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Finite-time control protocol for uniform allocation of second-order agents*

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Abstract— **The paper addresses the problem of uniform finite-time robust allocations of second-order agents on a straight line. A decentralized homogeneous control protocol is proposed that uses only agent's states and local interactions (distances between two closest neighbors). Sufficient conditions of finite-time (input-to-state) stability are proposed in the form of Linear Matrix Inequalities. The theoretical results are illustrated via numerical simulations.**

I. INTRODUCTION

Formation control is one of the most important and rapidly developing fields in multi-agent systems theory (see, e.g., [1], [2], [3]). Uniform allocation of agents on a straight line is a typical problem of formation control. When developing control protocols, it is desirable to strive the decentralization of the system and to ensure the autonomy of agents that allows one to avoid difficulties in case of network reorganizations and it does not require a central controller providing flexibility and overall cheapness. In the case of the mentioned uniform allocation problem, the described principles correspond to the control protocol realization based on local interactions between agents (see, e.g., [4], [5], [6]).

A finite-time control (i.e., a control that ensures completion of all transients in a finite time) can provide useful properties: faster convergence, higher accuracy, and better disturbance rejection (see, for example, [7], [8], [9]). In [5] the problem of fixed-time (the partial case of finite-time) uniform allocation is solved via the use of control protocol of a polynomial type that is a nonlinear extension of well known linear protocol [4]. However, this result is based on a very restrictive assumption that all agents do not have own dynamics (i.e., each agent has model of single integrator with control). Such an assumption in practice can lead to undesirable consequences up to the instability of the system. In [6] a linear control protocol is proposed for more realistic second order model of agents. In [10, Section 7.3] the fixed-time control protocol is provided for agents of second order, but the class of tolerable perturbations and the settling time are not evaluated.

In this paper we provide a new finite-time robust control protocol for the system with second order dynamics of agents. The proposed stability conditions are presented in the form of Linear Matrix Inequalities (LMIs) and based on the use of homogeneity property and the Implicit Lyapunov Function (ILF) method. Note that due to homogeneity the proposed control protocol provides such practically important robustness properties as Input-to-State Stability (ISS), non-Lipschitz disturbance rejection and delay robustness (see, e.g., [7], [11], [12], [13], etc.). In comparison with [10] the proposed result provides quantitative performance analysis and it allows to estimate the asymptotic gain and settling time, tune them based on LMIs; the class of disturbances that can be rejected is evaluated as well. Another advantage of the presented approach is that it can be straightforwardly extended to the agents of higher dimension and distributed observers.

The structure of this paper is as follows. Notations used in the work are introduced in Section II. Section III presents preliminaries used in the paper. The problem formulation is presented in Section IV. The robust finite-time control protocol for equidistant allocation of agents on a line is proposed in Section V. The results of numerical simulations are presented in Section VI. Finally, conclusions are given in Section VII.

II. NOTATION

Through the paper the following notation will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, where $\mathbb{R}$ is the field of real numbers;
- $0_{m\times n}$ denotes zero matrix with dimension $m \times n$;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix;
- $\mathrm{diag}\{\lambda_i\}_{i=1}^n$ is a diagonal matrix with elements λ_i;
- the symbol $\overline{1,m}$ is used to denote a sequence of integers $1,...,m$;
- the symbol $\otimes$ denotes the Kronecker product;
- $\mathscr{L}_\infty(\mathbb{R}^p)$ denotes the set of essentially bounded measurable functions $f : \mathbb{R}_+ \to \mathbb{R}^p$;
- a continuous function $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$.
- a continuous function $\beta : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to the class $\mathscr{KL}$ if $\beta(\cdot,r) \in \mathscr{K}$ and $\beta(r,\cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$;
- $e_i(s) = \left(\underbrace{0 \cdots 0 \overbrace{1}^{i^{\text{th}}} 0 \cdots 0}_{s \text{ components}}\right) \in \mathbb{R}^s$, $s \geq 1$, is a vector of the canonical basis of $\mathbb{R}^s$;
- for any real number $\alpha \geq 0$ and for all real x we define $\lfloor x \rceil^\alpha = \mathrm{sign}(x)|x|^\alpha$;

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- the order relation $P > 0$ (< 0; ≥ 0; ≤ 0) for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive (negative) definite (semidefinite);
- $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote maximum and minimum eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$.

