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On Small-Gain Theorem for Interconnected Finite/Fixed-Time Input-to-State Stable Systems*

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Abstract— **The paper addresses the problem of input-to-state stability analysis for interconnected systems with accelerated convergence, namely, for interconnected nonlinear systems composed of two finite-time or (nearly) fixed-time input-to-state stable subsystems. Under additional mild restrictions, small-gain theorems are proposed for a class of such systems that guarantee the preservation of accelerated convergence for the interconnection.**

I. INTRODUCTION

The problem of stability and robustness analysis with respect to external inputs is in the focus of many research works (see, e.g., [1], [2], [3], [4], [5], [6] references therein). Input-to-State Stability (ISS) concept, proposed by E. Sontag over 30 years ago, offers a comprehensive set of conditions, extends the Lyapunov function method, and provides diverse concepts applicable to various stability analysis, control and estimation problems (see, e.g., books and surveys on the ISS concept framework [6], [7], [8], [9]). One important application of the ISS concepts is in interconnected and networked systems stability analysis. Indeed, under certain conditions (small-gain theorems) the ISS property is preserved for systems connected in feedback loops (see, e.g., [10], [11], [12], [13], [14]). Such results are based on one of two approaches: estimating interconnected subsystems solutions (as, e.g., in [10]) or using the concept of ISS Lyapunov functions (e.g., [12]).

Over the past decades, systems with accelerated, i.e., finite-time and fixed-time convergence, have attracted considerable attention (see, e.g., [15], [16], [17]). This is because finite/fixed-time controllers and observers provide fast convergence and high precision as well as significant robustness properties. The trajectories of the finite-time stable systems, initiated in a neighbourhood of the origin, settle at that equilibrium after a finite-time transient, while for fixed-time dynamics their approach the zero in a bounded time uniformly on initial conditions. Extensions of the ISS concept to the systems with accelerated convergences (namely, Finite-Time Input-to-State Stability (FT-ISS), (nearly) Fixed-Time Input-to-State Stability ((n)FxT-ISS)) are presented in [18], [19], [20]. There are several results devoted to small-gain theorem for interconnected FT-ISS systems formulated in [13], [21] that use the Lyapunov function method. To the best of the authors' knowledge, there are no small-gain theorem results for FT/FxT-ISS systems based on the estimate on solutions and not on existence of Lyapunov functions (the paper [19] contains a claim that such a result should be valid for FT-ISS systems, but without a detailed proof).

In this paper we propose a (nearly) fixed-time version of small-gain theorem for interconnected nonlinear systems composed of two (n)FxT-ISS subsystems. A new small-gain theorem for FT-ISS interconnected subsystems is presented as well in this paper. It is noteworthy that to guarantee accelerated convergence rates, we need to impose additional constraints on the gains of the system together with the conventional small-gain condition $\gamma_1 \circ \gamma_2(s) < s$ for $s \neq 0$, where $\gamma_1, \gamma_2 \in \mathscr{K}_\infty$ are asymptotic gain functions of interconnected subsystems.

The paper is organized in the following way. Notation used in the work is introduced in Section II. Section III presents preliminaries used in the paper. Small-gain theorems for (n)FxT-ISS and FT-ISS interconnected subsystems are proposed in Section IV. An illustrative example is given in Section V. Finally, conclusions are given in Section VI.

II. NOTATION

Through the paper the following notation will be used:

- $\mathbb{R}$ ($\mathbb{R}_+$) is the set of real (nonnegative) numbers;
- The class of continuous functions $X \to Y$ is denoted by $\mathscr{C}(X,Y)$ for two metric spaces X and Y;
- $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$;
- $\|d\|_{[t_0,t_1)} = \operatorname{ess\,sup}_{s\in[t_0,t_1)} \|d(s)\|$ for $[t_0,t_1) \subset \mathbb{R}_+$ and $\|d\|_\infty = \operatorname{ess\,sup}_{s\geq 0} \|d(s)\|$;
- $\mathscr{L}_\infty^m$ is the set of essentially bounded measurable functions $\mathbb{R} \to \mathbb{R}^m$ with the norm $\|\cdot\|_\infty$;
- For $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$, the expression $\gamma_1 \circ \gamma_2$ represents their composition function, i.e., $\gamma_1 \circ \gamma_2(s) = \gamma_1(\gamma_2(s))$ for any $s \in \mathbb{R}_+$.

