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Condition for Sensitivity Unidentifiability of Linear Systems with Affinely Parameter-Dependent Coefficient Matrices

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Abstract— **In this paper, we analyze "sensitivity identifiability" of initial states and parameters in affinely parametrized linear systems. If the true parameter is sensitivity unidentifiable (non-SI), optimization-based estimation algorithms may face computational problems. Thus, it is important to detect whether the parameter is non-SI a priori. To this aim, we address a problem to find the condition that the parameter is non-SI for any initial state and input. Then, we obtain a sufficient condition for the problem. The condition is given by algebraic equations and is expected to be the foundation of structural conditions. We show systems that satisfy the condition and are observable and controllable.**

I. Introduction

Parameter estimation methods from input-output data and its applications have been extensively studied, with the extended Kalman filter [1] being one representative method. The application of parameter estimation can be seen in many fields, including model-based predictive control [2], inspection of systems [3], and modeling biological systems [4].

The fundamental problem for parameter estimations is the uniqueness of a parameter with respect to input-output data. This uniqueness is called parameter identifiability [5]. When the true parameter is unidentifiable, detecting it is important for parameter estimations. For example, if we know the parameter is unidentifiable a priori, we can avoid wasting time and effort on impossible tasks for parameter estimations and/or assuming incorrectly that the estimated parameter is true.

In inspection of systems, we generally have to estimate initial states and parameters simultaneously. This is because the initial state is unknown from the input-output data of systems in operation [3]. Thus, the identifiability of initial states and parameters of a system must be considered.

In this paper, we analyze the local identifiability of initial states and parameters in the sense of "sensitivity identifiability" [6]. Sensitivity identifiability is defined based on linear independence of sensitivity of output data with respect to parameters and also initial states [6], and is a sufficient condition for the local identifiability. Moreover, it is known that the sensitivity identifiability guarantees quadratic convergence of Newton's method for the minimization of the output error. For linear systems, the sensitivity identifiability of initial states and parameters can be written by an algebraic representation [6].

It is important to detect that the true parameter is sensitivity unidentifiable (non-SI) for any initial state and input. This is because, in this case, we cannot estimate the true parameter in the sense of SI, unless the structure of the system is changed. If the initial state is fixed to zero, there exists research [7], [8] that deal with similar analysis using Markov parameters. However, there is no research to find conditions in the sense of sensitivity identifiability.

In this paper, we consider linear systems with affinely parameterized coefficient matrices to derive a condition so that the given parameter is non-SI for any initial state and input. Affinely parametrized linear systems can express complex network structured systems [9], where some results have been reported recently, for example, global identifiability conditions [10], [11] and observability conditions [12]–[14].

Our main contributions are summarized as the following two points: First, we define linear systems with affinely parameter-dependent coefficient matrices and address a problem to find conditions so that the given parameter is non-SI for any initial state and input. Second, we obtain a sufficient condition for this problem. The condition is written by algebraic linear equations. Thus, we can easily verify whether a given parameter satisfies the condition.

This paper is organized as follows. In section II, we explain local identifiability and sensitivity identifiability and define the problem. In section III we demonstrate the obtained condition that answers the problem. In section IV, we show example systems that are non-SI for any initial state and input to show how the condition works.

The definitions of symbols and operators used in this paper are as follows.
$L_2(t_0, t_f, \mathbb{R}^{m\times n})$: Function space made of $m \times n$ real matrix-valued functions that are square integrable on $[t_0, t_f]$.
$\langle f, g\rangle := \int_{t_0}^{t_f} f^T(t)g(t)dt$ $(f, g \in L_2(t_0, t_f, \mathbb{R}^{n\times m}))$.
$\|f\|_2 := \langle f, f\rangle$ $(f \in L_2(t_0, t_f, \mathbb{R}^{n\times m}))$.