III. PRELIMINARIES

A. Stability notions

Consider the following system:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous vector field and $f(0) = 0$.

Definition 1 [14], [15] *The origin of (1) is said to be globally finite-time stable if it is globally uniformly asymptotically stable and there exists a locally bounded function $T : \mathbb{R}^n \to [0,+\infty)$ such that any solution $x(t,x_0)$ of the system (1) for all $x_0 \in \mathbb{R}^n$ satisfies $x(t,x_0) = 0, \forall t \geq T(x_0)$. The function T is called a settling-time estimate.*

The following theorem presents the ILF method for finite-time stability analysis.

Theorem 1 [9] *If there exists a continuous function*

$$\begin{aligned} Q &: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \\ (V,x) &\mapsto Q(V,x) \end{aligned}$$

such that
C1) $Q(V,x)$ is continuously differentiable for $\forall x \in \mathbb{R}^n \backslash \{0\}$ and $\forall V \in \mathbb{R}_+$;
C2) for any $x \in \mathbb{R}^n \backslash \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$:

$$Q(V^-,x) < 0 < Q(V^+,x); \tag{2}$$

C3) for $\Omega = \{(V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0\}$

$$\lim_{\substack{x\to 0\\(V,x)\in\Omega}} V = 0^+, \ \lim_{\substack{V\to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \ \lim_{\substack{\|x\|\to\infty\\(V,x)\in\Omega}} V = +\infty;$$

C4) the inequality

$$-\infty < \frac{\partial Q(V,x)}{\partial V} < 0$$

holds for $\forall V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n \backslash \{0\}$;
C5) the inequality

$$\frac{\partial Q(V,x)}{\partial x} f(x) \leq \sigma V^{1+\nu} \frac{\partial Q(V,x)}{\partial V}$$

holds $\forall (V,x) \in \Omega$, where $\nu \in [-1,0)$ and $\sigma \in \mathbb{R}_+$ are some constants.
Then the origin of the system (1) is globally finite-time stable with the following settling time estimate

$$T(x_0) \leq \frac{V_0^{\mu}}{\sigma\mu},$$

where $V_0 \in \mathbb{R}_+ : Q(V_0,x_0) = 0$.

The conditions $C1) - C5)$ and the implicit function theorem [16] imply that the equation $Q(V,x) = 0$ implicitly defines a unique function $V : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ such that $Q(V(x),x) = 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and the function V satisfies the necessary Lyapunov function conditions described in [14] for finite-time stability analysis.

Let us give definition of input-to-state stability notion that is widely used for robustness analysis of nonlinear systems. Consider the system

$$\dot{x}(t) = f(x(t), d(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{3}$$

where $x \in \mathbb{R}^n$, $d \in \mathscr{L}_\infty(\mathbb{R}^p)$ is a disturbance and $f \in \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuous or discontinuous vector field that satisfies Filippov conditions [17].

Definition 2 [18] *The system (3) is called Input-to-state stable (ISS), if there exist functions $\zeta \in \mathscr{KL}$ and $\vartheta \in \mathscr{K}$ such that for any $d \in \mathscr{L}_\infty(\mathbb{R}^p)$ and any $x_0 \in \mathbb{R}^n$*

$$\|x(t,t_0,d)\| \leq \zeta(\|x_0\|,t) + \vartheta(\|d\|_{[0,t)}), \quad \forall t \geq 0.$$

B. Weighted homogeneity

For $r_i \in \mathbb{R}_+$, $i = \overline{1,n}$, and $\lambda > 0$ define vector of weights $r = [r_1,\ldots,r_n]^T$, dilation matrix $D_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$.

