2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

On delay-independent stability and guaranteed convergence rate for linear time-varying continuous-time delay systems

Vittorio De Iuliis, Costanzo Manes

Abstract— **This work deals with linear continuous-time systems with delays, in the general case of time-varying matrices. We first consider the special case of positive systems of such class, introducing two different conditions of delay-independent global exponential stability formulated by means of linear inequalities. Moreover, guaranteed bounds on the exponential convergence rate are given as a function of the largest admissible delay. Then, employing a state-bounding approach built on the properties of positive systems, we extend the analysis to systems with no sign constraints. Due to the time-varying nature of the systems, all such conditions involve infinite-many tests. Hence, we discuss the significant special case of switching systems with delays, for which the conditions can be finitely tested.**

Index Terms— **Time-delay systems, Time-varying systems, Positive systems, Linear Systems.**

I. Introduction

Describing reality, in all its nuances, has always been one of the main goals of applied sciences. In many engineering applications, such as the ones investigated in systems and control, accurate modeling of complex processes cannot be achieved neglecting the inevitable time-delays arising due to transport phenomena, data communication, and heterogeneous kinds of physical constraints. Accurate modeling usually comes at a price: analyzing complex models is a difficult task. This is especially true when time-varying models are investigated, as is the case of this work, which aims at analyzing the stability properties of a very general class of continuous-time linear models with time-varying matrices and multiple time-varying delays. Most of the well-established results of the literature about time-delay systems, in fact, deals with the analysis and control of time-invariant delay systems, i.e. systems with both constant matrices and constant delays (see, e.g. [1], [2]). Even in this apparently simple special case, it is well-known that analysis and control problems are far from being trivially solvable, and huge complexities arise when time-varying behaviors are investigated. Analyzing the stability of delay systems of time-varying nature is a very difficult problem, which in most cases asks for conservative approaches based on popular extensions of the well-known classical Lyapunov theory, established by Krasovskii and Razumikhin (see, e.g., the recent [3], [4], [5] and references therein); other recent approaches exploit trajectory-based methodologies or Halanay-like inequalities (see, e.g. [6], [7] and related works), or seek for a continuous-time counterpart of the popular discrete-time approach based on lifted switched representations (see, e.g. [8], [9], [10], [11]).

Quite remarkably, analyzing stability of time-delay systems can be much simpler for positive systems [12], [13], i.e. systems whose dynamics can only be non-negative at all times provided that non-negative inputs and initial conditions are given. For linear positive delay systems with constant matrices, even in the usually difficult case of time-varying delays, analyzing stability is trivial: straightforward necessary and sufficient conditions exist on the system matrices (see, e.g., [14]). Moreover, for various classes of linear positive delay systems with constant matrices a remarkable property holds: the stability is always delay-independent, meaning that if the stability holds for a given value of the delays, it holds for all possible values of the delays, even time-varying (see, e.g., the notable works [15], [16], [17], [18], [19], [20], [21]). Hence, it is natural to investigate how positivity impacts on the stability analysis of delay systems with time-varying matrices.

This is indeed the first contribution of this work: we derive two different conditions of delay-independent global exponential stability for continuous-time positive delay systems with time-varying matrices and time-varying delays. The conditions, albeit requiring infinite-many tests at least in the most general case where no periodicity arises, are remarkably simple, being formulated via linear inequalities. Moreover, for both such stability conditions we also illustrate how a guaranteed exponential bound can be derived as a function of the maximum allowable delay on the system. The problem of finding a guaranteed bound on the exponential convergence rate for positive delay systems is a well-studied problem in the literature (see, e.g., the recent [22], [23] and references therein), yet in this work we address it in the general case of system with time-varying matrices.

The second main contribution of the paper deals with continuous-time general linear systems with time-varying matrices and delays, dropping the positivity assumption. We show that one of the aforementioned delay-independent conditions can be applied to this general case, yielding sufficient conditions of delay-independent global exponential stability. Similarly to the positive case, also for general systems we provide a guaranteed bound on the exponential convergence rate as a function of the largest admissible delay. The key tool exploited to apply results from positive delay systems

The research leading to these results was supported by the Centre of EXcellence on Connected, Geo-Localized and Cybersecure Vehicles (EX-Emerge), funded by the Italian Government under Grant CIPE 70/2017 (7 August 2017).

The authors are with Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli studi dell'Aquila, Via Vetoio, 67100 Coppito (AQ), Italy (corresponding author's e-mail: vittorio.deiuliis@univaq.it)

also to general systems is the usage of so-called comparison systems (see, e.g. [22], [24], [25]), here realized by means of the Internally Positive Representation methodology, see e.g. [26], [27], [28], [29], [30], [31] for recent applications.

Finally, we describe a significant special case in which the aforementioned results become largely more treatable (indeed, they provide finite tests): linear switching systems with time-varying delays.

