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Event-triggered robust stabilization by using fast-varying square wave dithers

Jin Zhang, Zhihao Zhang and Emilia Fridman

***Abstract*— This paper is concerned with event-triggered robust static output-feedback stabilization of the second-order linear uncertain systems by a fast-varying square wave with high gain. Recently, a constructive time-delay approach for designing such a fast-varying output-feedback controller was suggested by using continuous measurements. In the present paper, we employ an event-trigger (ET) based on switching approach that determines the measurement transmission instants for this design. For the resulting switching system, we construct appropriate coordinate transformations that cancel the high gains and apply the time-delay approach to periodic averaging of the system in new coordinates. By employing appropriate Lyapunov functionals, we derive linear matrix inequalities (LMIs) for finding an efficient upper bound on the square wave frequency that guarantees the stability of the original systems. Numerical examples illustrate the efficiency of the method.**

I. Introduction

Stabilization of linear/nonlinear systems by employing vibrational control with a fast-varying dither (with zero mean value) that depends on a small parameter was studied in [1], [2], [3], [4]. The latter works rely on coordinate transformation that allows to transform the system to a standard form for application of averaging. A related to vibrational control is Brockett's problem of stabilization by static output-feedback with a time-varying gain in [5], where the system is not stabilizable by constant gain. Some solutions to this problem were provided by [6], [7]. However, the above results do not provide an efficient upper bound on the small parameter that guarantees the stability.

Recently, robust stabilization of linear uncertain systems by using a static time-varying output-feedback was studied in [8]. This was done by employing the coordinate transformation [1] that cancels the high gain and leads to a stable averaged system, and the time-delay approach to periodic averaging [9] that allows to present constructive quantitative results for finding an upper bound on the small parameter while ensuring the exponential stability. Besides, the time-delay approach to averaging was applied to power systems [10] and extended to L_2-gain analysis [11], extremum seeking [12], [13], [14] as well as discrete-time counterpart [15].

It should be noticed that the controller in [8] depends on the continuous measurement leading to "redundant" transmissions. For a reduction of network load, several event-triggers (ETs) were suggested in the literature, e.g. continuous ET [16], periodic ET [17], [18], [19] and ET via switching approach [20]. Note that the switching ET allows to i) avoid Zeno phenomenon in continuous ET; and ii) reduce transmission of measurement in periodic sampling.

In this paper, we study event-triggered robust stabilization of the second-order linear uncertain systems by a fast-varying square wave with high gain. First, we employ ET based on switching approach from [20] to determine the measurement transmission instants. Based on the triggered measurements, we design a new controller that leads to a switching system. Following [8], we employ the coordinate transformation to cancel the high gain, and apply the time-delay approach to periodic averaging of the system in new coordinates. Note that extension to ET based on switching approach is not straightforward since in the Lyapunov analysis we have to compensate additional errors due to ET based on switching approach. To compensate these errors, we construct additional terms for the corresponding Lyapunov functionals that lead to LMI conditions. Finally, we present numerical examples to illustrate the efficiency of the method.

II. Problem formulation

Consider a linear uncertain system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + Bu(t), \quad y(t) = Cx(t), \quad (1)$$

where $x(t) \in \mathbb{R}^2$ is the state, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the measurement, and

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}. \quad (2)$$

Here $a_1 \geq 0$ (implying that A is not Hurwitz), $a_2 < 0$, b, c_1 and $c_2 \neq 0$ are constants. The time-varying uncertainty $\Delta A(t) \in \mathbb{R}^{n\times n}$ satisfies the following inequality

$$\|\Delta A(t)\| \leq \sigma_0 \quad \forall t \geq 0 \quad (3)$$

with a small constant $\sigma_0 > 0$.