Definition 3 [19] *A function $g\colon \mathbb{R}^n \to \mathbb{R}$ (vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$) is said to be weighted homogeneous of degree $\nu \in \mathbb{R}$ if*

$$\begin{gathered} g(D_r(\lambda)x) = \lambda^\nu g(x) \\ (f(D_r(\lambda)x) = \lambda^\nu D_r(\lambda) f(x)) \end{gathered}$$

for fixed r, all $\lambda > 0$ and $x \in \mathbb{R}^n$.

Note that the weighted homogeneity is a partial case of the linear geometric one (see, e.g., [20], [21]), and from [20, Theorem 2] for $0 < P \in \mathbb{R}^{n\times n}$ such that

$$P\mathrm{diag}\{r_i\}_{i=1}^n + \mathrm{diag}\{r_i\}_{i=1}^n P > 0$$

one can obtain

$$\begin{aligned} \lambda^{\rho_1} \leq \|D_r(\lambda)\|_P \leq \lambda^{\rho_2} \quad if \quad \lambda \leq 1, \\ \lambda^{\rho_2} \leq \|D_r(\lambda)\|_P \leq \lambda^{\rho_1} \quad if \quad \lambda \geq 1, \end{aligned} \tag{4}$$

where $\|x\|_P = \sqrt{x^T P x}$, $\|D_r(\lambda)\|_P = \sup_{x\neq 0} \frac{\|D_r(\lambda)x\|_P}{\|x\|_P}$ and

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}\mathrm{diag}\{r_i\}_{i=1}^n P^{-\frac{1}{2}} + P^{-\frac{1}{2}}\mathrm{diag}\{r_i\}_{i=1}^n P^{\frac{1}{2}}\right), \\ \rho_2 &= \tfrac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}\mathrm{diag}\{r_i\}_{i=1}^n P^{-\frac{1}{2}} + P^{-\frac{1}{2}}\mathrm{diag}\{r_i\}_{i=1}^n P^{\frac{1}{2}}\right). \end{aligned} \tag{5}$$

The homogeneity property implies robustness to external perturbations, measurement noise (see, for example, [11]) and delays [12], [13].

IV. PROBLEM FORMULATION

Let us consider a group of n mobile agents, where each agent is described by linear model

$$\dot{x}_i = Ax_i + Bu_i + d_i(t,x), \quad i = \overline{1,n}, \tag{6}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$x_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2$ is the state vector of i-th agent, $u_i \in \mathbb{R}$ is the control input, $d_i(t,x) \in \mathbb{R}^2$ describes exogenous disturbances and uncertainties. Let $x_{01}, x_{(n+1)1} \in \mathbb{R}$ denote the fixed endpoints of the segment.

The main goal is to design a decentralized control protocol which:

- provides finite-time equidistant allocation of the agents on a given line between the points x_{01}, $x_{(n+1)1}$;
- exploits own states and the information only about the distances between "successor" and "predecessor" in the formation, i.e.,

$$u_i = u_i(x_i, x_{(i-1)1} - x_{i1}, x_{(i+1)1} - x_{i1}), \quad i = \overline{1,n};$$

- provides robust properties with respect to exogenous disturbances.

V. MAIN RESULT

Consider the following nonlinear control protocol

$$u_i = -k_{i1} \lfloor x_{i2} \rceil^{1+\nu} - k_{i2} \lfloor \phi_i(x) \rceil^{\frac{1+\nu}{1-\nu}}, \quad i = \overline{1,n}, \tag{7}$$

where $\phi_i(x) = \frac{1}{2}(x_{i1} - x_{(i-1)1}) + \frac{1}{2}(x_{i1} - x_{(i+1)1})$, $k_{i1}, k_{i2} \in \mathbb{R}_+$ and $\nu \in (-1, 0)$. Note that for $\nu = 0$ the provided control protocol became the linear one considered in [6].