The work is structured as follows: the Introduction is closed by a brief overview of the Notation used throughout the work. Section II introduces the class of systems investigated in this paper and their stability analysis, both in the positive case and in the general case. Section III illustrates the special case of switching systems with time-varying delays. Conclusions follow.

Notation: The following notation is used throughout the work. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively denote the set of non-negative and positive real numbers. $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) is the non-negative (positive) orthant of $\mathbb{R}^n$. $\mathbb{R}^{m\times n}_+$ is the cone of non-negative $m \times n$ matrices. $\mathbb{1}_n$ is the column vector consisting of n ones, I_n is the $n \times n$ identity matrix and $0_{m\times n}$ is the $m\times n$ matrix consisting of all zeros. Inequalities among vectors and matrices of the same dimensions are meant to be understood component-wise, i.e. $M \leq N$ ($M < N$) if $[M]_{i,j} \leq [N]_{i,j}$ ($[M]_{i,j} < [N]_{i,j}$) for all i, j, where $[M]_{i,j}$ denotes the i, j entry of matrix M. In this sense, a non-negative matrix $M \in \mathbb{R}^{m\times n}_+$ can also be denoted by $M \geq 0_{m\times n}$, or briefly $M \geq 0$. We will also denote $|M|$ for the component-wise absolute value of matrix M. A square matrix M is *Metzler* if all its off-diagonal components are non-negative, i.e. $[M]_{i,j} \geq 0$ for $i \neq j$. $\bar{\lambda}(M)$ denotes the spectral abscissa of the square matrix M, i.e. $\bar{\lambda}(M) = \max_i\{\mathrm{Re}(\lambda_i)\}$, λ_i being the eigenvalues of M.

II. Continuous time systems with time-varying delays

This work analyzes the stability properties of continuous-time linear time-delay systems, with multiple time-varying delays and time-varying matrices, described by:

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{j=1}^{r} A_j(t)x(t-\delta_j(t)), \quad t \geq 0, \\ x(t) &= \phi(t), \qquad\quad t \in [-\bar{\delta}, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\delta_j(t)$ are time-varying delays taking values in $[0, \bar{\delta}]$, for $j = 1, \dots, r$, and $\bar{\delta} \in \mathbb{R}_+$. The function $\phi : [-\bar{\delta}, 0] \mapsto \mathbb{R}^n$, bounded and integrable, defines the initial conditions of system (1). All matrices $A_j(t)$, $j = 0, \dots, r$ are assumed to be integrable, so that the solution $x(t)$, $t \geq 0$, exists.

For compactness, a dummy delay $\delta_0(t) = 0$, $\forall t \geq 0$, can be defined, so that the term $A_0(t)x(t)$ in the system equation (1) can be included in the summation:

$$\dot{x}(t) = \sum_{j=0}^{r} A_j(t)x(t-\delta_j(t)), \quad t \geq 0. \tag{2}$$

This compact notation for the state equation of system (1) will be used throughout the work, even though explicit references to the system in statements may point to the equivalent form (1).

Throughout the work, we will employ the usual definitions of stability for system (1). We refer the reader to [3] for further details. In particular, we will briefly write GES to denote the global exponential stability. The terminology 'delay-independent' stability will be used to mean that the stability of (1) holds regardless of the magnitude of the bounded time-varying delays, that is, for any $\delta_j(t) \in [0, \bar{\delta}]$ and any given $\bar{\delta} \in \mathbb{R}_+$, $j = 1, \dots, r$.

The remainder of this section will provide a stability analysis for the above-defined system. We will approach the problem exploiting positivity-based results, and then generalize the analysis removing any sign constraint.

A. Positive Systems

In order to provide a delay-independent stability condition for system (1), we first provide a sufficient stability condition that holds under the positivity assumption for the system.

The usual definition of positivity applies to system (1). Namely, such system is positive if $x(t) \geq 0$ at all times provided that $\phi(t) \geq 0$ for all $t \in [-\bar{\delta}, 0]$. Trivial necessary and sufficient conditions do exist on system matrices $A_j(t)$ to ensure that the aforementioned definition is fulfilled (see, e.g., Lemma 1 in [32]).

Proposition 1: System (1) is positive if and only if $A_0(t)$ is Metzler for all $t \geq 0$ and $A_j(t)$ is non-negative for all $t \geq 0$, $j = 1, \dots, r$.

We are now in a position to state the following delay-independent stability criterion for a positive system (1).