III. PRELIMINARIES

A. Comparison functions

Through the paper the following comparison functions will be used:

- A function $\sigma \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}_+)$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathscr{K}_\infty$ if it is also unbounded.

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- A function $\beta \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ belongs to class $\mathscr{KL}$ if $\beta(\cdot, r) \in \mathscr{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r > 0$.
- A function $\chi \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ belongs to class $\mathscr{FKL}$ if $\chi(\cdot, 0) \in \mathscr{K}$, $\chi(r, \cdot)$ is decreasing to zero for any fixed $r > 0$ and there is a so called settling time function $\mathbb{T} \in \mathscr{K}$ such that $\chi(r,t) = 0$ for all $t \geq \mathbb{T}(r)$. A function $\chi \in \mathscr{FKL}$ with $\sup_{r \geq 0} \mathbb{T}(r) < +\infty$ belongs to the class $\mathscr{F}x\mathscr{KL}$.
- A function $\chi \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ belongs to class $n\mathscr{F}x\mathscr{KL}$ if $\chi(\cdot, 0) \in \mathscr{K}$, $\chi(r, \cdot)$ is decreasing to zero for any fixed $r > 0$, and there is $\mathscr{T} \in \mathscr{K}$ such that $\chi(r,t) \leq \rho$ for any $\rho > 0$, for all $t \geq \mathscr{T}(\rho^{-1})$ and all $r \in \mathbb{R}_+$.

B. Stability notions

Consider the system:

$$\dot{x}(t) = f(x(t), d(t)), \quad t > 0, \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $d(t) \in \mathbb{R}^m$, $d \in \mathscr{L}_\infty^m$, $f \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ ensures forward existence and uniqueness of solutions of the system at least locally in time, $f(0,0) = 0$. For any $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$ the respective solution is denoted by $x(t, x_0, d)$ with $x(0, x_0, d) = x_0$.

Definition 1 [2] *System (1) has the asymptotic gain (AG) property, if there exists $\gamma \in \mathscr{K}_\infty$, such that for all initial states $x_0 \in \mathbb{R}^n$ and all $d \in \mathscr{L}_\infty^m$ the estimate*

$$\limsup_{t \to \infty} \|x(t, x_0, d)\| \leq \gamma(\|d\|_\infty)$$

holds.

The function γ is called AG function.

Definition 2 *The system (1) is called ISS, if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that*

$$\|x(t, x_0, d)\| \leq \max\left\{\beta(\|x_0\|, t), \gamma(\|d\|_{[0,t)})\right\} \tag{2}$$

for all $x_0 \in \mathbb{R}^n$, $d \in \mathscr{L}_\infty^m$ and $t \geq 0$. It is called FT-ISS or (n)FxT-ISS if $\beta \in \mathscr{FKL}$ or $\beta \in \mathscr{F}x\mathscr{KL}$ ($\beta \in n\mathscr{F}x\mathscr{KL}$), respectively.

The above properties are called local if they are satisfied for a restricted set of initial conditions and inputs, i.e., $\|x_0\| + \|d\|_\infty \leq \kappa$ for some $\kappa > 0$.

The results on FT-ISS, (n)FxT-ISS and its Lyapunov characterizations can be found in [18], [19], [20]. When the input d is set to zero, the condition (2) implies that the system (1) is globally finite-time or (nearly) fixed-time stable, respectively.