II. Problem Formulation

Consider a parameter-dependent linear system

$$\Sigma(\theta)\begin{cases}\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t),\\ y(t) = C(\theta)x(t),\end{cases}$$
$$x(t_0) = x_0,$$

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where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, and $u(t) \in \mathbb{R}^m$. The system matrices are assumed to be parameterized affinely as

$$\begin{aligned} A(\theta) &= A_0 + \theta_1 A_1 + \cdots + \theta_q A_q, \\ C(\theta) &= C_0 + \theta_1 C_1 + \cdots + \theta_q C_q, \\ B(\theta) &= [b_1(\theta) \ \cdots \ b_m(\theta)], \\ b_i(\theta) &= b_{i,0} + \theta_1 b_{i,1} + \cdots + \theta_q b_{i,q}, \end{aligned}$$

where $A_j \in \mathbb{R}^{n\times n}$, $C_j \in \mathbb{R}^{p\times n}$, and $b_{i,j} \in \mathbb{R}^{n\times 1}$ are known matrices and $\theta \in \mathbb{R}^q$ is the parameter.

In this paper, we consider estimating $\psi = [x_0^T \ \theta^T]^T$ based on the input-output data. Since we are interested in the characteristics of the system itself, we consider the case of data without noise. For any given $u \in L_2(t_0, t_f; \mathbb{R}^m)$ and $\psi \in \mathbb{R}^{n+q}$, the output of $\Sigma(\theta)$ is denoted by $y(t; \psi, u)$. Since $\Sigma(\theta)$ is a linear system, $\boldsymbol{y}(\psi) \in L_2(t_0, t_f; \mathbb{R}^p)$ holds, where $(\boldsymbol{y}(\psi))(t) = y(t; \psi, u)$.

Let ψ^* be the true parameter. The output data of $\Sigma(\theta^*)$ is nothing but $\boldsymbol{y}(\psi^*)$. Then, if $\psi = \psi^*$ holds,

$$J(\psi) := \|\boldsymbol{y}(\psi) - \boldsymbol{y}(\psi^*)\|_2^2 = 0 \tag{1}$$

holds. Therefore, we consider estimating ψ^* by finding ψ so as to minimize $J(\psi)$. However, ψ satisfying (1) may not be unique, and (1) is only a necessary condition for $\psi = \psi^*$. The uniqueness of ψ is called identifiability. Practically, local identifiability is more important than global identifiability. Local identifiability is defined as follows:

Definition 1 (Local identifiability): If there exists $r > 0$ such that the following condition holds:

$$\forall \psi \in \{\psi \in \mathbb{R}^{n+q} \mid \|\psi - \psi^*\|_2 < r\} \quad J(\psi) = 0 \Rightarrow \psi = \psi^*,$$

then ψ^* is locally identifiable (LI).
If ψ^* is LI, then by minimizing $J(\psi)$, $\psi = \psi^*$ is expected. However, local identifiability does not guarantee that the minimization is easy nor that the estimation is accurate. It follows that another condition that is more suitable for the minimization is desired.

To this aim, we consider a notion of sensitivity identifiability [6].

Definition 2 (Sensitivity identifiability [6]): Let $\psi^* \in \mathbb{R}^{n+q}$ and $u \in L_2(t_0, t_f; \mathbb{R}^m)$ be given. Let

$$\left(\frac{\partial \boldsymbol{y}(\psi)}{\partial \psi_i}\right)(t) = \frac{\partial y(t; \psi, u)}{\partial \psi_i} \quad (i \in \{1, \cdots, n+q\}) \tag{2}$$

be given for all $t \in [t_0, t_f]$. If

$$\left\{\frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_1}, \cdots, \frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_{n+q}}\right\}$$

is linearly independent, ψ^* is sensitivity identifiable (SI).
Definition 2 assumes that the derivative in the RHS of (2) exists. This is the case for $\Sigma(\theta)$, since $\eta(t)$ is of class C^2 with respect to ψ for any $t \in [t_0, t_f]$, where

$$(\eta(t))(\psi) = y(t; \psi, u).$$

Since $(\boldsymbol{y}(\psi))(t)$ is of class C^2 with respect to ψ for any $t \in [t_0, t_f]$, the first order Taylor expansion of $\boldsymbol{y}(\psi) : \mathbb{R}^{n+q} \to L_2(t_0, t_f; \mathbb{R}^n)$ at ψ^* is possible. Then, $\boldsymbol{y}(\psi)$ is expanded as

$$\boldsymbol{y}(\psi) = \boldsymbol{y}(\psi^*) + \frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi}(\psi - \psi^*) + o(\|\psi - \psi^*\|).$$