Theorem 1: Consider system (1), where $A_0(t)$ is Metzler and $A_j(t)$ are non-negative, for $j = 1, \dots, r$. It there exist $\beta \in \mathbb{R}_{++}$ and $q \in \mathbb{R}^n_{++}$ such that

$$\Big(\sum_{j=0}^{r} A_j(t)\Big)q \leq -\beta\, q, \quad \forall t \geq 0, \tag{3}$$

then the system is delay-independent GES. Moreover, for any given maximum delay $\bar{\delta} \in \mathbb{R}_+$ the following exponential bound holds true

$$x(t) \leq \bar{q}\|\phi\|_\infty e^{-\alpha t}, \quad \forall t \geq -\bar{\delta}, \tag{4}$$

where α is any positive real satisfying

$$\big(\alpha I_n + A_0(t)\big)q \leq -e^{\alpha\bar{\delta}}\Big(\sum_{j=1}^{r} A_j(t)\Big)q, \tag{5}$$

and

$$\bar{q} = \frac{1}{q_{\min}}q, \quad \text{with} \quad q_{\min} = \min_{i\in[1,n]}(q_i). \tag{6}$$

Proof. Note first that if the inequality (3) holds true, then for any $\bar{\delta}$ there exists $\bar{\alpha}_{\bar{\delta}} \in (0, \beta)$, such that (5) is verified for all $\alpha \in [0, \bar{\alpha}_{\bar{\delta}}]$. This can be easily proved by rewriting (3) as

$$\big(\beta I_n + A_0(t)\big)q \leq -\Big(\sum_{j=1}^{r} A_j(t)\Big)q. \tag{7}$$

The inequality (3), written in the form (7), clearly implies that (5) is *strictly* satisfied for $\alpha = 0$. Since both terms of inequality (5) are continuous functions of α, it follows that there exists a positive $\bar{\alpha}_{\bar{\delta}} < \beta$ such that the strict inequality holds true for any $\alpha \in [0, \bar{\alpha}_{\bar{\delta}})$, and the non-strict version (5) holds in $[0, \bar{\alpha}_{\bar{\delta}}]$.

Given the assumptions on the matrices $A_j(t)$, for any non-negative initial condition ϕ, the solution $x(t)$ of system (1) is non-negative for all $t \geq -\bar{\delta}$. Let us define the variable $\eta(t) \in \mathbb{R}^n$ as follows

$$\eta(t) = \bar{q}\|\phi\|_\infty e^{-\alpha t} - x(t), \quad t \geq -\bar{\delta}. \tag{8}$$

Note that $\bar{q}$ defined in (6) is such that $\bar{q} \geq \mathbb{1}_n$, and therefore $\phi(t) \leq \bar{q}\|\phi\|_\infty$, $\forall t \in [-\bar{\delta}, 0]$. Thus, the inequality (4) is satisfied in $[-\bar{\delta}, 0]$ and the function $\eta(t)$ is non-negative in the same interval $[-\bar{\delta}, 0]$. Computing the time derivative of $\eta(t)$, for $t \geq 0$, we get

$$\begin{aligned} \dot{\eta}(t) &= -\alpha\bar{q}\|\phi\|_\infty e^{-\alpha t} - \dot{x}(t) \\ &= -\alpha\bar{q}\|\phi\|_\infty e^{-\alpha t} - \sum_{j=0}^{r} A_j(t)x(t-\delta_j(t)). \end{aligned} \tag{9}$$

By the definition (8) of $\eta(t)$ we have

$$x(t-\delta_j(t)) = \bar{q}\|\phi\|_\infty e^{-\alpha(t-\delta_j(t))} - \eta(t-\delta_j(t)). \tag{10}$$

Substitution into (9) gives

$$\begin{aligned} \dot{\eta}(t) = -\alpha\bar{q}\|\phi\|_\infty e^{-\alpha t} &- \|\phi\|_\infty \sum_{j=0}^{r} A_j(t)\bar{q}\, e^{-\alpha(t-\delta_j(t))} \\ &+ \sum_{j=0}^{r} A_j(t)\eta(t-\delta_j(t)). \end{aligned} \tag{11}$$

Let us define the function $v(t)$, for $t \geq 0$, as

$$\begin{aligned} v(t) &= -\alpha\,\bar{q}\|\phi\|_\infty\, e^{-\alpha t} - \|\phi\|_\infty \sum_{j=0}^{r} A_j(t)\bar{q}\, e^{-\alpha(t-\delta_j(t))} \\ &= \Big(-\alpha\,\bar{q} - \sum_{j=0}^{r} A_j(t)\bar{q}\, e^{\alpha\delta_j(t)} \Big)\|\phi\|_\infty e^{-\alpha t}. \end{aligned} \tag{12}$$