Remark 1 Note that finite-time stability implies fast convergence near the origin, while nearly fixed-time stability implies accelerated convergence outside the vicinity of the origin. Therefore, combining local finite-time stability with global asymptotic stability ensures global finite-time stability, and combining global finite-time stability with nearly fixed-time stability ensures global fixed-time stability. Similar conclusions can be drawn for ISS versions (the convergence rates are inherited from disturbance-free scenarios):

- ISS and local FT-ISS properties imply FT-ISS;
- FT-ISS and nFxT-ISS properties imply FxT-ISS.

Definition 3 *The set M is said to be finite-time attractive for (1) if any solution $x(t, x_0, d)$ of (1) reaches M in a finite instant of time $t \leq T_M(\|x_0\|)$ and remains there $\forall t \geq T_M(\|x_0\|)$, where $T_M \in \mathscr{K}$ is a settling-time function. The set M is called fixed-time attractive if $\sup_{x_0 \in \mathbb{R}^n} T(\|x_0\|) < +\infty$.*

C. ISS small-gain theorem

Consider an interconnected system composed of two subsystems (see the scheme in Fig. 1)

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), d_1(t)), \tag{3a}$$

$$\dot{x}_2(t) = f_2(x_2(t), x_1(t), d_2(t)), \tag{3b}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $i = 1, 2$ is the state vector, $d_i(t) \in \mathbb{R}^{m_i}$, $d_i \in \mathscr{L}_\infty^{m_i}$ represents the external disturbance input, and $f_i : \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is a continuous function satisfying $f_i(0,0,0) = 0$.

Define $x = \left(x_1^T, x_2^T\right)$, $f = \left(f_1^T, f_2^T\right)$, $d = \left(d_1^T, d_2^T\right)$ and the complete system by

$$\dot{x} = f(x, d). \tag{4}$$

Theorem 1 [10] (Small-gain theorem) *Assume each subsystem of (3) is ISS with the corresponding AG functions $\gamma_i, \gamma_{d_i} \in \mathscr{K}_\infty$, $i = 1, 2$: there exists $\beta_i \in \mathscr{KL}$ such that for all $x_i(t_0) \in \mathbb{R}^{n_i}$ and $d_i \in \mathscr{L}_\infty^{m_i}$ the following estimate holds for all $t \geq t_0 \geq 0$:*

$$\|x_i(t, x_i(t_0), d_i)\| \leq \max\left\{\beta_i(\|x_i(t_0)\|, t - t_0), \gamma_i(\|x_{3-i}\|_{[t_0,t)}), \gamma_{d_i}(\|d_i\|_{[t_0,t)})\right\}.$$

If the small-gain condition $\gamma_1 \circ \gamma_2(s) < s$ for all $s > 0$ is satisfied, then the system (4) is ISS.

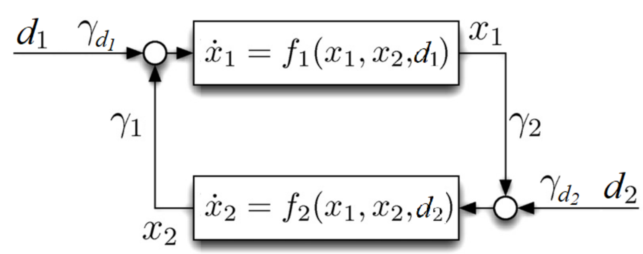

Fig. 1. Feedback interconnection of two ISS systems with AG functions γ_1, γ_2, γ_{d_1}, γ_{d_2}

IV. MAIN RESULT

The main goal of the paper is to provide small-gain theorems for FT-ISS and (n)FxT-ISS analysis of the interconnected system (3) utilizing the estimates of the form (2) and the restrictions on the AGs.

