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Locally Homogeneous Finite-time Stabilization of Quasi-Linear Systems

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***Abstract*— In this paper, an algorithm of a local finite-time control design is developed for a class of quasi-linear systems. The design procedure is essentially based on the concept of generalized homogeneity and the convex embedding technique. The adjustment of control parameters is realized by solving a system of linear matrix inequalities (LMIs). Theoretical results are supported by numerical simulations.**

I. Introduction

A. State of the art

A symmetry with respect to a dilation is known as homogeneity [18], [7], [4]. In control theory, the homogeneity is utilized for stability/controlability analysis, controller and observer design (e.g. [5], [13], [2], [12] and references therein). The standard (Euler) homogeneity means that a function $f(x)$ is symmetric with respect to the scaling of its argument $f(e^s x) = e^{\nu s} f(x), \forall s, x \in \mathbb{R}$, where the constant ν is called the homogeneity degree. The generalized homogeneity is introduced by V.I. Zubov in 1958 [18] using the weighted dilation $(x_1, x_2, .., x_n) \to (e^{r_1 s} x_1, e^{r_2 s} x_2, ..., e^{r_n s} x_n), s \in \mathbb{R}$, where the positive numbers $r_1, r_2, ..., r_n$ are the weights specifying the dilation rate of each coordinate. Nonlinear (geometric) dilations are studied in [8], [6], [14]. This paper deals with the *linear (geometric) dilation* [11] given by $x \to e^{G_{\mathbf{d}} s} x$, where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is anti-Hurwitz matrix[1]. Being a relaxation of linearity, the homogeneity can provide an extra degree of freedom for for advanced control design. In particular, the finite/fixed-time stabilization can be guaranteed by a proper selection of the homogeneity degree [3], [17].

As shown in [9], the finite stabilization of affine-in-control systems can be realised using the Sontag's universal formula [15]. However, it requires a design of an appropriate control Lyapunov function. The convex embedding approach for homogeneous affine-in-control systems has been introduced recently [16]. It essentially uses the homogeneity (symmetry) of the vector field to define a stability condition in terms of LMIs. This paper generalizes the latter approach to a special class of locally homogeneous (quasi-linear) systems, which consideration is important for control applications in fluid mechanics. To the best of our knowledge, the homogeneous finite-time controllers have never been designed for such a class of systems.

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[1]The matrix $G_{\mathbf{d}} \in \mathbb{R}^n$ is anti-Hurwitz, if $-G_{\mathbf{d}}$ is Hurwitz.

B. Motivating Example

A quasi-linear Ordinary Differential Equation (ODE) may appear as reduced order model of mathematical physics. For example, the finite difference discretization of the viscous Burgers or Navier-Stokes equations with controlled boundary conditions provides nonlinear reduced order models (ODEs), that are affine in control. Let us consider one dimensional viscous Burgers equation

$$\partial_t v + v\partial_x v = \nu \partial_{xx} v, \quad t > 0, \quad x \in (0, L) \tag{1}$$

$\nu > 0$, with the initial condition $v(0, x) = f(x)$ and time-varying Dirichlet boundary conditions:

$$v(t, 0) = v_0 + \xi(t), \quad v(t, L) = v_L \tag{2}$$

where v_0, v_L are positive constants and ξ is assumed to be a positive control input. The positivity of the control inputs is a specific feature of many real control problems. For the models of fluid dynamics, the positive control may be inspired by the use of air-blowers as actuators.

Let N be strictly positives integers, $h_x = L/N$ be the step of spatial discretization and $x_i = ih_x$, with $i \in \{0, \ldots, N\}$.

Let denote $v_i(t)$ respectively the approximation of $v(t, x_i)$. The system (1) can be discretized as follows:

$$\dot{v}_i(t) + v_i(t)\tfrac{v_{i+1}(t) - v_{i-1}(t)}{2h_x} = \nu \tfrac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_x^2}, \tag{3}$$

with $v_i(0) = f_i$, and $v_0(t) = v_0 + \xi(t), v_N(t) = v_L$. The obtained system has a rather specific structure: it is affine in control and the finite dimensional vector field has linear and bilinear/quadratic components. Notice that the above approximates of partial derivatives have the second order of approximation $O(h_x^2)$. The use of more accurate approximates of partial derivatives (e.g., $O(h_x^3)$, $O(h_x^4)$, etc) would lead to a system of the similar structure. In this paper, we consider such a class of control systems, which covers the above and many other examples of reduced order models of fluid mechanics.

C. Contributions and organization

In this paper the problem of the finite-time set-point tracking is studied for a class of quasi-linear systems. The stabilizing feedback is designed using the concept of linear (geometric) homogeneity and the convex embedding approach. The control parameters tuning is based on solving of the system of LMIs, which is feasible in the case of local controlability of the system.

The paper is organized as follows. First, the problem statement and basic assumptions are given. Next, preliminaries about homogeneous systems are discussed. After

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that, the finite-time control for the quasi-linear system is designed. Finally, the numerical simulation results and some conclusions are presented.