Let $x = [x_1, x_2, \ldots, x_n]^T$ be the state vector of the multiagent system. Then the dynamics of the overall system can be written in compact form as

$$\dot{x} = A_0 x + A_1 f(Ux - b) + d(t, x),$$

where $A_0 = I_n \otimes A$, $A_1 = \mathrm{diag}\{\tilde{A}_i\}_{i=1}^n$, $\tilde{A}_i = \begin{bmatrix} 0 & 0 \\ -k_{i2} & -k_{i1} \end{bmatrix}$, $d(\cdot,\cdot) = [d_1^T(\cdot,\cdot), d_2^T(\cdot,\cdot), \ldots, d_n^T(\cdot,\cdot)]^T \in \mathbb{R}^{2n}$,

$$U = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -0.5 & 0 & 1 & 0 & -0.5 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & -0.5 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

$b = [0.5x_{01}, 0, 0, \ldots, 0, 0.5x_{(n+1)1}, 0]^T \in \mathbb{R}^{2n}$, $f(x) = [f_1(x), f_2(x), \ldots, f_{2n}(x)]^T$, $f_i(x) = \lfloor e_i(2n)(Ux+b) \rceil^{\gamma_i}$, $\gamma_i = \begin{cases} \frac{1+\nu}{1-\nu}, & \text{if } i \text{ is odd} \\ 1+\nu, & \text{if } i \text{ is even} \end{cases}$ for $i = \overline{1,2n}$.

Note that the matrix U is invertible. Then let us introduce the new variable

$$z = x - U^{-1} b.$$

Since $A_0 = A_0 U$ (i.e., $A_0 U^{-1} = A_0$) we have

$$\dot{z} = A_0 z + A_1 f(Uz) + d(t, z + U^{-1} b), \quad z_0 = z(0), \tag{8}$$

where the origin is an equilibrium point for the system (8).

Note that the system (8) is homogeneous of degree ν with the vector of weights $r = [1-\nu, 1, 1-\nu, 1, \ldots, 1-\nu, 1]^T \in \mathbb{R}^{2n}$ and corresponding dilation matrix $D_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^{2n}$.

Let us introduce an ILF candidate in the form

$$\begin{aligned} Q(V,z) = {} & z^T D_r(V^{-1}) P D_r(V^{-1}) z \\ & + 2\sum_{i=1}^{2n} \frac{\Lambda_i}{\gamma_i + 1} \left| e_i(2n) U D_r(V^{-1}) z \right|^{\gamma_i + 1} - 1, \end{aligned} \tag{9}$$

where $\Lambda = \mathrm{diag}\{\Lambda_i\}_{i=1}^{2n} \in \mathbb{D}_+^{2n}$, $P \in \mathbb{R}^{2n \times 2n}$ is a symmetric positive semi-definite matrix.

Denote the matrix $H = \mathrm{diag}\{r_i\}_{i=1}^{2n}$ and $\tilde{\Lambda} = \Lambda \mathrm{diag}\{\frac{1}{\gamma_i + 1}\}_{i=1}^{2n}$. Now we are ready to present our main result.

Theorem 2 *Let $\alpha_1, \alpha_2, \iota_1, \iota_2, \beta, \psi \in \mathbb{R}_+$ be chosen such that the LMI system*

$$P \geq U \tilde{P} U; \tag{10a}$$

$$\iota_1 P + U \Lambda U > 0; \tag{10b}$$

$$PH + HP \geq 0; \tag{10c}$$

$$U \Lambda U + \iota_2 (PH + HP) > 0; \tag{10d}$$

$$S \geq \mathrm{diag}\left\{ (1+\psi)^{\frac{1}{1-\gamma_i}} \right\}_{i=1}^{2n}; \tag{10e}$$

$$\beta \geq \mathrm{trace}\left(S \,\mathrm{diag}\left\{ \left(\frac{\gamma_i}{\psi}\right)^{\frac{2}{1-\gamma_i}} \left(\frac{\psi}{\gamma_i} - \psi\right)^2 \right\}_{i=1}^{2n} \right); \tag{10f}$$