Assumption 1 *Systems (3a) and (3b) are FT-ISS (FxT-ISS, nFxT-ISS) with the corresponding AG functions $\gamma_1, \gamma_2, \gamma_{d_1}, \gamma_{d_2} \in \mathscr{K}_\infty$:*

$$\|x_i(t, x_i(t_0), d_i)\| \leq \max\left\{\beta_i(\|x_i(t_0)\|, t - t_0), \gamma_i(\|x_{3-i}\|_{[t_0,t)}), \gamma_{d_i}(\|d_i\|_{[t_0,t)})\right\}, \tag{5}$$

for $i = 1,2$ and some $\beta_i \in \mathscr{FKL}$ ($\beta_i \in \mathscr{F}x\mathscr{KL}$, $\beta_i \in n\mathscr{F}x\mathscr{KL}$), for all $x_i(t_0) \in \mathbb{R}^{n_i}$ and $d_i \in \mathscr{L}_\infty^{m_i}$ and $t \geq t_0 \geq 0$.

Assuming that each of the subsystems in (3) to be (n)FxT-ISS (FT-ISS), we are interested in conditions guaranteeing that the system (4) is (n)FxT-ISS (FT-ISS).

First, let us consider the case $\beta_i \in \mathscr{FKL}$. For $\beta_i \in \mathscr{FKL}$, denote by $\mathbb{T}_i \in \mathscr{K}$ the settling time function satisfying $\beta_i(s, \mathbb{T}_i(s)) = 0$ for any $s \in \mathbb{R}_+$. The function $\beta_i \in \mathscr{FKL}$ can be (locally) upper bounded by $\chi_i \circ \max\{0, \bar{\mathbb{T}}_i(s) - t\}$ for some $\chi_i \in \mathscr{K}_\infty : \chi_i(s) \geq s$ and $\bar{\mathbb{T}}_i \in \mathscr{K} : \mathbb{T}_i(s) \leq \bar{\mathbb{T}}_i(s) \leq 2\mathbb{T}_i(s)$, $\forall s \in \mathbb{R}_+$ (see [20, page 130] for more details), and further in this work we may assume that β_i are in this canonical form. Let us define the function $\beta \in \mathscr{FKL}$ as

$$\beta(s,t) = \max_{i=1,2}(\beta_i(s,t)) \leq \chi \circ \max\{0, \mathbb{T}(s) - t\}, \qquad (6)$$

for some $\chi \in \mathscr{K}_\infty$ (for example, $\chi(s) = \max_{i=1,2}(\chi_i(s))$) and $\mathbb{T} \in \mathscr{K} : \max_{i=1,2}(\mathbb{T}_i(s)) \leq \mathbb{T}(s) \leq 2\max_{i=1,2}(\mathbb{T}_i(s))$, $\forall s \in \mathbb{R}_+$.

Theorem 2 *Let the FT-ISS part of Assumption 1 be valid with the corresponding AG functions γ_1, γ_2 satisfying small-gain condition*

$$\gamma_i \circ \gamma_{3-i}(s) < s \quad \textit{for all } s > 0, \qquad (7a)$$

$$\gamma_i \circ \gamma_{3-i}(s) \leq \mathbb{T}^{-1} \circ \chi^{-1}(0.5s) \quad \textit{for } s \in (0, \hat{s}), \qquad (7b)$$

$$\gamma_i(2\chi(\mathbb{T}(s))) \leq s \quad \textit{for } s \in (0, \hat{s}), \qquad (7c)$$

for some $\hat{s} > 0$ and $i = 1,2$. Then the system (4) is FT-ISS.