D. Notation

$\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$; $\bar{\mathbb{R}}_+ = \{a \in \mathbb{R} : a \geq 0\}$; $|z| = \sqrt{z^\top z}$ is Euclidean norm of $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$; $\|z\|_1 = \sum_{k=1}^n |z_i|$; we write $P \succ 0$ (resp, $\prec 0, \succeq 0, \preceq 0$) if the symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is positive (resp., negative,semi- positive , semi-negative) definite; $P^{1/2}$ means that $P = P^{1/2}P^{1/2}$ (always exists for symmetric positive definite matrices);

II. Problem statement

Let us consider the system

$$\dot{x} = (L + W(x))x + (F + Q(x))\xi + g, \quad x(0) = x_0, \quad (4)$$

where $x \in \mathbb{R}^n$ is the system state, $\xi \in \bar{\mathbb{R}}^m_+$ is the control input, which is assumed to be positive (due to the physical constraints), $g \in \mathbb{R}^n$ is a known constant vector (inspired, for instance, by the boundary conditions in the above motivating example), $L \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times m}$ are known constant matrices and $W(x) \in \mathbb{R}^{n\times n}$ are state dependent matrices, whose values can be computed for any $x \in \mathbb{R}^n$.

Notice that the matrix-valued functions $W : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ and $Q : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ are linear. The latter implies that $\exists \tilde{W} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ and $\tilde{Q} : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ such that

$$W(x)y = \tilde{W}(y)x, \;\; Q(x)\xi = \tilde{Q}(\xi)x, \;\; \forall x, y \in \mathbb{R}^n, \;\; \forall \xi \in \mathbb{R}^m. \quad (5)$$

Assumption 1: Given $x^* \in \mathbb{R}^n$ let there exists $\xi^* \in \mathbb{R}^m_+$ such that

$$(L + W(x^*))x^* + (F + Q(x^*))\xi^* + g = 0.$$

The state $x = x^*$ is an equilibrium (possibly unstable) of the system (4) with the control $\xi = \xi^*$. We consider α^* as a desired set point of for tracking by the system.

Given $r > 0$, $x^* \in \mathbb{R}^n, \xi^* \in \mathbb{R}^m_+$, the *control aim* is to design a finite-time stabilizing feedback $\xi = \xi(x)$ such that for any $x_0 \in \mathbb{R}^n : |x_0 - x^*| \leq r$ the trajectory of the closed-loop system satisfies

$$x(t) \to x^* \quad \text{as} \quad t \to T(x_0), \quad (6)$$

$$\xi(x(t)) \in \bar{\mathbb{R}}^m_+, \quad \forall t \geq 0, \quad (7)$$

where $T : \mathbb{R}^n \mapsto \bar{\mathbb{R}}_+$ is the settling-time function.

III. Preliminaries on homogeneity

A. Linear dilations

Let us recall that *a family of operators* $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$ with $s \in \mathbb{R}$ is a *group* if $\mathbf{d}(0)x = x$, $\mathbf{d}(s) \circ \mathbf{d}(t)x = \mathbf{d}(s+t)x$, $\forall x \in \mathbb{R}^n, \forall s, t \in \mathbb{R}$. A *group* $\mathbf{d}$ is

- *continuous* if the mapping $s \mapsto \mathbf{d}(s)x$ is continuous, $\forall x \in \mathbb{R}^n$;
- *linear* if $\mathbf{d}(s)$ is a linear mapping (i.e., $\mathbf{d}(s) \in \mathbb{R}^{n\times n}$), $\forall s \in \mathbb{R}$;
- a *dilation* in $\mathbb{R}^n$ if $\liminf\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ and $\limsup\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$, $\forall x \neq \mathbf{0}$.

A continuous linear group in $\mathbb{R}^n$ admits the representation [10]:

$$\mathbf{d}(s) = e^{sG_\mathbf{d}} = \sum_{j=0}^{\infty} \tfrac{s^j G^j_\mathbf{d}}{j!}, \quad s \in \mathbb{R}. \quad (8)$$

where $G_\mathbf{d} \in \mathbb{R}^{n\times n}$ is a generator of $\mathbf{d}$. A continuous linear group (8) is a dilation in $\mathbb{R}^n$ if and only if $G_\mathbf{d}$ is an anti-Hurwitz matrix [11]. In this paper we deal only with continuous linear dilations. A *dilation* $\mathbf{d}$ in $\mathbb{R}^n$ is

- *monotone* if the function $s \mapsto \|\mathbf{d}(s)x\|$ is strictly increasing, $\forall x \neq \mathbf{0}$;
- *strictly monotone* if $\exists \beta > 0$ such that $\|\mathbf{d}(s)x\| \leq e^{\beta s}\|x\|$, $\forall s \leq 0$, $\forall x \in \mathbb{R}^n$.

The following result is the straightforward consequence of the existence of the quadratic Lyapunov function for asymptotically stable LTI systems.