$$Q_1 + \alpha_1 Q_2 + Q_3 + \alpha_2 Q_4 \leq 0 \tag{10g}$$

be feasible for $\Lambda, \tilde{P}, \Upsilon \in \mathbb{D}_+^{2n}$, $P, \Gamma \geq 0$, $S = \tilde{P} + 2\tilde{\Lambda}$, where for $\tilde{A}_0 = A_0 + \psi A_1 U$ and $X = P + \psi U^T \Lambda U$

$$Q_1 = \begin{bmatrix} X\tilde{A}_0 + \tilde{A}_0^T X & XA_1 + \tilde{A}_0^T U \Lambda & X \\ \Lambda U \tilde{A}_0 + A_1^T X & \Lambda A_1 + A_1^T \Lambda & \Lambda U \\ X & U\Lambda & -\Gamma \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} XH + HX & U\Lambda H & O_{2n\times 2n} \\ H\Lambda U & O_{2n\times 2n} & O_{2n\times 2n} \\ O_{2n\times 2n} & O_{2n\times 2n} & O_{2n\times 2n} \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} \beta(P + 2\psi U\tilde{\Lambda}U) - 2U\Upsilon U & \beta U\tilde{\Lambda} + U\Upsilon & O_{2n\times 2n} \\ \beta\tilde{\Lambda}U + \Upsilon U & -S & O_{2n\times 2n} \\ O_{2n\times 2n} & O_{2n\times 2n} & O_{2n\times 2n} \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} P + 2U\tilde{\Lambda}U & U\tilde{\Lambda} & O_{2n\times 2n} \\ \tilde{\Lambda}U & O_{2n\times 2n} & O_{2n\times 2n} \\ O_{2n\times 2n} & O_{2n\times 2n} & O_{2n\times 2n} \end{bmatrix},$$

and let

$$d^T D_r(V^{-1}) \Gamma D_r(V^{-1}) d \leq \alpha_2 V^{2\nu} \tag{11}$$

for $V \in \mathbb{R}_+ : Q(V,x) = 0$.

Then the protocol (7) establishes finite-time allocation of the agents (6) along the line segment with endpoints x_{01} and $x_{(n+1)1}$ and $T(z_0) \leq -\frac{V_0^{-\nu}}{\alpha_1 \nu}$, where $V_0 \in \mathbb{R}_+ : Q(V_0, z_0) = 0$.

Remark 1 In order to satisfy the condition (10) the matrix X serve as a solution of the Lyapunov equation for the matrix $\tilde{A}_0$. Note that since A_0 is not Hurwitz, in line with the paper [22], the following representation of (8)

$$\begin{aligned} & \dot{z} = (A_0 + A_1 \Psi U) z + A_1 \hat{f}(Uz) + d(t, z + U^{-1} b), \\ & \hat{f}(Uz) = f(Uz) - \Psi Uz, \; X = P + U^T \Lambda \Psi U, \\ & \Psi = \psi I_{2n}, \; \psi \in \mathbb{R}_+ \end{aligned} \tag{12}$$

is used here to obtain stable linear term $\tilde{A}_0 z$ (e.g., in [6, Theorem 1] it is shown that the matrix $\tilde{A}_0$ is Hurwitz for the case $\psi = 1$, $k_{i2} = 1$ and $k_{i1} > 0$).

Remark 2 The implicit restriction (11) to the system disturbances and uncertainties can be replaced by explicit (but more conservative) inequality

$$\|d\|^2 \leq \frac{\alpha_2}{\kappa} \begin{cases} \|z\|^{(2+2\nu)/\rho_1} & \text{if } \|z\| \geq 1 \\ \|z\|^{2/\rho_2} & \text{if } \|z\| < 1 \end{cases} \tag{13}$$

under additional condition

$$PH + HP > 0$$

and $\Gamma = \kappa I_{2n}$, $\kappa \in \mathbb{R}_+$, where $\rho_1, \rho_2 \in \mathbb{R}_+$ are defined in (5). Indeed, according to (9) we have

$$\|z\|^2 \leq \begin{cases} V^{2\rho_1} & \text{if } V \geq 1 \\ V^{2\rho_2} & \text{if } V < 1 \end{cases}$$

and taking into account that the inequality

$$\|d\|^2 \leq \frac{\alpha_2}{\kappa} \begin{cases} V^{2+2\nu} & \text{if } V \geq 1 \\ V^{2} & \text{if } V < 1 \end{cases}$$

implies (11), we obtain that (13) implies (11).