Remark 2 Due to $\beta(s,0) = \chi \circ \mathbb{T}(s)$ the small-gain condition (7) can be rewritten in the form

$$\begin{array}{ll} \gamma_i \circ \gamma_{3-i}(s) < s & \text{for all } s > 0, \\ \beta(\gamma_i \circ \gamma_{3-i}(s), 0) \leq 0.5s & \text{for } s \in (0, \hat{s}), \\ \gamma_i(2\beta(s,0)) \leq s & \text{for } s \in (0, \hat{s}). \end{array} \qquad (8)$$

Hence, (7a) (or the first inequality in (8)) is the standard small-gain condition, while two other requirements, (7b) and (7c), are needed to guarantee the preservation by the interconnection (4) of the convergence rate originated in the subsystems (3). As we can conclude from (8), the condition (7b) requires that the common gain $\gamma_i \circ \gamma_{3-i}$ should be not simply smaller than the identity function, but ensures a sufficient decay on the trajectories of the system. The condition (7c), looking on its counterpart in (8), can be interpreted as the requirement for each gain γ_i to be sufficiently small, in addition to its combination to be small as in (7a). In this way, (7c) may implicitly imply (7a).

Remark 3 Note that in [19] the conditions (7b) and (7c) are not introduced, and it is claimed that under (7a) the finite-time convergence rates can be preserved for (4) having their analogues in (3). The proof in [19] just states that the result follows the conventional small-gain analysis arguments. Therefore, further investigation of the necessity of the additional restrictions introduced in Theorem 2 is required.

FT-ISS (FxT-ISS) systems provide fast convergence in the vicinity (and away of the vicinity as well) of the origin. In the presence of a high level of disturbances and sufficiently big AG functions, the rate of convergence in the vicinity of the origin may not be so important. In this sense, the nFxT-ISS property that provides fast convergence out of the vicinity of the origin (e.g., fixed-time attractiveness of a compact set) may be of most interest. As the following result shows, if subsystems of an interconnected system are nFxT-ISS, one can obtain the same property for (4) under small-gain conditions similar to (8).

Let the nFxT-ISS part of Assumption 1 hold, i.e., $\beta_i(s,t) \in n\mathscr{F}x\mathscr{KL}$. Define $\beta(s,t) = \max_{i=1,2}(\beta_i(s,t)) \in n\mathscr{F}x\mathscr{KL}$. According to [20] $\beta \in n\mathscr{F}x\mathscr{KL}$ can be bounded as follows

$$\beta(s,t) \leq \frac{\chi(s)}{1 + \mathscr{T}^{-1}(t)\chi(s)} \qquad (9)$$

for some $\chi \in \mathscr{K}_\infty : \chi(s) \geq s$, $\mathscr{T} \in \mathscr{K}_\infty$ and the argument of $s \in \mathbb{R}_+$ sufficiently big and $\beta(s,0) \leq \chi(s)$.

Theorem 3 *Let nFxT-ISS part of Assumption 1 hold with the corresponding AG functions γ_1, γ_2 satisfying small-gain condition*

$$\gamma_i \circ \gamma_{3-i}(s) < s \quad \textit{for all } s > 0, \qquad (10a)$$

$$\gamma_i \circ \gamma_{3-i}(2s) \leq \chi^{-1}(s) \quad \textit{for } s \in (\bar{s}, +\infty), \qquad (10b)$$

$$\gamma_i(2s) \leq \chi^{-1}(s) \quad \textit{for } s \in (\bar{s}, +\infty), \qquad (10c)$$

for some $\bar{s} > 0$, $i = 1,2$. Then the system (4) is nFxT-ISS.

Remark 4 Due to $\beta(s,0) = \chi(s)$ the small-gain condition (10) can be rewritten in the form

$$\begin{array}{ll} \gamma_i \circ \gamma_{3-i}(s) < s & \text{for all } s > 0, \\ \beta(\gamma_i \circ \gamma_{3-i}(2s), 0) \leq s & \text{for } s \in (\bar{s}, +\infty), \\ \beta(\gamma_i(2s), 0) \leq s & \text{for } s \in (\bar{s}, +\infty), \end{array}$$

which is close to (8).