Lemma 1: $J(\psi)$ can be written by

$$\|\boldsymbol{y}(\psi) - \boldsymbol{y}(\psi^*)\|_2^2 = (\psi - \psi^*)^T H (\psi - \psi^*) + o(\|\psi - \psi^*\|^2),$$

where H is the Hessian of $\|\boldsymbol{y}(\psi) - \boldsymbol{y}(\psi^*)\|_2^2$ at $\psi = \psi^*$ and expressed as

$$H = \left\langle \frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi}, \frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi} \right\rangle,$$

where

$$\left(\frac{\partial \boldsymbol{y}(\psi)}{\partial \psi}\right)(t) = \left[\left(\frac{\partial \boldsymbol{y}(\psi)}{\partial \psi_1}\right)(t) \quad \cdots \quad \left(\frac{\partial \boldsymbol{y}(\psi)}{\partial \psi_{n+q}}\right)(t) \right]. \tag{3}$$

Then, H is positive definite, iff ψ^* is SI.
Suppose that ψ^* is SI. Lemma 1 implies that H is positive definite, and so is $J(\psi)$. This means ψ^* is LI. Moreover, Newton's method for the minimization accomplishes quadratic convergence [15]. On the other hand, if ψ^* is non-SI, H is not positive definite and the quadratic convergence is not guaranteed. In some cases, optimization algorithms may face computational difficulties.

In this paper, we consider the following problem:

Problem 1: Find the condition of $\Sigma(\theta^*)$ so that ψ^* is non-SI for any x_0^* and u, i.e.,

$$\forall x_0^* \in \mathbb{R}^n \ \forall u \in L_2(t_0, t_f; \mathbb{R}^m) \quad \mathrm{rank}(H) < n + q. \tag{4}$$

Problem 1 is a generalization of unobservability analysis.

If ψ^* is non-SI for a given x_0^* and u, there may exist x_0^* and u such that ψ^* is SI. On the other hand, if (4) holds, we cannot make ψ^* SI by changing x_0^* or u. In this case, we cannot estimate ψ^* in the sense of SI, unless the structure of $\Sigma(\theta)$ is changed. This difficulty can be detected a priori if we know the condition so that (4) holds.

Similar problems have been addressed for the case of $x_0^* = 0$. For example, in [7], Markov parameters are used for the globally identifiable condition. In [8], derivatives of Markov parameters are used for the locally identifiable condition. However, these results are given for only the case of $x_0^* = 0$. Moreover, no analysis has been done in the sense of sensitivity identifiability.

Note that Problem 1 is defined for a given θ^*. Since θ^* is the true parameter, it is not known a priori in general. However, if we are interested in conditions so that the parameters are structurally non-SI, a set of θ^* may be dealt with. In those cases, Problem 1 plays a role of a fundamental problem to analyze structural characteristics.

III. Main Results

The Jacobian matrix-valued function $\frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi}$ in (3) can be written more explicitly. The following lemma is a result of Theorem 3 of [16] specialized for $\Sigma(\theta)$.

Lemma 2: For any given $\psi \in \mathbb{R}^{n+q}$ and $u \in L_2(t_0, t_f; \mathbb{R}^m)$, the following equation holds:

$$\begin{aligned} \left(\frac{\partial \boldsymbol{y}(\psi)}{\partial \psi}\right)(t) &= \begin{bmatrix} (f(t) \otimes I_p) & (g(t) \otimes I_p) \end{bmatrix} M(x_0, \theta), \\ M(x_0, \theta) &= \begin{bmatrix} M_o(\theta) & M_p(x_0, \theta) \\ 0 & M_u(\theta) \end{bmatrix} \in \mathbb{R}^{(2np+2npm)\times(n+q)}, \end{aligned} \tag{5}$$

where $f(t)$ and $g(t)$ are given as follows:

$$\begin{aligned} f(t) &= \begin{bmatrix} f_1(t) & \cdots & f_{2n}(t) \end{bmatrix} = \beta(t)\Lambda^{-1}, \\ \beta(t) &= \begin{bmatrix} e^{\lambda_1 t} \cdots t^{2n_1-1}e^{\lambda_1 t} \cdots e^{\lambda_r t} \cdots t^{2n_r-1}e^{\lambda_r t} \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} \bar{\lambda}_1 & \frac{d}{d\lambda_1}\bar{\lambda}_1 & \cdots & \frac{d^{2n_1-1}}{d\lambda_1^{2n_1-1}}\bar{\lambda}_1 & \cdots & \frac{d^{2n_r-1}}{d\lambda_r^{2n_r-1}}\bar{\lambda}_r \end{bmatrix}, \\ \bar{\lambda}_i &= \begin{bmatrix} 1 & \lambda_i & \cdots & \lambda_i^{2n-1} \end{bmatrix}^T, \\ g(t) &= \begin{bmatrix} g_1(t) & \cdots & g_m(t) \end{bmatrix}, \\ g_i(t) &= \begin{bmatrix} \int_{t_0}^t f_1(t-\tau)u_i(\tau)d\tau & \cdots & \int_{t_0}^t f_{2n}(t-\tau)u_i(\tau)d\tau \end{bmatrix}. \end{aligned}$$

The number of distinct eigenvalues of $A(\theta)$ is r. The i-th eigenvalue of $A(\theta)$ is λ_i, and n_i is the multiplicity of λ_i. Then, the set of functions $\{f_1, \cdots, f_{2n}\}$ is linearly independent. Matrices $M_o(\theta)$, $M_p(x_0, \theta)$, and $M_u(\theta)$ are defined by

$$\begin{aligned} M_o(\theta) &= \begin{bmatrix} C(\theta) \\ \vdots \\ C(\theta)A(\theta)^{2n-1} \end{bmatrix}, \\ M_p(x_0, \theta) &= \begin{bmatrix} \hat{C}(x_0) \\ \hat{C}(A(\theta)x_0) + C(\theta)\hat{A}(x_0) \\ \vdots \\ \begin{pmatrix} \hat{C}(A(\theta)^{2n-1}x_0) \\ +C(\theta)\left(\sum_{\ell=1}^{2n-1} A(\theta)^{\ell-1}\hat{A}(A(\theta)^{2n-1-\ell}x_0)\right) \end{pmatrix} \end{bmatrix}, \\ \hat{A}(z) &= \begin{bmatrix} A_1 z & \cdots & A_q z \end{bmatrix} \; (z \in \mathbb{R}^n), \\ \hat{C}(z) &= \begin{bmatrix} C_1 z & \cdots & C_q z \end{bmatrix} \; (z \in \mathbb{R}^n), \\ M_u(\theta) &= \begin{bmatrix} M_{u_1}(\theta) \\ \vdots \\ M_{u_m}(\theta) \end{bmatrix}, \\ M_{u_i}(\theta) &= \begin{bmatrix} \hat{C}(b_i(\theta)) + C(\theta)\tilde{B}_{i,:} \\ \hat{C}(A(\theta)b_i(\theta)) + C(\theta)A(\theta)\tilde{B}_{i,:} + C(\theta)\hat{A}(b_i(\theta)) \\ \begin{pmatrix} \hat{C}(A(\theta)^2 b_i(\theta)) + C(\theta)A(\theta)^2\tilde{B}_{i,:} \\ +C(\theta)\left(A(\theta)\hat{A}(b_i(\theta)) + \hat{A}(A(\theta)b_i(\theta))\right) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \hat{C}(A(\theta)^{n-1}b_i(\theta)) + C(\theta)A(\theta)^{2n-1}\tilde{B}_{i,:} \\ +C(\theta)\left(\sum_{\ell=1}^{2n-1} A(\theta)^{\ell-1}\hat{A}(A(\theta)^{2n-1-\ell}b_i(\theta))\right) \end{pmatrix} \end{bmatrix}, \\ \tilde{B}_{i,:} &= \begin{bmatrix} b_{i,1} & \cdots & b_{i,q} \end{bmatrix} \; (i \in \{1, \cdots, m\}). \end{aligned}$$