Corollary 1: A linear continuous dilation in $\mathbb{R}^n$ is strictly monotone with respect to the weighted Euclidean norm $\|x\| = \sqrt{x^\top Px}, 0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ if and only if

$$PG_\mathbf{d} + G^\top_\mathbf{d} P \succ 0, \quad P \succ 0. \quad (9)$$

Any dilation in $\mathbb{R}^n$ defines an alternative topology (balls, spheres, cones, etc) in $\mathbb{R}^n$.

B. Canonical homogeneous norm

Any linear continuous and monotone dilation in a normed vector space introduces also an alternative norm topology defined by the so-called canonical homogeneous norm [11].

Definition 1 (Canonical homogeneous norm): Let a linear dilation $\mathbf{d}$ in $\mathbb{R}^n$ be continuous and monotone with respect to a norm $\|\cdot\|$. A function $\|\cdot\|_\mathbf{d} : \mathbb{R}^n \mapsto [0, +\infty)$ defined as follows: $\|\mathbf{0}\|_\mathbf{d} = 0$ and

$$\|x\|_\mathbf{d} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (10)$$

is said to be a canonical $\mathbf{d}$-homogeneous norm in $\mathbb{R}^n$

For standard dilation $\mathbf{d}_1(s) = e^s I_n$ we have $\|x\|_{\mathbf{d}_1} = \|x\|$. In other cases, $\|x\|_\mathbf{d}$ with $x \neq \mathbf{0}$ is implicitly defined by a nonlinear algebraic equation, which always have a unique solution due to monotonicity of the dilation. In some particular cases [12], this implicit equation has explicit solution even for non-standard dilations. Since $\|\mathbf{d}(-\ln\|x\|_\mathbf{d})x\| = 1$ for $x \neq 0$, then the operator

$$\pi_\mathbf{d}(x) := \begin{cases} \mathbf{d}(-\ln\|x\|_\mathbf{d})x & \text{if} \quad x \neq \mathbf{0}, \\ \mathbf{0} & \text{if} \quad x = \mathbf{0}, \end{cases} \quad (11)$$

is a projector of a nonzero vector x on the unit sphere $\|x\| = 1$. For the standard dilation, such a projector is defined as $\pi_{\mathbf{d}_1}(x) = \mathbf{d}_1(-\ln\|x\|_{\mathbf{d}_1})x = \frac{x}{\|x\|}$ for $x \neq \mathbf{0}$. We have $\pi_\mathbf{d}(x) = \text{sign}(x)$ in the scalar case $n = 1$.

Lemma 1: [11] If a linear continuous dilation $\mathbf{d}$ in $\mathbb{R}^n$ is strictly monotone with respect to a norm $\|x\| = \sqrt{x^\top Px}$ then

1) $\|\cdot\|_{\mathbf{d}}:\mathbb{R}^n\mapsto\mathbb{R}_+$ is single-valued, continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n\backslash\{\mathbf{0}\}$:

$$\frac{\partial\|x\|_{\mathbf{d}}}{\partial x}=\frac{\|x\|_{\mathbf{d}}x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})P\mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})PG_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x},\quad x\neq\mathbf{0}. \tag{12}$$

2) for $\alpha=\frac{\lambda_{\max}(P^{1/2}G_{\mathbf{d}}P^{-1/2}+P^{-1/2}G_{\mathbf{d}}^\top P^{1/2})}{2}$ and $\beta=\frac{\lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2}+P^{-1/2}G_{\mathbf{d}}^\top P^{1/2})}{2}$ it holds

$$\|x\|_{\mathbf{d}}^\alpha\leq\|x\|\leq\|x\|_{\mathbf{d}}^\beta,\qquad\forall x\in\mathbb{R}^n:\|x\|\leq 1; \tag{13}$$

$$\|x\|_{\mathbf{d}}^\beta\leq\|x\|\leq\|x\|_{\mathbf{d}}^\alpha,\qquad\forall x\in\mathbb{R}^n:\|x\|\geq 1. \tag{14}$$

C. Homogeneous functions and vector fields

Below we study systems which are symmetric on homogeneous cones with respect to a linear dilation $\mathbf{d}$. The dilation symmetry introduced by the following definition is known as a generalized homogeneity [18], [7], [13], [3].

Definition 2: [7] Let $\mathbf{d}$ be dilation in $\mathbb{R}^n$. A function $h:\mathbb{R}^n\mapsto\mathbb{R}$ (resp., a vector field $f:\mathbb{R}^n\mapsto\mathbb{R}^n$) is $\mathbf{d}$-homogeneous of degree $\nu\in\mathbb{R}$ if

$$h(\mathbf{d}(s)x)=e^{\nu s}h(x),\quad\forall x\in\mathbb{R}^n,\quad\forall s\in\mathbb{R}.$$

(resp., if $f(\mathbf{d}(s)x)=e^{\mu s}\mathbf{d}(s)f(x),\ \forall s\in\mathbb{R},\ \forall x\in\mathbb{R}^n$.) Formally, to avoid a collision in the above definition for $n=1$, a vector field g should be defined as $g:\Xi\mapsto T\Xi$, where $T\Xi$ is the tangent space for Ξ. Since the tangent space of $\mathbb{R}^n$ can be associated with $\mathbb{R}^n$, we simply write $g:\mathbb{R}^n\mapsto\mathbb{R}^n$.

