and C. Dornhege, "Real-time Localization and Elevation Mapping within Urban Search and Rescue Scenarios," *Journal of Field Robotics*, vol. 24, no. 8-9, pp. 723–745, 2007.

[3] Y. Diao, Z. Lin, and M. Fu, "A Barycentric Coordinate Based Distributed Localization Algorithm for Sensor Networks," *IEEE Trans. Signal Processing*, vol. 62, no. 18, pp. 4760–4771, 2014.

[4] Y. Xia, C. Yu, and C. He, "An Exploratory Distributed Localization Algorithm Based on 3D Barycentric Coordinates," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 8, pp. 702–712, 2022.

[5] S. Zhao and D. Zelazo, "Localizability and distributed protocols for bearing-based network localization in arbitrary dimensions," *Automatica*, vol. 69, pp. 334–341, 2016.

[6] K. Cao, Z. Han, Z. Lin, and L. Xie, "Bearing-only distributed localization: A unified barycentric approach," *Automatica*, vol. 133, p. 109834, 2021.

[7] B. Wang and Y.-P. Tian, "Distributed Network Localization: Accurate Estimation With Noisy Measurement and Communication Information," *IEEE Trans. Signal Processing*, vol. 66, no. 22, pp. 5927–5940, 2018.

[8] X. Li, X. Luo, and S. Zhao, "Globally Convergent Distributed Network Localization Using Locally Measured Bearings," *IEEE Transactions on Control of Network Systems*, vol. 7, no. 1, pp. 245–253, 2019.

[9] L. Chen, "Triangular angle rigidity for distributed localization in 2D," *Automatica*, vol. 143, p. 110414, 2022.

[10] X. Fang, X. Li, and L. Xie, "Angle-Displacement Rigidity Theory With Application to Distributed Network Localization," *IEEE Trans. Automat. Contr.*, vol. 66, no. 6, pp. 2574–2587, 2020.

[11] G. Jing, C. Wan, and R. Dai, "Angle-Based Sensor Network Localization," *IEEE Trans. Automat. Contr.*, vol. 67, no. 2, pp. 840–855, 2021.

[12] M. Cominelli, P. Patras, and F. Gringoli, "Dead on Arrival: An Empirical Study of The Bluetooth 5.1 Positioning System," in *Proceedings of the 13th international workshop on wireless network testbeds, experimental evaluation & characterization*, 2019, pp. 13–20.

[13] L. Chen, K. Cao, L. Xie, X. Li, and M. Feroskhan, "3-D Network Localization Using Angle Measurements and Reduced Communication," *IEEE Trans. Signal Processing*, vol. 70, pp. 2402–2415, 2022.

[14] L. Chen, L. Xie, X. Li, X. Fang, and M. Feroskhan, "Simultaneous localization and formation using angle-only measurements in 2D," *Automatica*, vol. 146, p. 110605, 2022.

[15] L. Chen, M. Cao, and C. Li, "Angle Rigidity and Its Usage to Stabilize Multiagent Formations in 2-D," *IEEE Trans. Automat. Contr.*, vol. 66, no. 8, pp. 3667–3681, 2020.

[16] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. Anderson, and P. N. Belhumeur, "A Theory of Network Localization," *IEEE Transactions on Mobile Computing*, vol. 5, no. 12, pp. 1663–1678, 2006.

[17] U. Helmke and B. D. Anderson, "Equivariant morse theory and formation control," in *2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE, 2013, pp. 1576–1583.

[18] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han, "Necessary and Sufficient Graphical Conditions for Affine Formation Control," *IEEE Trans. Automat. Contr.*, vol. 61, no. 10, pp. 2877–2891, 2015.

[19] A. Shapiro and M. K. Fan, "On Eigenvalue Optimization," *SIAM Journal on Optimization*, vol. 5, no. 3, pp. 552–569, 1995.

[20] F. Jarre, "An Interior-Point Method for Minimizing the Maximum Eigenvalue of a Linear Combination of Matrices," *SIAM Journal on Control and Optimization*, vol. 31, no. 5, pp. 1360–1377, 1993.

[21] M. L. Overton, "Large-scale optimization of eigenvalues," *SIAM Journal on Optimization*, vol. 2, no. 1, pp. 88–120, 1992.

[22] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz, "An Interior-Point Method for Semidefinite Programming," *SIAM Journal on optimization*, vol. 6, no. 2, pp. 342–361, 1996.

[23] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.