Equation (5) implies $\left(\frac{\partial \boldsymbol{y}(\psi^*)}{\partial x_0}\right)(t) = (f(t) \otimes I_p)M_o(\theta^*)$, where $M_o(\theta^*)$ is the observability matrix of $\Sigma(\theta^*)$. $M_o(\theta^*)$ does not have full column rank, iff $\Sigma(\theta^*)$ is unobservable. If $M_o(\theta^*)$ does not have full column rank, there exists a non-zero vector $v \in \mathbb{R}^n \setminus \{0\}$ such that $M_o(\theta^*)v = 0$ holds. Then, for all $t \in [t_0, t_f]$,

$$\begin{bmatrix} \left(\frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_1}\right)(t) & \cdots & \left(\frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_{n+q}}\right)(t) \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} = 0.$$

This means $\left\{\frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_1}, \cdots, \frac{\partial \boldsymbol{y}(\psi^*)}{\partial \psi_{n+q}}\right\}$ is not linearly independent. It follows that the unobservability of $\Sigma(\theta^*)$ is a sufficient condition for (4).

The following theorem gives a sufficient condition for (4) and generalizes the unobservability:

Theorem 1: Let $\theta^* \in \mathbb{R}^q$ be given. If there exists $W \in \mathbb{R}^{n\times n} \setminus \{0\}$ or $\phi \in \mathbb{R}^q \setminus \{0\}$ such that

$$\begin{aligned} &-WB(\theta^*) = B(\phi) - B_0, \; C(\theta^*)W = C(\phi) - C_0, \\ &A(\theta^*)W - WA(\theta^*) = A(\phi) - A_0 \end{aligned} \tag{6}$$

holds, then (4) holds.

Theorem 1 is a sufficient condition so that $M(x_0^*, \theta^*)$ is column rank-deficient. If there exist W and ϕ in Theorem 1, the null space of $M(x_0^*, \theta^*)$ is spanned by $[(-Wx_0)^T \; \phi^T]^T$. In addition, Theorem 1 is a linear equation in terms of W and ϕ.

Corollary 1: Equation (6) is equivalent to

$$Z\begin{bmatrix} \mathrm{vec}(W) \\ \phi \end{bmatrix} = 0, \tag{7}$$

where

$$\begin{aligned} Z = &\begin{bmatrix} -(B^T(\theta^*) \otimes I_n) & -\mathrm{vec}(\tilde{B}_{:,1}) & \cdots & -\mathrm{vec}(\tilde{B}_{:,q}) \\ (I_n \otimes C(\theta^*)) & -\mathrm{vec}(C_1) & \cdots & -\mathrm{vec}(C_q) \\ \begin{pmatrix} (I_n \otimes A(\theta^*)) \\ -(A(\theta^*)^T \otimes I_n) \end{pmatrix} & -\mathrm{vec}(A_1) & \cdots & -\mathrm{vec}(A_q) \end{bmatrix} \\ &\in \mathbb{R}^{(nm+np+nn)\times(nn+q)}, \end{aligned}$$

and $\tilde{B}_{:,i} = [b_{1,i} \; \cdots \; b_{m,i}] \in \mathbb{R}^{n\times m}$.

Owing to Corollary 1, we can easily check the existence of W and ϕ where $\Sigma(\theta^*)$ satisfies the condition of Theorem 1. Note that (7) has the trivial solution $(W, \phi) = (0, 0)$, while Theorem 1 demands the search for the solution with $W \neq 0$ or $\phi \neq 0$. Moreover, Corollary 1 leads to the following corollary:

Corollary 2: Let $\theta^* \in \mathbb{R}^q$ be given. If

$$q > n(p+m), \tag{8}$$

then (4) holds.

Corollary 2 gives a sufficient condition for the condition given by Theorem 1. This condition gives an upper bound $n(p+m)$ for the number of parameters that can be SI. In other words, if (8) holds, (4) holds for any $\theta^* \in \mathbb{R}^q$.

Theorem 1 gives us a foundation to find more detailed non-SI conditions for applications. For example, there exist many results that connect network structures of systems to observability [12]–[14]. Similarly, we can expect to obtain network-based structural conditions based on Theorem 1.

Note that Theorem 1 is not sufficient for the unobservability of $\Sigma(\theta^*)$. We will show later an example that is observable and satisfies the condition of Theorem 1.