The homogeneity of a mapping is inherited by other mathematical objects induced by this mapping. In particular, solutions of $\mathbf{d}$-homogeneous system[2]

$$\dot{x}=g(x),\quad t>0,\quad x(0)=x_0\in\mathbb{R}^n \tag{15}$$

are symmetric with respect to the dilation $\mathbf{d}$ in the following sense [18], [7], [3]:

$$x(t,\mathbf{d}(s)x_0)=\mathbf{d}(s)x(e^{\mu s}t,x_0), \tag{16}$$

where $x(\cdot,x_0)$ denotes a solution of (15) with $x(0)=x_0\in\mathbb{R}^n$ and $\mu\in\mathbb{R}$ is the homogeneity degree of g. The solution symmetry (16) implies that any asymptotically stable homogeneous systems of negative degree are finite-time stable [3]. The latter property holds for locally homogeneous systems as well [2]. We follow the idea of local homogeneity for the finite-time stabilizing feedback design and we adapt the control design procedure, developed originally for linear plants in [17], to the quasi-linear system (4) .

IV. Control Design

Let us derive first the error equation and denote

$$\xi=\xi^*+u,\quad z=x-x^*, \tag{17}$$

[2]A system is homogeneous if its is governed by a $\mathbf{d}$-homogeneous vector field

where $u\in\mathbb{R}^m$ is a new (virtual) control such that $\xi^*+u\in\bar{\mathbb{R}}^m_+$. We derive

$$\begin{aligned}\dot{z}&=(L+W(x))x+F\xi^*+Fu+Q(x)\xi^*+Q(x)u+g\\&=Lx+W(x)x-Lx^*-W(x^*)x^*+Fu\\&\quad-Q(x^*)\xi^*+Q(x)\xi^*+Q(x)u\\&=Lz+W(x-x^*)x+W(x^*)z+(F+Q(x))u\\&\quad+Q(x-x^*)\xi^*.\end{aligned} \tag{18}$$

Therefore, from (18) and using (5) we obtain the following bilinear system:

$$\dot{z}=Az+Bu+D(z,u)z, \tag{19}$$

with

$$A=L+\tilde{W}(x^*)+W(x^*)+\tilde{Q}(\xi^*), \tag{20}$$

$$B=F+Q(x^*), \tag{21}$$

and

$$D(z,u)=W(z)+\tilde{Q}(u). \tag{22}$$

A finite-time stabilizing feedback for a linear plant can be designed if and only if the pair $\{A,B\}$ is controllable. The homogeneous controllers for this case can be found in [17], where, in particular, it is shown that for any controllable $\{A,B\}$, there exists a solution $Y_0\in\mathbb{R}^{m\times n}$, $G_0\in\mathbb{R}^{n\times n}$ of the linear algebraic equation

$$AG_0-G_0A+BY_0=A,\quad G_0B=\mathbf{0} \tag{23}$$

such that the matrix $G_{\mathbf{d}}:=I_n+\mu G_0$ is anti-Hurwitz for any $\mu\in[-1,1/\tilde{n}]$, where $\tilde{n}$ *is a minimal natural number such that* $\operatorname{rank}[B,AB,...,A^{\tilde{n}-1}B]=n$, and the matrix $A_0=A+BK_0$ with $K_0:=Y_0(G_0-I_n)^{-1}$ satisfies the identity

$$A_0G_{\mathbf{d}}=(G_{\mathbf{d}}+\mu I_n)A_0,\quad G_{\mathbf{d}}B=B. \tag{24}$$

The above property of the matrix A_0 guarantees that the vector field $z\mapsto A_0z$ is $\mathbf{d}$-homogeneous of the degree μ, so the feedback K_0z "homogenize" (in the generalized sense) the linear part of the system [17].

Theorem 1: Let the pair $\{A,B\}$ be controllable, the matrices K_0, $G_{\mathbf{d}}$ be defined by solving the equation (23) and $\mu\in[-1,0)$. If

1) $G_{\mathbf{d}}Q(z)=Q(z)$ for all $s\in\mathbb{R}$ and all $z\in\mathbb{R}^n$, where $\mathbf{d}(s)=e^{sG_{\mathbf{d}}}$ is the linear dilation.
2) if for some $\rho,\beta,\omega_1,\omega_2\in\mathbb{R}_+$ the matrices $X=X^\top\in\mathbb{R}^{n\times n}$ and $Y\in\mathbb{R}^{m\times n}$ satisfy the LMIs

$$A_0X+XA_0^\top+BY+Y^\top B^\top+\rho(G_{\mathbf{d}}X+XG_{\mathbf{d}})+(\omega_1+\omega_2)X\preceq 0, \tag{25}$$