System (1), (2) with $\Delta A(t) = 0$ may be not stabilizable by a static time-invariant output-feedback. Note that robust stabilization of system (1), (2) was studied in [8] by using a static time-varying output-feedback

$$u(t) = \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) y(t), \quad t \geq 0, \quad (4)$$

where the square wave dither has the form

$$\mathrm{sq}(\tfrac{t}{\varepsilon}) = \mathrm{sgn}\cos(\tfrac{2\pi t}{\varepsilon}) = \begin{cases} 1, & \tfrac{t}{\varepsilon} \in [j, j+\tfrac{1}{4}), \\ -1, & \tfrac{t}{\varepsilon} \in [j+\tfrac{1}{4}, j+\tfrac{3}{4}), \quad j \in \mathbb{N}_0. \\ 1, & \tfrac{t}{\varepsilon} \in [j+\tfrac{3}{4}, j+1), \end{cases} \quad (5)$$

Here k is a scalar controller gain and $\varepsilon > 0$ is a small parameter that is inverse of the dither frequency. Note that

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Jin Zhang and Zhihao Zhang are with the School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200444, China (e-mail address: zhangjin1116@126.com; zhangzhihao@shu.edu.cn).

Emilia Fridman is with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail address: emilia@eng.tau.ac.il).

system (1), (2) can be exponentially stabilizable by (4) with appropriate k and small enough $\varepsilon > 0$ iff the following holds [21]: $a_2 < 0$ and

$$a_1c_2^2 - a_2c_1c_2 - c_1^2 < 0. \tag{6}$$

The static output-feedback (4) depends on the continuous measurement $y(t)$. For practical application of the static output-feedback (4), we consider its implementation by employing an ET via switching approach from [20]. Then the measurement $y(t)$ is available only at the discrete-time instants t_ℓ ($\ell \in \mathbb{N}_0$), where $t_0 = 0$ and t_ℓ is determined by

$$t_{\ell+1} = \min_{t \geq t_\ell + \varepsilon^2 h}\{|y(t) - y(t_\ell)|^2 \geq \varsigma\varepsilon^2|y(t)|^2\} \tag{7}$$

with scalars $\varsigma \geq 0$ and $h > 0$. Clearly, the minimum inter-event interval is $\varepsilon^2 h$ implying that there is no Zeno phenomenon. Based on the triggered measurement $y(t_\ell)$, in this paper we design the following controller

$$u(t) = \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) y(t_\ell), \quad t \in [t_\ell, t_{\ell+1}), \quad \ell \in \mathbb{N}_0. \tag{8}$$

The resulting closed-loop system has the following form

$$\begin{aligned}\dot{x}(t) = [A + \Delta A(t)]x(t) + \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) By(t_\ell),\\ t \in [t_\ell, t_{\ell+1}), \quad \ell \in \mathbb{N}_0.\end{aligned} \tag{9}$$

We present

$$\begin{aligned}&y(t_\ell) = y(t) - \textstyle\int_{t_\ell}^t \dot{y}(s)ds = Cx(t) - C\int_{t-\tau(t)}^t \dot{x}(s)ds,\\ &\tau(t) = t - t_\ell \in [0, \varepsilon^2 h], \quad t \in [t_\ell, t_\ell + \varepsilon^2 h), \quad \ell \in \mathbb{N}_0,\end{aligned} \tag{10}$$

and

$$\begin{aligned}y(t_\ell) = y(t) - \underbrace{[y(t) - y(t_\ell)]}_{e(t)} = Cx(t) - e(t),\\ t \in [t_\ell + \varepsilon^2 h, t_{\ell+1}), \quad \ell \in \mathbb{N}_0.\end{aligned} \tag{11}$$

Thus, similar to [20] we rewrite the closed-loop system (9) as the following switched system

$$\begin{aligned}\dot{x}(t) = {}&[A + \Delta A(t)]x(t) + \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) BCx(t)\\ &-\chi(t)\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) BC\textstyle\int_{t-\tau(t)}^t \dot{x}(s)ds\\ &+(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) Be(t), \quad t \geq 0,\end{aligned} \tag{12}$$

where

$$\chi(t) = \begin{cases}1, & t \in [t_\ell, t_\ell + \varepsilon^2 h),\\ 0, & t \in [t_\ell + \varepsilon^2 h, t_{\ell+1}),\end{cases} \quad \ell \in \mathbb{N}_0. \tag{13}$$