Since stable homogeneous systems are ISS with respect to additive perturbations and measurement noises the following corollary can be obtained:

Corollary 1 *Let $\beta, \psi, \iota_1, \iota_2, \in \mathbb{R}_+$ be chosen such that the LMI system*

$$P \geq U\tilde{P}U; \quad (14a)$$

$$\iota_1 P + U\Lambda U > 0; \quad (14b)$$

$$PH + HP \geq 0; \quad (14c)$$

$$U\Lambda U + \iota_2(PH + HP) > 0; \quad (14d)$$

$$S \geq \operatorname{diag}\left\{(1+\psi)^{\frac{1}{1-\gamma_i}}\right\}_{i=1}^{2n}; \quad (14e)$$

$$\beta \geq \operatorname{trace}\left(S \operatorname{diag}\left\{\left(\frac{\gamma_i}{\psi}\right)^{\frac{2}{1-\gamma_i}}\left(\frac{\psi}{\gamma_i} - \psi\right)^2\right\}_{i=1}^{2n}\right); \quad (14f)$$

$$Q_1 + Q_2 < 0 \quad (14g)$$

be feasible for $\Lambda, \tilde{P}, \Upsilon \in \mathbb{D}_+^{2n}$, $P \geq 0$, $S = \tilde{P} + 2\tilde{\Lambda}$, where for $\tilde{A}_0 = A_0 + \psi A_1 U$ and $X = P + \psi U^T \Lambda U$

$$Q_1 = \begin{bmatrix} X\tilde{A}_0 + \tilde{A}_0^T X & XA_1 + \tilde{A}_0^T U\Lambda \\ \Lambda U\tilde{A}_0 + A_1^T X & \Lambda A_1 + A_1^T \Lambda \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} \beta(P + 2\psi U\tilde{\Lambda}U) - 2U\Upsilon U & \beta U\tilde{\Lambda} + U\Upsilon \\ \beta\tilde{\Lambda}U + \Upsilon U & O_{2n\times 2n} \end{bmatrix}.$$

Then the system (8) is ISS with respect to $d \in \mathscr{L}_\infty(\mathbb{R}^{2n})$ and measurement noise. For the noise- and disturbance-free case the origin is globally finite-time stable.

Remark 3 Since the proposed finite-time protocol is coordinate-wise decoupled, the results are applicable for the case where the state of each agent is multidimensional (i.e., for "multidimensional agents").

Remark 4 Note that the system (8) with $U = I_{2n}$ is relevant for some other multi-agent systems applications (see, e.g., [23], [24] on finite-time consensus algorithms). In this case, the presented control and finite-time stability analysis can be adapted for such applications also, providing advantages with respect to [23], [24] such as it allows to consider unmatched disturbances; to tune settling time; and to provide stability analysis based on LMIs.

VI. Numerical Example

To demonstrate the efficiency of the proposed finite-time control protocol we consider the multi-agent system (6), where $n = 3$, $x_{01} = 0$, $x_{41} = 5$. Fig. 1 presents the simulation results in the disturbance-free case ($d_i \equiv 0$) for the linear [6] and the proposed finite-time control protocols, where $k_{i1} = k_{i2} = 1$, $\nu = -0.5$. The stability property has been checked on the basis of Corollary 1 with $\beta = 1, \psi = 5$, $\iota_1 = \iota_2 = 1$. The initial conditions were chosen $x_1(0) = [5\ 1]^T$, $x_2(0) = x_3(0) = [0\ 0]^T$. Fig.2 depicts $V(t)$ in a logarithmic scale in order to demonstrate finite-time convergence. In order to calculate the function $V(z)$ implicitly defined by (9) the bisection method was utilized (see, e.g., [9, Algorithm 1]).