Recalling that $\delta_0(t) = 0$ (dummy delay) and noting that

$$e^{\alpha\bar{\delta}} \geq e^{\alpha\delta_j(t)}, \quad \forall \delta_j(t) \in [0, \bar{\delta}], \tag{13}$$

from inequality (5), which holds true by assumption, it easily follows that

$$\begin{aligned} 0 &\leq \Bigg(-\alpha I_n - A_0(t) - e^{\alpha\bar{\delta}} \Big(\sum_{j=1}^{r} A_j(t) \Big) \Bigg) \bar{q} \\ &\leq \Big(-\alpha\,\bar{q} - \sum_{j=0}^{r} A_j(t)\bar{q}\, e^{\alpha\delta_j(t)} \Big), \end{aligned} \tag{14}$$

so that the function $v(t)$ defined in (12) is non-negative. Thus, function $\eta(t)$ obeys the following differential equation:

$$\begin{aligned} &\dot{\eta}(t) = \sum_{j=0}^{r} A_j(t)\eta(t-\delta_j(t)) + v(t), \quad t \geq 0 \\ &\eta(t) = \bar{q}\|\phi\|_\infty\, e^{-\alpha t} - \phi(t) \geq 0, \quad t \in [-\bar{\delta}, 0], \end{aligned} \tag{15}$$

which defines a positive system, with non-negative initial condition and non-negative input $v(t)$. Thus, $\eta(t) \geq 0$, $\forall t \geq 0$, which implies the thesis (4), thanks to the definition (8) of $\eta(t)$. □

Remark 1: Note that (3) is a sufficient condition of delay-independent exponential stability of positive time-varying delay systems, while any α satisfying (5) is a guaranteed exponential convergence rate. It is clear that there is a trade-off between the maximum delay $\bar{\delta}$ and the size of α: the smaller is $\bar{\delta}$, the larger can be α that satisfies (5).

Consider now the dual condition of (3):
$\exists \beta \in \mathbb{R}_{++},\ \exists h \in \mathbb{R}^n_{++}$ such that

$$h^T \Big(\sum_{j=0}^{r} A_j(t) \Big) \leq -\beta\, h^T, \quad \forall t \geq 0, \tag{16}$$

with $A_0(t)$ Metzler and $A_j(t)$ non-negative for $j = 1, \ldots, r$. If a perfect analogy would hold with respect to the case of positive systems with constant matrices (i.e., $A_j(t) = A_j$ in (1), for $j = 0, \ldots, r$), one would expect that also (16) may prove delay-independent GES for (1) with time-varying matrices, since in the special case of constant matrices conditions (3) and (16) can be shown equivalent (indeed, they are both equivalent to showing that $\bar{\lambda}(\sum_{j=0}^{r} A_j) < 0$ which is necessary and sufficient for the delay-independent GES [27]). Remarkably, for positive systems with time-varying matrices condition (16) is in general not sufficient to prove the delay-independent GES of (1). A stronger condition is needed, as shown by the following theorem, whose proof is omitted due to space limitation.

Theorem 2: Consider system (1), where $A_0(t)$ is Metzler and $A_j(t)$ are non-negative for $j = 1, \ldots, r$, and bounded initial state $\phi : [-\bar{\delta}, 0] \mapsto \mathbb{R}^n_+$. If there exist $\beta \in \mathbb{R}_{++}$, $h \in \mathbb{R}^n_{++}$, and non-negative functions $\tilde{\alpha}_j(t)$, $j = 0, \ldots, r$ (possibly constant), such that $\forall t \geq 0$

$$h^T A_0(t) \leq -\tilde{\alpha}_0(t)\, h^T, \tag{17}$$

$$h^T A_j(t) \leq \tilde{\alpha}_j(t)\, h^T, \quad j = 1, \ldots, r, \tag{18}$$

$$\tilde{\alpha}_0(t) \geq \beta + \sum_{j=1}^{r} \tilde{\alpha}_j(t), \tag{19}$$

then, the system is delay-independent GES, and any $\alpha > 0$ such that

$$\tilde{\alpha}_0(t) \geq \alpha + \Big(\sum_{j=1}^{r} \tilde{\alpha}_j(t) \Big) e^{\alpha\bar{\delta}} \tag{20}$$

is a guaranteed exponential rate. More specifically:

$$\|x(t)\|_1 \leq \frac{h_{max}}{h_{min}} \bar{\phi}\, e^{-\alpha t}, \quad t \geq 0, \tag{21}$$

where

$$h_{max} = \max_{i \in [1,n]} (h_i), \quad h_{min} = \min_{i \in [1,n]} (h_i), \quad \bar{\phi} = \sup_{t \in [-\bar{\delta}, 0]} \|\phi(t)\|_1. \tag{22}$$

Remark 2: Note that conditions (17)–(19) imply condition (16), but the converse is not true.

Remark 3: An analogous remarkable dissimilarity among seemingly dual conditions of stability was recently discussed also for the discrete-time case of delay systems with time-varying matrices, in [11] and [33]. It is even more remarkable that in the discrete-time case one of the two conditions is delay-independent (Theorem 2 in [11]), while the other one is intrinsically delay-dependent (Theorem 3 in [11]).

B. General Systems

Condition (3) of Theorem 1 can be adapted to provide a sufficient criterion of delay-independent exponential stability for general delay systems of the type (1), without any positivity assumption. This result is obtained exploiting the technique of Internally Positive Representation (IPR) of linear delay systems, presented in [27] for systems as in (1). We remark that, while [27] introduced the IPR methodology for delay systems with time-varying matrices, the stability analysis reported in that work only considers the special case of constant matrices. Hence, in this section we directly generalize the stability results of [27] to the way more general case of time-varying matrices. In order to briefly present the IPR construction of [27] we need to recall some operators that apply to square matrices.