To cancel the large term $\frac{k}{\varepsilon}\mathrm{sq}(\frac{t}{\varepsilon})BCx(t)$ in (12), inspired by [1] we consider the following generating equation

$$\tfrac{d}{dt}\phi\left(\tfrac{t}{\varepsilon}\right) = \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) BC\phi\left(\tfrac{t}{\varepsilon}\right), \quad t \geq 0. \tag{14}$$

The fundamental matrix $\Phi(\frac{t}{\varepsilon}, 0)$ of equation (14) is obtained as

$$\begin{aligned}\Phi\left(\tfrac{t}{\varepsilon}, 0\right) &= e^{k\rho(\frac{t}{\varepsilon})BC}\\ &= \begin{bmatrix}1 & 0\\ \frac{c_1}{c_2}e^{kbc_2\rho(\frac{t}{\varepsilon})} - \frac{c_1}{c_2} & e^{kbc_2\rho(\frac{t}{\varepsilon})}\end{bmatrix}, \quad t \geq 0,\end{aligned} \tag{15}$$

where

$$\rho\left(\tfrac{t}{\varepsilon}\right) = \begin{cases}\frac{t}{\varepsilon} - j, & \frac{t}{\varepsilon} \in [j, j + \frac{1}{4}),\\ -\frac{t}{\varepsilon} + j + \frac{1}{2}, & \frac{t}{\varepsilon} \in [j + \frac{1}{4}, j + \frac{3}{4}),\\ \frac{t}{\varepsilon} - j - 1, & \frac{t}{\varepsilon} \in [j + \frac{3}{4}, j + 1),\end{cases} \quad j \in \mathbb{N}_0. \tag{16}$$

It is clear that $\rho(\frac{t}{\varepsilon})$ (and thus $\Phi(\frac{t}{\varepsilon}, 0)$ in (15)) is ε-periodic. Note that $\|\Phi(\frac{t}{\varepsilon}, 0)\|$ and $\|\Phi^{-1}(\frac{t}{\varepsilon}, 0)\|$ are uniformly bounded for all $t \geq 0$. Assume that

A1 there exists positive constants ψ_i $(i = 1, 2)$ (independent of $\varepsilon \in (0, \varepsilon^*]$) satisfying

$$|C\Phi\left(\tfrac{t}{\varepsilon}, 0\right)| \leq \psi_1, \quad |\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) B| \leq \psi_2. \tag{17}$$

Introduce next the coordinate transformation

$$x(t) = \Phi\left(\tfrac{t}{\varepsilon}, 0\right)\zeta(t), \quad t \geq 0. \tag{18}$$

This coordinate transformation is stability preserving since $\|\Phi(\frac{t}{\varepsilon}, 0)\|$ and $\|\Phi^{-1}(\frac{t}{\varepsilon}, 0)\|$ are uniformly bounded for all $t \geq 0$. Taking the derivative with respect to t in (18) and using the relation $\frac{d}{dt}\Phi(\frac{t}{\varepsilon}, 0) = \frac{k}{\varepsilon}\mathrm{sq}(\frac{t}{\varepsilon})BC\Phi(\frac{t}{\varepsilon}, 0)$ for all $t \geq 0$, we obtain

$$\dot{x}(t) = \tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right) BC\Phi\left(\tfrac{t}{\varepsilon}, 0\right)\zeta(t) + \Phi\left(\tfrac{t}{\varepsilon}, 0\right)\dot{\zeta}(t), \quad t \geq 0. \tag{19}$$