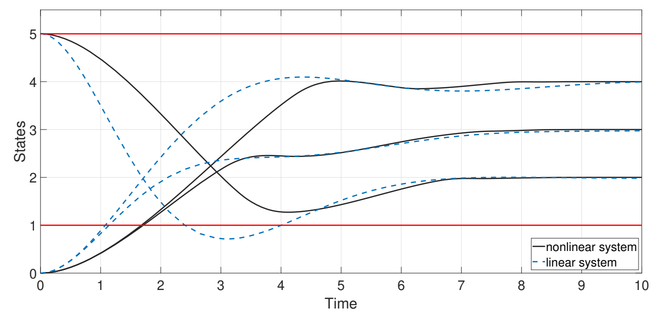

Fig. 1. Trajectories of the system states x_{i1} subject to the linear and nonlinear control protocols (disturbance-free case)

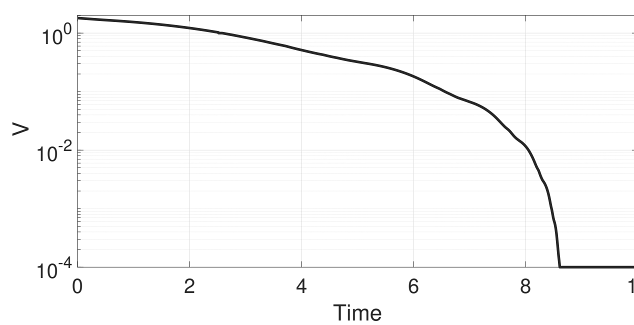

Fig. 2. Lyapunov function for the finite-time control protocol (disturbance-free case)

As mentioned in Remark 3, the proposed finite-time protocol is applicable for the space of arbitrary dimension. Fig. 3 (disturbance-free case) and Fig. 4 (the case with bounded additive disturbances and measurement noises) present the simulation results for the problem of uniform deployment for agents on the plane

$$\ddot{\xi}_i = u_i + d_i(t), \quad \xi_i = [x_i\ y_i]^T \in \mathbb{R}^2, \quad i = \overline{1,5}$$

with $\xi_{01} = [-2\ 2]^T$, $\xi_{61} = [2\ 3]^T$, $d_i(t) = 0.1\sin(t)$ and measurement band limited noise of power 10^{-3}.

VII. Conclusion

This paper is devoted to robust finite-time equidistant allocation of second-order agents on a straight line. The

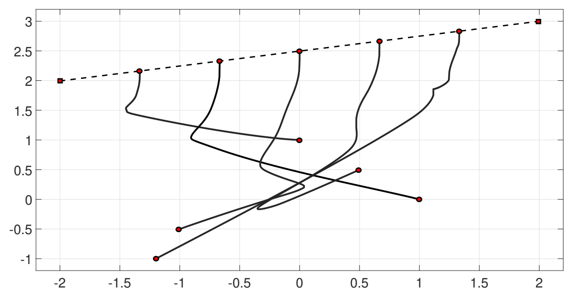

Fig. 3. Agent trajectories (disturbance-free case)

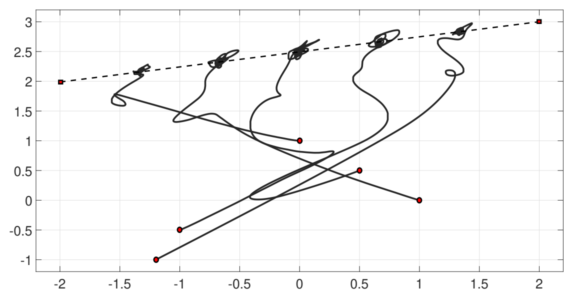

Fig. 4. Agent trajectories in the presense of bounded additive disturbances and measurement noises

protocol allows to reject a class of exogenous disturbances satisfying (11). Also it provides ISS properties with respect to additive disturbances and measurement noises. The protocol is applicable for multidimensional agents as well. The theoretical results were successfully tested through several numerical experiments. In comparison with other papers the proposed result provides quantitative performance analysis (e.g., asymptotic gain and settling time estimation, tolerable perturbations evaluation) and LMI-based stability analysis.