$$\begin{pmatrix}\omega_1X & rY^\top Q(h_k)^\top\\ rQ(h_k)Y & \omega_1X\end{pmatrix}\succeq 0,\quad k=\overline{1,n} \tag{26}$$

$$\begin{pmatrix}\omega_2X & rX(W(h_k)+Q(h_k)K_0)^\top\\ r(W(h_k)+Q(h_k)K_0)X & \omega_2X\end{pmatrix}\succeq 0, \tag{27}$$

$$\begin{pmatrix}0.5\xi_j^*X & XK_0^\top e_j\\ e_j^\top K_0X & 0.5\xi_j^*\end{pmatrix}\succeq 0,\quad\begin{pmatrix}0.5\xi_j^*X & Y^\top e_j\\ e_j^\top Y & 0.5\xi_j^*\end{pmatrix}\succeq 0,\quad j=\overline{1,m}, \tag{28}$$

$$2(2\beta-\mu)X\succ G_{\mathbf{d}}X+XG_{\mathbf{d}}^\top\succeq 2\beta X, \tag{29}$$

$$\tfrac{r^2}{n}I_n\succeq X\succ 0, \tag{30}$$

where $\lambda_j \geq 0$, $h_k = (0, \ldots, 1, \ldots, 0)^\top$ is the unit Euclidean vector in $\mathbb{R}^n$, $e_j = (0, \ldots, 1, \ldots, 0)^\top$ is the unit Euclidean vector in $\mathbb{R}^m$ and ξ_j^* is the j-th element of the vector $\xi^* \in \mathbb{R}_+^m$, then the system (19) with the feedback control

$$u = K_0 z + \|z\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z, \qquad (31)$$
$$K_0 := Y_0 G_{\mathbf{d}}^{-1}, \quad K := Y X^{-1},$$

is locally finite-time stable with the invariant ellipsoid

$$\mathcal{E}(X) := \{z \in \mathbb{R}^n : z^\top X^{-1} z \leq 1\}; \qquad (32)$$

belonging to the attraction domain. Moreover, for $|z(0)| \leq r$, we have

$$0 \leq \xi^* + u(z(t)), \quad \forall t \geq 0,$$
$$z(t) \to \mathbf{0} \quad \text{as} \quad t \to T(z(0)),$$

where $T(z(0)) \leq \frac{\|z(0)\|_{\mathbf{d}}^{-\mu}}{(-\mu\rho)}$ with the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by the weighted Euclidean norm $\|z\| = \sqrt{z^\top X^{-1} z}$.

Remark 1: Given $r > 0$, the system of LMIs (25)-(29) is feasible for μ sufficiently close to 0 provided that $|rW(h_i)|$ and $|rQ(h_i)|$ are small enough but $\xi_j^* \in \mathbb{R}_+^n$ is large enough. Indeed, taking $\beta = -3/4$ and any $X \succ 0$ once can see that the LMI (29) is fulfilled for μ sufficiently close to 0. The controllability of $\{A, B\}$ implies the feasibility of (25) (see, [17] for mode details). For any fixed $\omega_1, \omega_2 \in \mathbb{R}_+$ and $X \succ 0$, the LMIs (26), (27) are feasible if $|rW(h_i)|$ and $|rQ(h_i)|$ are small enough. For any fixed $X \succ 0$, the LMI (28) is feasible if $\xi_j^* \in \mathbb{R}_+$ is large enough.

Therefore, the only critical restriction in the above theorem is $G_{\mathbf{d}} Q(z) = Q(z)$, but it may be fulfilled in some practical cases, for example, for a discretization of the one-dimensional viscous Burgers equation controlled on the boundary.

Proof. Using (12), for $V = \|z\|_{\mathbf{d}}$ we derive

$$\dot{V} = \|z\|_{\mathbf{d}} \frac{z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}})(Az + Bu + D(z,u)z)}{z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z}. \qquad (33)$$

Using the homogeneity (namely, $A_0 \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) A_0$ and $\mathbf{d}(s) B = e^s B$) we derive (see, [17] for mode details)

$$\begin{aligned} &z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}})(Az + Bu) \\ &= \|z\|_{\mathbf{d}}^{\mu} z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1}(A_0 + BK) \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z, \end{aligned} \qquad (34)$$

which in conjunction with condition (25), we obtain from (33) the following estimate:

$$\dot{V} \leq -\rho V^{1+\mu} + \frac{-(\omega_1+\omega_2)\|z\|_{\mathbf{d}}^{\mu} + z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) D(z,u) z)}{z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z}. \qquad (35)$$

Notice that the term $D(z, u)z$ can be rewritten as follows:

$$\begin{aligned} D(z, u)z &= W(z)z + Q(z)u \\ &= W(z)z + Q(z)K_0 z \\ &\quad + \|z\|_{\mathbf{d}}^{1+\mu} Q(z) K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z. \end{aligned} \qquad (36)$$

Hence, using $\mathbf{d}(s)Q(z) = e^s Q(z)$ (since $G_{\mathbf{d}} Q(z) = Q(z)$) we obtain

$$\begin{aligned} &z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) D(z, u) z \\ &= z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}})(W(z) + Q(z)K_0) z \\ &\quad + \|z\|_{\mathbf{d}}^{\mu} z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} Q(z) K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z. \end{aligned} \qquad (37)$$

Taking into account $z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z = 1$, and using Cauchy-Swartz inequality we derive

$$z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} Q(z) K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z \leq \omega_1 \qquad (38)$$

provided that $K^\top Q^\top(z) X^{-1} Q(z) K \leq \omega_1^2 X^{-1}$.