The first operator is denoted $(\cdot)^{\mathcal{M}}$, and transforms a square matrix A into a Metzler matrix: for any given $A \in \mathbb{R}^{n\times n}$, $A^{\mathcal{M}}$ is the Metzler matrix obtained from A by replacing all its off-diagonal entries with their absolute values, i.e.

$$A^{\mathcal{M}} = \mathrm{d}(A) + |A - \mathrm{d}(A)|, \tag{23}$$

where $\mathrm{d}(A)$ is a diagonal matrix with the same diagonal of matrix A. Moreover, let us denote with A^+ and A^- the component-wise non-negative and non-positive parts of A, respectively defined as:

$$[A^+]_{i,j} = \max\{[A]_{i,j}, 0\}, \tag{24}$$

$$[A^-]_{i,j} = \max\{-[A]_{i,j}, 0\}. \tag{25}$$

Then we can define the following operators:

$$\tilde{\mu}(A) = \mathrm{d}(A) + \big(A - \mathrm{d}(A)\big)^+, \tag{26}$$

$$\tilde{\nu}(A) = \big(A - \mathrm{d}(A)\big)^-, \tag{27}$$

where we note that $\tilde{\mu}(A)$ is Metzler, $\tilde{\nu}(A)$ is non-negative, and they allow to write

$$A^{\mathcal{M}} = \tilde{\mu}(A) + \tilde{\nu}(A), \tag{28}$$

$$A = \tilde{\mu}(A) - \tilde{\nu}(A). \tag{29}$$

Also, we define:

$$\Gamma(A) = \begin{bmatrix} \tilde{\mu}(A) & \tilde{\nu}(A) \\ \tilde{\nu}(A) & \tilde{\mu}(A) \end{bmatrix}, \tag{30}$$

$$\Pi(A) = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}. \tag{31}$$

Note that $\Gamma(A)$ is a $2n \times 2n$ Metzler matrix, while $\Pi(A)$ is $2n \times 2n$ non-negative.

The following operator $\pi : \mathbb{R}^n \mapsto \mathbb{R}^{2n}$, and matrix Δ_n are also needed

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \quad \Delta_n = \begin{bmatrix} I_n & -I_n \end{bmatrix}. \tag{32}$$

It is easy to see that they are such that $x = \Delta_n \pi(x)$. Note also that

$$|x| = x^+ + x^- = [I_n \;\; I_n]\pi(x). \tag{33}$$

We refer the reader to Definition 3 in [27] for a precise definition of Internally Positive Representation for the class of systems under investigation. Namely, an IPR consists of:

- A novel system, of larger dimension, which is positive by construction;
- a set of transformations which allow to shift from the original system to its positive representation, and viceversa.

The following theorem is a simplified version of Theorem 2 of [27] and introduces a simple IPR for system (1).

Theorem 3: Consider system (1), and let $x(t)$ be the state evolution for a given initial state function $\phi(t)$, $t \in [-\bar{\delta}, 0]$. The following system

$$\begin{aligned} \dot{\mathcal{X}}(t) &= \Gamma\big(A_0(t)\big)\mathcal{X}(t) + \sum_{j=1}^{r} \Pi\big(A_j(t)\big)\mathcal{X}(t-\delta_j(t)), \;\; t \geq 0, \\ \mathcal{X}(t) &= \pi\big(\phi(t)\big), \qquad\qquad t \in [-\bar{\delta}, 0], \end{aligned} \tag{34}$$

is such that

$$x(t) = \Delta_n \mathcal{X}(t), \quad t \geq -\bar{\delta}. \tag{35}$$

Theorem 3 states that system (34), together with forward and backward state transformations $T_X^f(x) = \pi(x)$ and $T_X^b(\mathcal{X}) = \Delta_n \mathcal{X}$, is an IPR for system (1), and is a straightforward consequence of Theorem 2 of [27]. Indeed, Theorem 3 is a simplified version of Theorem 2 of [27], since the integral term considered in [27] is missing in (34). Moreover, inputs and outputs are not considered in (34), because in this work we are only interested in exploiting the IPR (34) for the analysis of internal stability of (1).

Now we can give two preliminary Lemmas that will be used to prove a delay-independent stability condition for system (1) exploiting its above-defined IPR.