IV. Numerical Example

Consider $\Sigma(\theta)$ whose $A(\theta)$ and θ are given by a graph Laplacian of an undirected graph [17] and the edge weights of the graph, respectively. For example, if the graph is represented by Fig. 1, the state matrix is given by

$$A(\theta) = -\begin{bmatrix} \theta_1 + \theta_2 & -\theta_1 & -\theta_2 \\ -\theta_1 & \theta_1 + \theta_3 & -\theta_3 \\ -\theta_2 & -\theta_3 & \theta_2 + \theta_3 \end{bmatrix}.$$

Let B and C of $\Sigma(\theta)$ be given by $(0, 1)$ matrices indicated by input and output vertices. If the graph is represented by Fig. 1, and v_1 and v_3 are input and output vertices, respectively,

$$B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Some identifiability analyses have been conducted with this system [10], [11]. Since $A(\theta), B, C$ in this system are affine with respect to θ, we use it as an example system class to show how the condition of Theorem 1 works.

The first example is a system that satisfies the condition of Corollary 2. Let $\Sigma_1(\theta)$ be given by the graph shown in Fig. 2. Since n,p,m, and q of $\Sigma_1(\theta)$ satisfy the condition of Corollary 2, $\Sigma_1(\theta)$ satisfies the condition of Theorem 1 for any $\theta^* \in \mathbb{R}^{13}$. Let

$$\theta^* = [1\ 3\ 1\ 3\ 2\ 2\ 3\ 2\ 1\ 1\ 1\ 3\ 2]^T.$$

Then,

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 546 & 12537 & -4472 & -3222 & -5389 & 0 \\ 1921 & -3691 & 1401 & -2345 & 2714 & 0 \\ 413 & -4861 & 2283 & 1870 & 295 & 0 \\ -2257 & -5004 & 804 & 3061 & 3396 & 0 \\ -623 & 1019 & -16 & 636 & -1016 & 0 \end{bmatrix},$$

$$\phi = \begin{bmatrix} -340 & 1966 & -6479 & 0 & 1841 & -1060 \\ & 424 & -1796 & 3619 & 1825 & 0 & 0 & 0 \end{bmatrix}^T$$

are an answer of (6). Note that, in this example, $\Sigma_1(\theta^*)$ is observable and controllable. Therefore, the condition of Theorem 1 is neither sufficient for nor equivalent to the unobservability. Fig. 4 is $\boldsymbol{y}(\psi^*)$ of $\Sigma_1(\theta^*)$ and $\boldsymbol{y}(\tilde{\psi})$ of $\Sigma_1(\tilde{\theta})$, where

$$\begin{aligned} &\tilde{\theta} = \theta^* + \frac{1}{10000}\phi,\ \tilde{x}_0 = x_0^* - \frac{1}{10000}Wx_0^*, \\ &x_0^* = \begin{bmatrix} 6 & -9 & 3 & 5 & 0 & -5 \end{bmatrix}^T, \\ &u = 2\sin(t) + 5\sin(12t + 1). \end{aligned} \tag{9}$$

Although $\tilde{\psi}$ is not the true parameter, we can see that $\boldsymbol{y}(\tilde{\psi})$ is almost the same as $\boldsymbol{y}(\psi^*)$. Since $\Sigma_1(\theta^*)$ satisfies the condition of Theorem 1, (4) holds. Then, ψ^* is non-SI with different x_0^* and u. Fig. 5 is $\boldsymbol{y}(\psi^*)$ and $\boldsymbol{y}(\tilde{\psi})$, where x_0^* and u are given by

$$\begin{aligned} &x_0^* = \begin{bmatrix} -5 & 1 & 2 & -9 & -2 & -5 \end{bmatrix}^T, \\ &u = \sin(3t + 2) + 10\sin(20t). \end{aligned}$$

We can also see that $\boldsymbol{y}(\tilde{\psi})$ is almost the same as $\boldsymbol{y}(\psi^*)$ in Fig. 5. To make ψ^* SI, some structure of $\Sigma_1(\theta)$ must be

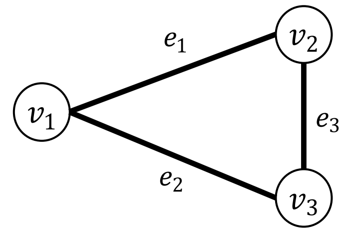

Fig. 1. Example of graph

changed. For example, let us define $\Sigma_1'(\theta)$ by replacing C in $\Sigma_1(\theta)$ with

$$C' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we can verify that ψ^* is SI for $\Sigma_1'(\theta)$ by calculating (5). Fig. 6 is $\boldsymbol{y}(\tilde{\psi})$ of $\Sigma_1'(\tilde{\theta})$, where u and $\tilde{\psi}$ are given by (9). We can see that the differences between the outputs in Fig. 6 are larger than Fig. 4, although C' is the only difference between Fig. 4 and Fig. 6.