Consider the case, where the FxT-ISS part of Assumption 1 is valid, i.e., $\beta_i \in \mathscr{F}x\mathscr{KL}$, $i = 1,2$. Define $\beta(s,t) = \max_{i=1,2}(\beta_i(s,t))$. According to (6), (9) the fuction β can be represented as

$$\beta(s,t) \leq \chi_{FT} \circ \max\{0, \mathbb{T}(s) - t\}$$

for $s \in (0, \hat{s})$ and some $\hat{s} > 0$, $\chi_{FT} \in \mathscr{K}_\infty : \chi_{FT}(s) \geq s$, and

$$\beta(s,t) \leq \frac{\chi_{FxT}(s)}{1 + \mathscr{T}^{-1}(t)\chi_{FxT}(s)}$$

for $s \in (\bar{s}, +\infty)$ and some $\bar{s} > 0$, $\chi_{FxT} \in \mathscr{K}_\infty : \chi_{FxT}(s) \geq s$. Since FT-ISS and nFxT-ISS jointly provide the FxT-ISS property, the following result can be obtained:

Theorem 4 *Let (3a) and (3b) are FxT-ISS with the corresponding AG functions γ_1, γ_2 satisfying small-gain conditions (7) with $\chi(\cdot) = \chi_{FT}(\cdot)$ and (10) with $\chi(\cdot) = \chi_{FxT}(\cdot)$.* Then the system (4) is FxT-ISS.

V. EXAMPLE

For a brief illustration, consider the following interconnected system

$$\dot{x}_1 = -|x_1|^{1/2}\text{sign}(x_1) - 3x_1^2\text{sign}(x_1) + \frac{1 + x_1}{2}x_2 + d_1^2, \quad (11a)$$

$$\dot{x}_2 = -2|x_2|^{1/3}\text{sign}(x_2) - 2x_2^3 + \frac{1}{2}x_1 + d_2, \quad (11b)$$

where $x_1, x_2, d_1, d_2 \in \mathbb{R}$.

Take $V_1(x_1) = x_1^2$ as a Lyapunov function candidate for the subsystem (11a). For $|x_1| \geq \frac{1}{2}|x_2|$ and $|x_1| \geq |d_1|$ we have

$$\begin{aligned} \dot{V}_1 &= -2|x_1|^{3/2} - 6|x_1|^3 + x_1(1+x_1)x_2 + 2x_1 d_1^2 \\ &\leq -0.9|x_1|^{3/2} - |x_1|^3 \\ &= -0.9V_1^{3/4} - V_1^{3/2}, \end{aligned}$$

i.e., for $x_2 \equiv 0$, $d_1 \equiv 0$ the subsystem (11a) is fixed-time stable with $\chi_{FT_1}(s) = \frac{0.9^2}{16}s^2$, $\mathbb{T}_1(s) = \frac{4s^{0.5}}{0.9}$ and $\chi_{FxT_1}(s) = s$. Similarly, for the subsystem (11b), $|x_2| \geq \frac{1}{2}|x_1|$, $|x_2| \geq |d_2|$ and $V_2 = x_2^2$ one can obtain

$$\begin{aligned} \dot{V}_2 &= -4|x_2|^{4/3} - 4x_2^4 + x_2 x_1 + 2x_2 d_2 \\ &\leq -|x_2|^{4/3} - x_2^4 \\ &= -V_2^{2/3} - V_2^2 \end{aligned}$$

and $\chi_{FT_2}(s) = \left(\frac{s}{3}\right)^{1.5}$, $\mathbb{T}_2(s) = 3s^{2/3}$, $\chi_{FxT_2}(s) = s$. Thus, the FxT-ISS part of Assumption 1 is valid with $\gamma_{d_1}(s) = \gamma_{d_2}(s) = s$, $\gamma_1(s) = \gamma_2(s) = \frac{1}{2}s$, and the condition $\gamma_1 \circ \gamma_2(s) < s$ for all $s > 0$ is satisfied. For $\beta_i(s,t) = \chi_{FT_i} \circ \max\{0, \mathbb{T}_i - t\}$ we have $\beta(s,t) = \max_{i=1,2}(\chi_{FT_i} \circ \max\{0, \mathbb{T}_i - t\})$ and $\beta(s,0) = s$, and according to Remark 2 the condition (7) is satisfied. Finally, for $\chi_{FxT}(s) = \chi_{FxT_1}(s) = \chi_{FxT_2}(s) = s$ the condition (10) is also satisfied, and the system (11) is FxT-ISS by Theorem 4.