Lemma 1: Consider the matrices $A_j(t)$ in (1). For a given $\beta \in \mathbb{R}_{++}$, the following conditions are equivalent:

1) $\exists p \in \mathbb{R}^{2n}_{++},\ \exists \beta \in \mathbb{R}_{++}:$

$$\Big(\Gamma\big(A_0(t)\big) + \sum_{j=1}^{r} \Pi\big(A_j(t)\big)\Big)p \leq -\beta\, p, \quad \forall t \geq 0. \tag{36}$$

2) $\exists q \in \mathbb{R}^{n}_{++},\ \exists \beta \in \mathbb{R}_{++}:$

$$\Big(A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r} |A_j(t)|\Big)q \leq -\beta\, q, \quad \forall t \geq 0. \tag{37}$$

Proof: (36) $\implies$ (37). If (36) holds, it is easy to check that (37) is verified with $q = [I_n \; I_n]p$, by simply left-multiplying both terms in (36) by $[I_n\ I_n]$ and noting that

$$\begin{aligned}&\begin{bmatrix}I_n & I_n\end{bmatrix}\Gamma\big(A_0(t)\big)p = A_0^{\mathcal{M}}(t)\begin{bmatrix}I_n & I_n\end{bmatrix}p,\\ &\begin{bmatrix}I_n & I_n\end{bmatrix}\Pi\big(A_j(t)\big)p = |A_j(t)|\begin{bmatrix}I_n & I_n\end{bmatrix}p, \quad j=1,\dots,r.\end{aligned} \tag{38}$$

(37) $\implies$ (36). If (37) holds true, then (36) is verified with $p=[q^T\ q^T]^T$, as it is easy to check, since

$$\begin{aligned}\Gamma\big(A_0(t)\big)\begin{bmatrix}q\\q\end{bmatrix} &= \begin{bmatrix}A_0^{\mathcal{M}}(t)\,q\\A_0^{\mathcal{M}}(t)\,q\end{bmatrix}\\ \Pi\big(A_j(t)\big)\begin{bmatrix}q\\q\end{bmatrix} &= \begin{bmatrix}|A_j(t)|\,q\\|A_j(t)|\,q\end{bmatrix} \quad j=1,\dots,r.\end{aligned} \tag{39}$$

□

Lemma 2: Assume that inequality (37) holds true for some $\beta\in\mathbb{R}_{++}$ and $q\in\mathbb{R}^n_{++}$. Then, for any $\bar{\delta}\in\mathbb{R}_+$ there exists $\bar{\alpha}_{\bar{\delta}}\in(0,\beta)$ such that $\forall\alpha\in[0,\bar{\alpha}_{\bar{\delta}}]$

$$\alpha\,q + A_0^{\mathcal{M}}(t)q + e^{\alpha\bar{\delta}}\Big(\sum_{j=1}^{r}|A_j(t)|\Big)q \le 0, \quad \forall t\ge 0. \tag{40}$$

Proof. The thesis is obtained following the same steps done in the first lines of the proof of Theorem 1. □

Remark 4: Thanks to Lemma 2, the inequality (37) also implies that for any set of bounded delays $\delta_j(t)\in[0,\bar{\delta}]$ $(j=1,\dots,r)$, $\forall\alpha\in[0,\bar{\alpha}_{\bar{\delta}}]$

$$\alpha\,q + A_0^{\mathcal{M}}(t)q + \Big(\sum_{j=1}^{r}|A_j(t)|e^{\alpha\delta_j(t)}\Big)q \le 0, \quad \forall t\ge 0, \tag{41}$$

see inequalities in (14).

Now the main stability theorem for general systems (1) can be given, as a direct generalization of the positive-only Theorem 1.

Theorem 4: Consider system (1), with bounded delays $\delta_j(t)\in[0,\bar{\delta}]$, $j=1,\dots,r$. If there exist $\beta\in\mathbb{R}_{++}$ and $q\in\mathbb{R}^n_{++}$ such that

$$\Big(A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r}|A_j(t)|\Big)q \le -\beta\,q, \quad \forall t\ge 0, \tag{42}$$

then the system is delay-independent GES. Moreover, let $\alpha>0$ be any value satisfying (40); then, the following exponential bound holds true:

$$|x(t)| \le 2\bar{q}\|\phi\|_\infty\,e^{-\alpha t}, \quad t\ge 0, \tag{43}$$

where $\bar{q}=q/q_{\min}$, and α is a guaranteed exponential convergence rate.

Proof. The proof is obtained by exploiting the IPR (34) for system (1). The inequality (42) is equivalent to (36), thanks to Lemma 1, and therefore the exponential stability of the IPR is proved, with convergence rate α. Since $\|x(t)\|\le\|\mathcal{X}(t)\|$ at all $t\ge-\bar{\delta}$, the exponential convergence of the IPR implies the exponential convergence of the system (1), with the same rate α. In detail, (43) is obtained as follows. Due to Theorem 1 the state of the IPR (34) satisfies:

$$[I_n\ I_n]\,\mathcal{X}(t) \le [I_n\ I_n]\begin{bmatrix}\bar{q}\\\bar{q}\end{bmatrix}\|\pi(\phi)\|_\infty e^{-\alpha t}, \ \forall t\ge-\bar{\delta}. \tag{44}$$

Since $\|\pi(\phi)\|_\infty=\|\phi\|_\infty$, and noting the bound

$$|x(t)| \le [I_n\ I_n]\mathcal{X}(t), \quad \forall t\ge-\bar{\delta}, \tag{45}$$

we obtain the desired:

$$|x(t)| \le 2\bar{q}\|\phi\|_\infty e^{-\alpha t}, \quad \forall t\ge-\bar{\delta}. \tag{46}$$

□

Remark 5: Also Theorem 2 lends itself to be extended to general systems similarly to what done above for Theorem 1. Yet, due to space constraints and for simplicity, we have chosen to only report an extension of Theorem 1.