Let us show that for $\|z\|_1 \leq r$, one has

$$K^\top Q^\top(z) X^{-1} Q(z) K \leq \omega_1^2 X^{-1}. \qquad (39)$$

Indeed, using the Schur complement, inequality (39) can be rewritten as follows:

$$\begin{pmatrix} \omega_1 X & rY^\top Q(z/r)^\top \\ rQ(z/r)Y & \omega_1 X \end{pmatrix} \succeq 0, \qquad (40)$$

By Carathéodory lemma on convex hull, for any z such that $\|z\|_1 \leq 1$, there exist scalars $\alpha_k^{\pm} \geq 0$ such that $\sum_{k=1}^n \alpha_k^+ + \alpha_k^- = 1$ and

$$z = \sum_{k=1}^n (\alpha_k^+ - \alpha_k^-) h_k. \qquad (41)$$

where $h_k = (0, \ldots, 1, \ldots, 0)^\top \in \mathbb{R}^n$ is the unit Euclidean vector. Hence, for $\|z\|_1 \leq r$ we derive

$$\begin{aligned} \begin{pmatrix} \omega_1 X & rYQ(z/r)^\top \\ rQ(z/r)Y & \omega_1 X \end{pmatrix} &= \sum_{k=1}^n \alpha_k^+ \begin{pmatrix} \omega_1 X & rYQ(h_k)^\top \\ rQ(h_k)Y & \omega_1 X \end{pmatrix} \\ &\quad + \alpha_k^- \begin{pmatrix} \omega_1 X & rYQ(-h_k)^\top \\ rQ(-h_k)Y & \omega_1 X \end{pmatrix}, \end{aligned} \qquad (42)$$

Taking into account $Q(-z) = -Q(z), \forall z \in \mathbb{R}^n$ we derive that (26) implies (38) for any $z : \|z\|_1 \leq r$.

Similarly, we derive

$$\frac{z^\top \mathbf{d}^\top(-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|z\|_{\mathbf{d}})(W(z) + Q(z)K_0) z}{\omega_2 \|z\|_{\mathbf{d}}^{\mu}} \leq 1 \qquad (43)$$

provided that

$$\left\| \frac{1}{\omega_2 \|z\|_{\mathbf{d}}^{-\mu}} \mathbf{d}(-\ln \|z\|_{\mathbf{d}})(W(z) + Q(z)K_0) z \right\| \leq 1.$$

The inequality $2(2\beta - \mu)X \succ G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top$ guarantees that the matrix $G_{\tilde{\mathbf{d}}} = (2\beta - \mu) I_n - G_{\mathbf{d}}$ is anti-Hurwits and the dilation $\tilde{\mathbf{d}}(s) = e^{sG_{\tilde{\mathbf{d}}}}$ is monotone with respect to the norm $\|\cdot\|$. Hence, we have

$$\left\| \|z\|_{\mathbf{d}}^{2\beta-\mu} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) \right\| = \|\tilde{\mathbf{d}}(\ln \|z\|_{\mathbf{d}})\| \leq 1 \qquad (44)$$

for all $z : \|z\|_{\mathbf{d}} \leq 1$ (which is equivalent to $\|z\| \leq 1$). Thus, the inequality

$$\|\omega_2^{-1}(W(z/\|z\|_{\mathbf{d}}^{\beta}) + Q(z/\|z\|_{\mathbf{d}}^{\beta}) K_0) z / \|z\|_{\mathbf{d}}^{\beta}\| \leq 1 \qquad (45)$$

implies (43). The LMI $G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top \succeq 2\beta X$ implies $\|z\| \leq \|z\|_{\mathbf{d}}^{\beta}$ (see Lemma 1) provided that $\|z\|_{\mathbf{d}} \leq 1$. In this case, for $\|z\| \leq 1$ the inequality (45) follows from $\|\omega_2^{-1}(W(z) + Q(z)K_0)z\| \leq \|z\|^2 \Leftarrow$

$$(W(z) + Q(z)K_0)^\top (W(z) + Q(z)K_0) \leq \omega_2 X^{-1}. \quad (46)$$

Repeating the above consideration, we derive that LMI (26) implies (43) provided that $\|z\| \leq 1$ and $\|z\|_1 \leq r$. Therefore,

$$\dot{V} \leq -\rho V^{1+\mu}, \quad (47)$$

where $\|z\|_1 \leq r$ and $\|z\| \leq 1$. Since $r^2 n^{-1} I_n \succeq X$, then

$$\|z\|_1 r^{-1} \leq \tfrac{\sqrt{n}}{r}|z| = \sqrt{r^{-2} n z^\top z} \leq \sqrt{z^\top X^{-1} z} \leq 1,$$

for $\|z\| \leq 1$. Therefore, the closed-loop system is locally finite-time stable and the ellipsoid $\mathcal{E}(X) = \{z : \|z\| \leq 1\}$ belongs to the domain of attraction. Moreover, since $|z| \leq \|z\|_1 \leq r$ for $\|z\| \leq 1$ then $|z(0)| \leq r \Rightarrow z(t) \to 0$ with in the finite time $T(z(0)) \leq (-\mu\rho)^{-1} V(z(0))^{-\mu}$.