Fig. 2, Fig. 3 present the simulation results for different initial conditions for the case $d_1 \equiv 0$, $d_2 \equiv 0$ that confirm fixed-time stability of the system (11). Fig. 4 presents the simulation results for the case $d_1 = \sin t$, $d_2 = 2\cos t$. The simulation results in Fig. 3, Fig. 4 are shown with using the logarithmic scale in order to demonstrate fast convergence rate.

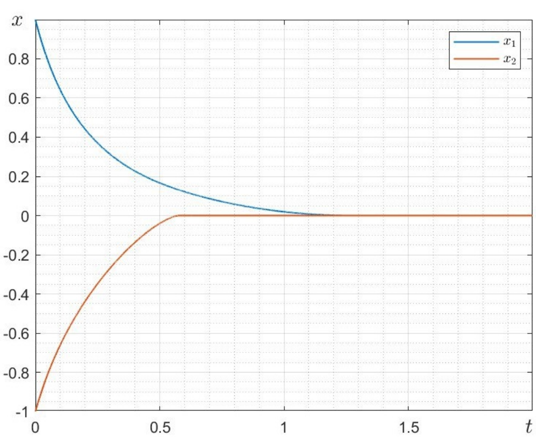

Fig. 2. States of the system (11) for the disturbance-free case and $x_0 = [1, -1]^T$

VI. CONCLUSIONS

In the paper small-gain theorems are presented for the interconnected system (3) composed of FT-ISS or (n)FxT-ISS subsystems. It is shown that to guarantee accelerated convergence rates, additional constraints on AG functions of the interconnected system, in line with the conventional small-gain condition, should be satisfied.

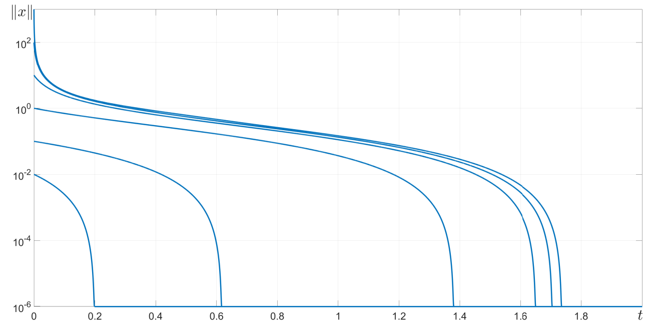

Fig. 3. State norm of the system (11) for the disturbance-free case

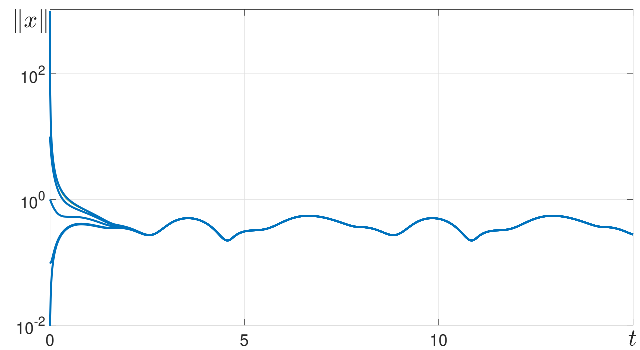

Fig. 4. State norm of the system (11) for the disturbed case

Possible directions for future research include relaxation of the small-gain condition (7), extension for network of FT-ISS/(n)FxT-ISS systems (considering also mixed systems with different kinds of convergences), Lypunov-based small-gain theorems design and extension of the results for hyper-exponential rates of convergence. Application of the obtained conditions for controller/observer designs is also of great interest.

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