III. SWITCHING SYSTEMS WITH DELAYS

For general systems with time-varying matrices described by (1) it is not obvious how to check the stability condition (42) of Theorem 4 for all $t\ge 0$, since it requires infinite-many tests. On the other hand, there are various significant cases in which the aforementioned result can be finitely tested. In particular, the analysis carried out in the previous sections can be easily extended to a quite interesting class of systems described by continuous-time switching models with time-delays, described by:

$$\begin{aligned}&\dot{x}(t) = A_{0,\sigma(t)}x(t) + \sum_{j=1}^{r}A_{j,\sigma(t)}x(t-\delta_j(t)), \quad t\ge 0,\\ &x(t)=\phi(t), \quad t\in[-\bar{\delta},0],\end{aligned} \tag{47}$$

where $\sigma(t)$ is a switching signal taking values on the set $\{1,\dots,m\}$. Note that the switching system (47), with r time-varying delays, is defined by $m(r{+}1)$ matrices $A_{j,i}$, with $j=0,\dots,r$, and $i=1,\dots,m$. Many interesting work appeared in the literature aiming at analyzing the stability properties of this class of systems, based on different techniques. When focusing the positive case, one should at least mention the contributions of [34], [35], [36].

In the general case with no sign constraint, a sufficient condition of delay-independent GES for (47) can be obtained as a special case of Theorem 4. Notice that the stability condition, in this special case, consists of a finite test.

Theorem 5: Consider a switching system with delays as in (47). If there exist $\beta\in\mathbb{R}_{++}$, and $q\in\mathbb{R}^n_{++}$ such that

$$\Big(A_{0,i}^{\mathcal{M}} + \sum_{j=1}^{r}|A_{j,i}|\Big)q \le -\beta\,q, \quad \forall i=1,\dots,m, \tag{48}$$

then the system is delay-independent GES.

As already remarked above, we again note that in principle also Theorem 2 could be generalized to yield conditions of delay-independent GES for positive and general switching systems (47), yet we leave the explicit statement of such generalizations to future work due to space limitation.

IV. Conclusion

This work introduced conditions of delay-independent global exponential stability and guaranteed convergence rate for linear continuous-time systems with time-varying matrices and delays. We started from the case of positive systems, and then generalized the analysis to systems with no sign constraints. Furthermore, we discussed in some detail a significant special case in which the proposed conditions can be finitely tested: switching systems with time-varying delays. Further investigation about implementation issues and other special cases in which the conditions may be finitely tested is left for future work.

In the near future, we will also address the possible extension of the proposed stability analysis to input-to-state stability, on the lines of the recent contribution [11] for the discrete-time case.

References

[1] S.-I. Niculescu, *Delay effects on stability: a robust control approach*. Springer Science & Business Media, 2001, vol. 269.

[2] E. Fridman, *Introduction to time-delay systems: Analysis and control*. Springer, 2014.

[3] B. Zhou and A. V. Egorov, "Razumikhin and Krasovskii stability theorems for time-varying time-delay systems," *Automatica*, vol. 71, pp. 281–291, 2016.

[4] F. Mazenc and M. Malisoff, "Extensions of Razumikhin's theorem and Lyapunov–Krasovskii functional constructions for time-varying systems with delay," *Automatica*, vol. 78, pp. 1–13, 2017.

[5] X. Li, X. Yang, and S. Song, "Lyapunov conditions for finite-time stability of time-varying time-delay systems," *Automatica*, vol. 103, pp. 135–140, 2019.

[6] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability analysis for systems with time-varying delay: Trajectory based approach," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 1811–1816.

[7] H. Li, B. Zhou, M. Hou, and G. Duan, "On the time-varying Halanay inequality with applications to stability analysis of time-delay systems," *Journal of the Franklin Institute*, vol. 358, no. 10, pp. 5488–5512, 2021.

[8] I. Haidar, "Non-coercive Lyapunov–Krasovskii functionals for exponential stability of time-varying delay systems: a switched system approach," in *2022 10th International Conference on Systems and Control (ICSC)*. IEEE, 2022, pp. 18–22.

[9] L. Hetel, J. Daafouz, and C. Iung, "Equivalence between the Lyapunov–Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 3, pp. 697–705, 2008.