The LMI (28) and the inequality $z^\top X^{-1} z \leq 1$ guarantees

$$z^\top K_0^\top e_j e_j^\top K_0 z \leq (\xi_j^*)^2/4$$

or, equivalently, $|e_j^\top K_0 z(t)| \leq \xi_j^*/2$ for all $t \geq 0$. Similarly, taking into account $z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln\|z\|_{\mathbf{d}}) z = 1$, we derive

$$z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}}) X^{-1} Y^\top e_j e_j^\top Y X^{-1} \mathbf{d}(-\ln\|z\|_{\mathbf{d}}) z \leq \tfrac{(\xi_j^*)^2}{4}$$

or, equivalently, $|e_j^\top K \mathbf{d}(-\ln\|z(t)\|_{\mathbf{d}}) z(t)| \leq \xi_j^*/2$ for all $t \geq 0$. Since $\|z\|_{\mathbf{d}} \leq 1$ then $\xi_j^* + e_j^\top K_0 z + \|z\|_{\mathbf{d}} e_j^\top K \mathbf{d}(-\ln\|z(t)\|_{\mathbf{d}}) z(t) \geq 0$ for all $t \geq 0$ The proof is complete.

The feasibility of the system (25)-(29) can be studied similarly to [17].

Remark 2: The closed-loop system (19), (31) is locally homogeneous and locally asymptotically stable so it is locally ISS in the view of results [2] with respect to a rather large class of perturbations (in particular with respect to measurement noises and additive disturbances).

V. Numerical Example

Let us consider the error system (19) with the parameters

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Q(z) = \varepsilon \begin{pmatrix} -0.5 z_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$W(z) = \varepsilon \cdot \mathrm{diag}\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} z \right\},$$

where $z = (z_1, z_2, z_3, z_4, z_5)^\top \in \mathbb{R}^5$ and $\mathrm{diag}\{v\}$ denotes the diagonal matrix with the components of the vector $v \in \mathbb{R}^n$ on the main diagonal.

Solving the algebraic equation (23) we derive

$$K_0 = \begin{pmatrix} 20 & -88 & 220 & -330 & 264 \end{pmatrix},$$

$$G_0 = \begin{pmatrix} 0 & -8 & 6 & -4 & 2 \\ 0 & -1 & -6 & 4 & -2 \\ 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

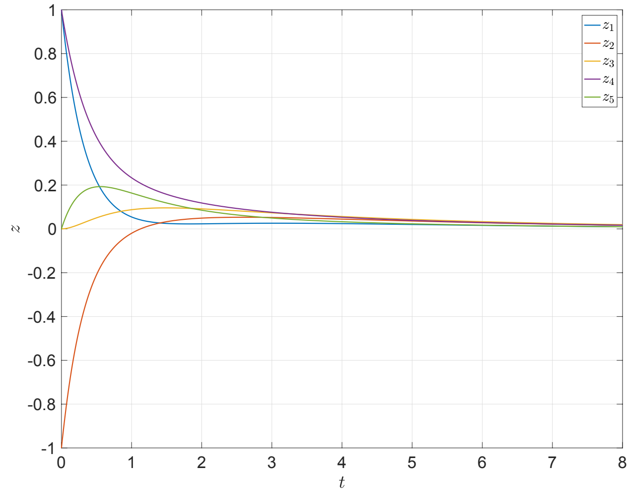

Fig. 1: Evolution of the states of the open-loop system ($u = 0$)

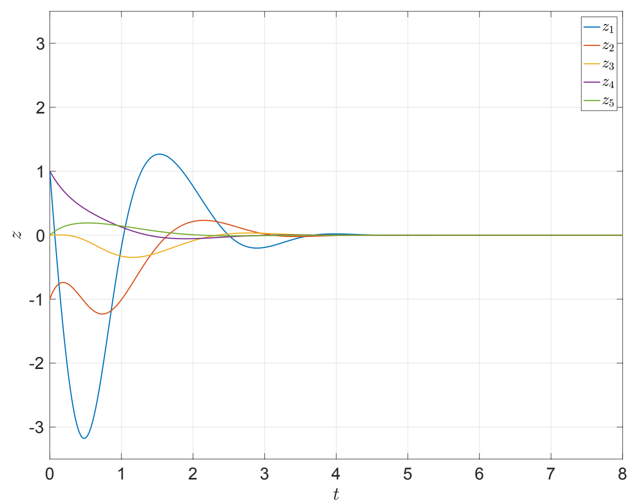

Fig. 2: Evolution of the states of the closed-loop system

The dilation is given by $G_{\mathbf{d}} = I_5 + \mu G_0$ where $\mu \in [-1, 0)$. Obviously, $G_{\mathbf{d}} Q(z) = Q(z)$ so the first condition of the main theorem is fulfilled. The pair $\{A, B\}$ is controllable, so the LMIs (25)-(27),(29),(30) feasible for small enough $\varepsilon > 0$ (in this example, the control restrictions are omitted). Solving (25),(29),(30) for $\mu = -0.1$ we derive