There exists $\Sigma(\theta^*)$ satisfying the condition of Theorem 1 but not the condition of Corollary 2. Let $\Sigma_2(\theta)$ be given by the graph shown in Fig. 3. Since $n = 6$, $m = 2$, $p = 2$, and $q = 5$, $\Sigma_2(\theta)$ does not satisfy the condition of Corollary 2. However, when

$$\theta^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T,$$

$\Sigma_2(\theta^*)$ satisfies (6). In addition, $\Sigma_2(\theta^*)$ is observable and controllable. Fig. 4 is $\boldsymbol{y}(\psi^*)$ and $\boldsymbol{y}(\tilde{\psi})$, where x_0^* and u are given by

$$\begin{aligned} &\tilde{\theta} = \theta^* + \frac{1}{10}\phi,\ \tilde{x}_0 = x_0^* - \frac{1}{10}Wx_0^*, \\ &W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}, \\ &\phi = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T, \\ &x_0^* = \begin{bmatrix} -9 & 1 & 0 & -7 & -7 & -5 \end{bmatrix}^T, \\ &u = \begin{bmatrix} 2\sin(4t) & 5\sin(6t + 3) \end{bmatrix}^T. \end{aligned}$$

We can see that $\boldsymbol{y}(\tilde{\psi})$ is almost the same as $\boldsymbol{y}(\psi^*)$.

We have to be careful that the condition of Theorem 1 cannot be the necessary and sufficient condition for (4). There exists $\Sigma(\theta)$ satisfying (4) but not the condition of Theorem 1. Let $\Sigma_3(\theta)$ be given by the graph shown in Fig. 1, with $B = 0$ and $C = I_3$. When $\theta^* = [1\ 1\ 1]^T$, $\Sigma_3(\theta^*)$ satisfies (4) [18]. For example, ψ^* is non-SI at $x_0^* = [-5\ 1\ 2]^T$ because there exists $\phi = [1/6\ \ -1/7\ \ 1]^T$ such that $M_p(x_0^*, \theta^*)\phi = 0$. However, there does not exist W such that (6) holds with this ϕ. Thus, $\Sigma_3(\theta^*)$ does not satisfy the condition of Theorem 1.

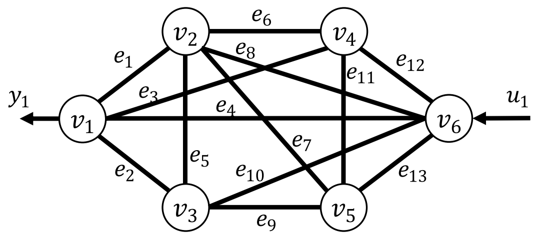

Fig. 2. Graph of a system satisfying the condition of Corollary 2

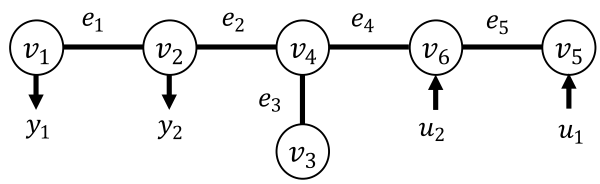

Fig. 3. Graph of a system satisfying the condition of Theorem 1

V. Conclusion

In this paper, we have analyzed the sensitivity identifiability of initial state x_0 and parameter θ of linear systems with affinely parameterized coefficient matrices. Then, we have found a sufficient condition so that $\psi^* = [(x_0^*)^T \ (\theta^*)^T]^T$ is non-SI for any x_0^* and input u. Thus, by checking whether $\Sigma(\theta^*)$ satisfies the condition, we can detect the possibility to face difficulties in parameter estimations.