The proposed ILF-based stability analysis is promising for other distributed homogeneity-based control applications, which is the main direction for future research.

References

[1] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues*. Springer, London, 2011.

[2] W. Dong and A. Farrell, "Decentralized cooperative control of multiple nonholonomic dynamic systems with uncertainty," *Automatica*, vol. 45, no. 3, pp. 706–710, 2009.

[3] A. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[4] I. Wagner and A. Bruckstein, "Row straightening via local interactions," *Circuits Systems Signal Processing*, vol. 16(2), pp. 287–305, 1997.

[5] S. Parsegov, A. Polyakov, and P. Shcherbakov, "Nonlinear fixed-time control protocol for uniform allocation of agents on a segment," *Doklady Mathematics*, vol. 87, pp. 133–136, 2013.

[6] S. Parsegov and A. Proskurnikov, "Uniform deployment of second-order agents on a line segment," *Proc. of 6th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT)*, pp. 631–636, 2014.

[7] A. Polyakov, *Generalized homogeneity in systems and control*. Springer, 2020.

[8] L. Rosier, "Homogeneous lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.

[9] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: Implicit lyapunov function approach," *Automatica*, vol. 51, pp. 332–340, 2015.

[10] A. Aleksandrov, D. Efimov, and S. Dashkovskiy, "Design of finite-/fixed-time iss-lyapunov functions for mechanical systems," *Math. Control Signals Syst.*, vol. 35, pp. 215–235, 2023.

[11] E. Bernuau, A. Polyakov, D. Efimov, and P. W., "Verification of ISS, iISS and IOSS properties applying weighted homogeneity," *System and Control Letters*, vol. 62, no. 12, pp. 1159–1167, 2013.

[12] D. Efimov, A. Polyakov, W. Perruquetti, and J.-P. Richard, "Weighted homogeneity for time-delay systems: Finite-time and independent of delay stability," *IEEE Trans. Automatic Control*, vol. 61, no. 1, pp. 210–215, 2016.

[13] K. Zimenko, D. Efimov, A. Polyakov, and W. Perruquetti, "A note on delay robustness for homogeneous systems with negative degree," *Automatica*, vol. 79, pp. 178–184, 2017.

[14] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[15] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43(4), pp. 1253–1271, 2004.

[16] R. Courant and F. John, *Introduction to calculus and analysis II/1*. New York: Springer, 2000.

[17] A. Filippov, *Differential equations with discontinuous right-hand sides*. Kluwer, Dordrecht, 1988.

[18] S. Dashkovskiy, D. Efimov, and E. Sontag, "Input to state stability and allied system properties," *Automation and Remote Control*, vol. 72, no. 8, pp. 1579–1614, 2011.

[19] V. I. Zubov, "Systems of ordinary differential equations with generalized-homogeneous right-hand sides," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, no. 1, pp. 80–88, 1958.

[20] A. Polyakov, "Sliding mode control design using canonical homogeneous norm," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 682–701, 2019.

[21] K. Zimenko, A. Polyakov, D. Efimov, and W. Perruquetti, "Robust feedback stabilization of linear MIMO systems using generalized homogenization," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5429–5436, 2020.

[22] K. Zimenko, D. Efimov, A. Polyakov, and A. Kremlev, "Finite-time stability analysis of homogeneous systems with sector nonlinearities," *Automatica*, submitted.

[23] S. Li, H. Du, and X. Lin, "Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics," *Automatica*, vol. 47, no. 8, pp. 1706–1712, 2011.

[24] Z. Zuo, B. Tian, M. Defoort, and Z. Ding, "Fixed-time consensus tracking for multiagent systems with high-order integrator dynamics," *IEEE Transactions on Automatic Control*, vol. 63, pp. 563–570, 2018.