$$X = \begin{pmatrix} 148.1977 & 5.4546 & -28.9205 & -2.0518 & 9.0251 \\ 5.4546 & 25.7547 & 0.2926 & -4.9821 & 0.0481 \\ -28.9205 & 0.2926 & 6.3759 & 0.1808 & -2.3111 \\ -2.0518 & -4.9821 & 0.1808 & 1.0161 & -0.1000 \\ 9.0251 & 0.0481 & -2.3111 & -0.1000 & 1.0000 \end{pmatrix},$$

$$K = \begin{pmatrix} -18.0000 & 65.0431 & -210.9065 & 272.6945 & -257.9888 \end{pmatrix}.$$

The simulations results for the open-loop ($u = 0$) and the closed-loop systems for $\varepsilon = 0.1$ are shown on the Figure 1 and 2. The simulation results are obtained using the explicit Euler method with the sampling period 10^{-3}. Application of the homogeneous control (31) requires a computation of the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ by means of the special numerical procedure [17]. Homogeneous Control Systems (HCS) Toolbox for MATLAB [1] is utilized for this purpose.

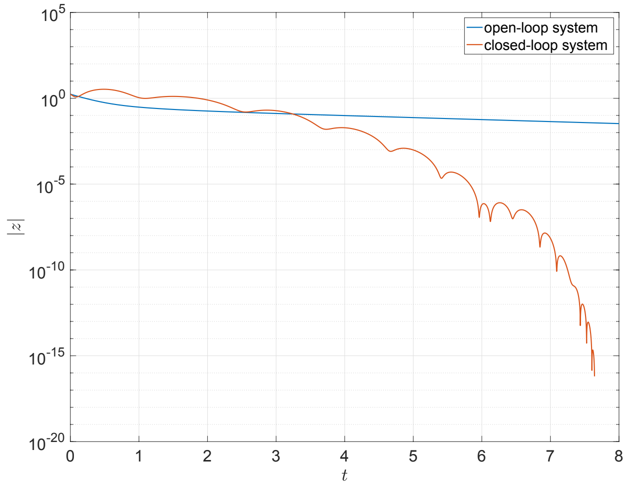

Fig. 3: Comparison of the decay rates of the open-loop and closed-loop systems in the logarithmic scale

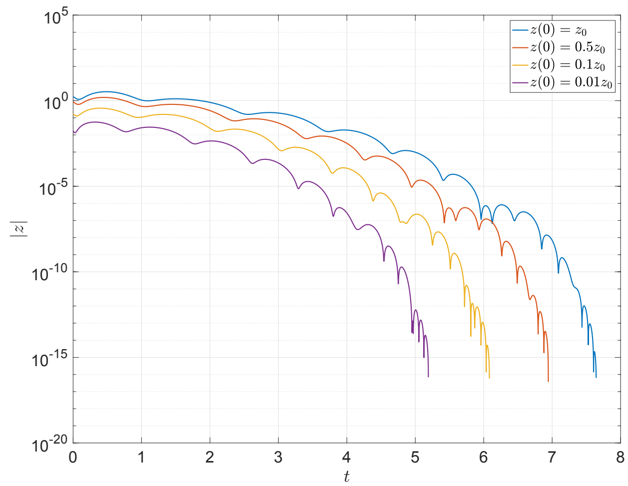

Fig. 4: Dependence of the settling time on the variation of the norm of the initial state

The comparison of the decay rates of the open-loop and the closed-loop systems is given on the Figure 3. The system with the homogeneous controller demonstrates the behavior typical for the finite-time stable system. According to simulation results the settling time for the chosen initial state $z(0) = z_0 := (1, -1, 0, 1, 0)^\top$ is about 8 units of time. The dependence of the settling time of the initial condition can be observed from the simulation results depicted on Figure 4.

VI. Conclusions

In this paper, an algorithm for the finite-time stabilizing control design is developed for the class of quasi-linear systems. The design procedure uses the linear approximation as the reference mode to design a homogeneous controller. The parameters of the controller are tuned in such a way that allow the stabilization of the original (quasi-linear) system as well. The convex embedding technique is utilized in order to derive the control parameters from linear matrix inequalities. The obtained LMIs are shown to be feasible under conditions of controllability of the linear approximation. However, they are still rather conservative. Further relaxation of the restriction to the quasi-linear system and derivation of less conservative LMIs are interesting problems for the future research.

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