We have shown that the condition is given by the algebraic equations. Since the condition is equivalent to the linear equations, we can easily check whether $\Sigma(\theta^*)$ satisfies the condition. Moreover, we can expect to obtain structural conditions based the condition.

We have demonstrated the examples that satisfy the condition and are observable and controllable. It follows that the condition is neither sufficient for nor equivalent to the unobservability.

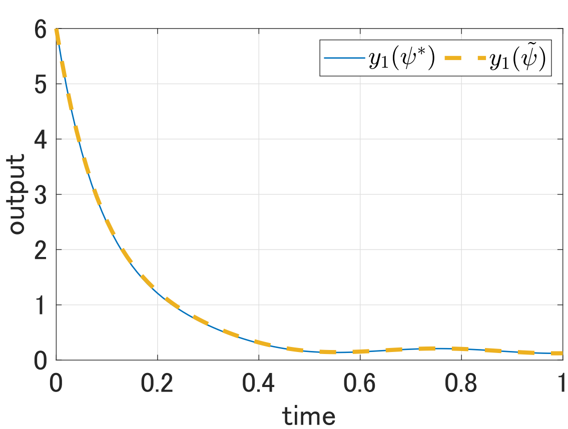

Fig. 4. Outputs of $\Sigma_1(\theta^*)$ and $\Sigma_1(\tilde{\theta})$

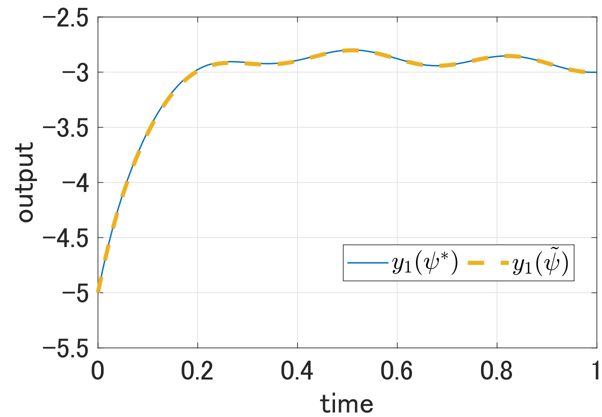

Fig. 5. Outputs of $\Sigma_1(\theta^*)$ and $\Sigma_1(\tilde{\theta})$ with different x_0^* and u

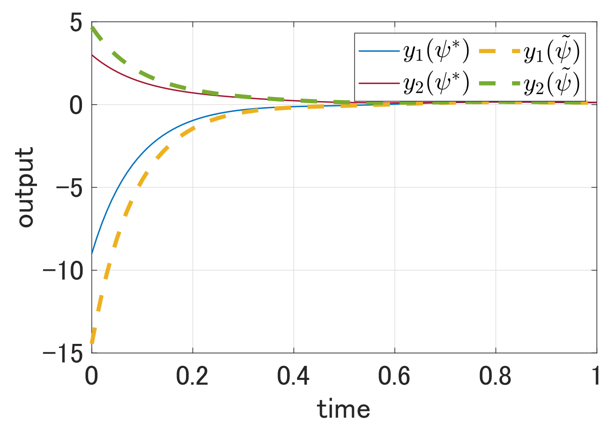

Fig. 6. Outputs of $\Sigma_1'(\theta^*)$ and $\Sigma_1'(\tilde{\theta})$

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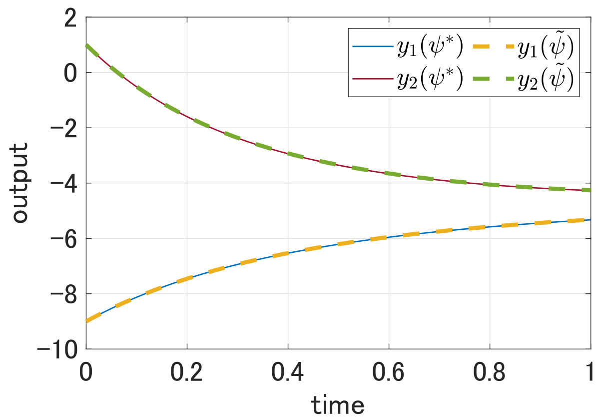

Fig. 7. Outputs of $\Sigma_2(\theta^*)$ and $\Sigma_2(\tilde{\theta})$

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