[10] V. De Iuliis, F. Smarra, C. Manes, and A. D'Innocenzo, "Stability analysis of switched ARX models and application to learning with guarantees," *Nonlinear Analysis: Hybrid Systems*, vol. 46, p. 101250, 2022.

[11] V. De Iuliis and C. Manes, "Conditions of stability and guaranteed convergence rate for linear time-varying discrete-time delay systems," *Systems & Control Letters*, vol. 177, p. 105546, 2023.

[12] L. Farina and S. Rinaldi, *Positive linear systems: theory and applications*. John Wiley & Sons, 2000, vol. 50.

[13] W. M. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and compartmental dynamical systems*. Princeton University Press, 2010.

[14] X. Liu, W. Yu, and L. Wang, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1024–1028, 2010.

[15] W. Kerscher and R. Nagel, "Asymptotic behavior of one-parameter semigroups of positive operators," *Positive semigroups of operators, and applications*, pp. 297–309, 1984.

[16] T. Kaczorek, "Stability of positive continuous-time linear systems with delays," in *2009 European Control Conference (ECC)*. IEEE, 2009, pp. 1610–1613.

[17] P. H. A. Ngoc, "Stability of positive differential systems with delay," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 203–209, 2012.

[18] M. Di Loreto and J. J. Loiseau, "On the stability of positive difference equations," *Time delay systems: Methods, applications and new trends*, pp. 19–33, 2012.

[19] J. Shen and W. X. Zheng, "Positivity and stability of coupled differential–difference equations with time-varying delays," *Automatica*, vol. 57, pp. 123–127, 2015.

[20] V. De Iuliis, A. Germani, and C. Manes, "Internally positive representations and stability analysis of coupled differential-difference systems with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2514–2521, 2018.

[21] C. Briat, "Stability and performance analysis of linear positive systems with delays using input–output methods," *International Journal of Control*, vol. 91, no. 7, pp. 1669–1692, 2018.

[22] P. T. Nam, T. N. Nguyen, H. Trinh *et al.*, "Comparison principle for positive time-delay systems: An extension and its application," *Journal of the Franklin Institute*, vol. 358, no. 13, pp. 6818–6834, 2021.

[23] T. Ngoc Nguyen, "Exponential state estimate of positive systems with time-varying delays: a Lyapunov–Razumikhin approach," *IMA Journal of Mathematical Control and Information*, vol. 40, no. 2, pp. 135–151, 2023.

[24] F. Mazenc, "Stability analysis of time-varying neutral time-delay systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 540–546, 2014.

[25] F. Mazenc and M. Malisoff, "Stability analysis for time-varying systems with delay using linear Lyapunov functionals and a positive systems approach," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 771–776, 2015.

[26] F. Conte, V. De Iuliis, and C. Manes, "Internally positive representations and stability analysis of linear delay systems with multiple time-varying delays," *Positive Systems: Theory and Applications (POSTA 2016) Rome, Italy, September 14-16, 2016 5*, pp. 81–93, 2017.

[27] V. De Iuliis, A. D'Innocenzo, A. Germani, and C. Manes, "Internally positive representations and stability analysis of linear differential systems with multiple time-varying delays," *IET Control Theory & Applications*, vol. 13, no. 7, pp. 920–927, 2019.

[28] V. De Iuliis, A. D'Innocenzo, A. Germani, and C. Manes, "Stability analysis of coupled differential-difference systems with multiple time-varying delays: A positivity-based approach," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 6085–6092, 2021.

[29] Q. Xiao, Y. Yang, and T. Huang, "Internally positive representation to stability of delayed timescale-type differential-difference equation," *IEEE Access*, vol. 9, pp. 34 660–34 666, 2021.

[30] V. De Iuliis, C. Manes, and A. D'Innocenzo, "First-moment stability of Markov jump linear systems with homogeneous and inhomogeneous transition probabilities," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 668–673.

[31] D. Li, J. Chang, W. Chen, and T. Raïssi, "IPR-based distributed interval observers design for uncertain LTI systems," *ISA transactions*, vol. 121, pp. 147–155, 2022.

[32] R. Li and P. Zhao, "Practical stability of time-varying positive systems with time delay," *IET Control Theory & Applications*, vol. 15, no. 8, pp. 1082–1090, 2021.

[33] V. De Iuliis and C. Manes, "Some remarks on the stability of time-varying discrete-time positive delay systems," in *2023 9th International Conference on Control, Decision and Information Technologies (CoDIT), Rome, Italy*, 2023.

[34] X. Liu and C. Dang, "Stability analysis of positive switched linear systems with delays," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1684–1690, 2011.

[35] X. Zhao, L. Zhang, and P. Shi, "Stability of a class of switched positive linear time-delay systems," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 5, pp. 578–589, 2013.

[36] M. Xiang and Z. Xiang, "Stability, l1-gain and control synthesis for positive switched systems with time-varying delay," *Nonlinear Analysis: Hybrid Systems*, vol. 9, pp. 9–17, 2013.