Substituting the right-hand of (19) for $\dot{x}(t)$ in (12) and taking into account that matrix $\Phi\left(\frac{t}{\varepsilon}, 0\right)$ is nonsingular for all $t \geq 0$, we obtain

$$\begin{aligned}\dot{\zeta}(t) &= \Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right)[A + \Delta A(t)]\Phi\left(\tfrac{t}{\varepsilon}, 0\right)\zeta(t)\\ &\quad -\chi(t)\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) BC\textstyle\int_{t-\tau(t)}^t \dot{x}(s)ds\\ &\quad +(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) Be(t)\\ &= [\mathcal{A}\left(\tfrac{t}{\varepsilon}\right) + \Delta\mathcal{A}(t)]\zeta(t) + \chi(t)[\delta_1(t) + \delta_2(t)]\\ &\quad +(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) Be(t), \quad t \geq 0,\end{aligned} \tag{20}$$

where

$$\begin{aligned}&\mathcal{A}\left(\tfrac{t}{\varepsilon}\right) = \Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) A\Phi\left(\tfrac{t}{\varepsilon}, 0\right),\\ &\Delta\mathcal{A}(t) = \Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right)\Delta A(t)\Phi\left(\tfrac{t}{\varepsilon}, 0\right),\\ &\delta_1(t) = -\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) BC\textstyle\int_{t-\tau(t)}^t \Phi\left(\tfrac{s}{\varepsilon}, 0\right)\dot{\zeta}(s)ds,\\ &\delta_2(t) = -\tfrac{k^2}{\varepsilon^2}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right)(BC)^2\\ &\qquad\times\textstyle\int_{t-\tau(t)}^t \mathrm{sq}\left(\tfrac{s}{\varepsilon}\right)\Phi\left(\tfrac{s}{\varepsilon}, 0\right)\zeta(s)ds.\end{aligned} \tag{21}$$

Note that $\delta_1(t)$ and $\delta_2(t)$ are of the order $O(\varepsilon h)$ and $O(h)$, respectively, provided ζ and $\dot{\zeta}$ are both of order $O(1)$.

The averaged system of (20) with $\Delta A(t) = 0$, $h \to 0$, $\varepsilon \to 0$ and $\sigma \to 0$ (i.e. $\delta_1(t) \to 0$, $\delta_2(t) \to 0$ and $e(t) \to 0$) has the following form

$$\dot{\zeta}_{av}(t) = \mathcal{A}_{av}\zeta_{av}(t), \quad t \geq 0, \tag{22}$$

where $\zeta_{av}(t) \in \mathbb{R}^2$ and

$$\begin{aligned}&\mathcal{A}_{av} = \tfrac{1}{\varepsilon}\textstyle\int_0^\varepsilon \mathcal{A}\left(\tfrac{s}{\varepsilon}\right) ds = \begin{bmatrix}\mathbf{a}_{11} & \frac{4}{kbc_2}\sinh\left(\frac{kbc_2}{4}\right)\\ \mathbf{a}_{21} & \mathbf{a}_{22}\end{bmatrix},\\ &\mathbf{a}_{11} = \tfrac{4c_1}{kbc_2^2}\sinh\left(\tfrac{kbc_2}{4}\right) - \tfrac{c_1}{c_2},\\ &\mathbf{a}_{21} = \tfrac{4(a_1c_2^2 - a_2c_1c_2 - 2c_1^2)}{kbc_2^3}\sinh\left(\tfrac{kbc_2}{4}\right) + \tfrac{a_2c_1}{c_2} + \tfrac{2c_1^2}{c_2^2},\\ &\mathbf{a}_{22} = a_2 - \tfrac{4c_1}{kbc_2^2}\sinh\left(\tfrac{kbc_2}{4}\right) + \tfrac{c_1}{c_2}.\end{aligned} \tag{23}$$

Then the following result holds:

Proposition 1: [8] Let $a_2 < 0$. Matrix $\mathcal{A}_{av}$ given by (23) is Hurtwiz iff

$$a_1 + \frac{2(a_1c_2^2 - a_2c_1c_2 - c_1^2)}{c_2^2} \sum_{i=2}^{\infty} \frac{1}{(2i)!}\left(\frac{kbc_2}{2}\right)^{2(i-1)} < 0 \tag{24}$$

Moreover, if (6) holds, then inequality (24) is always feasible for large enough $|k|$.

III. MAIN RESULTS

A. A time-delay model

Consider system (20), where $\zeta(t) \in \mathbb{R}^2$, $a_2 < 0$ and $c_2 \neq 0$. Let k be subject to (24) such that $\mathcal{A}_{av}$ given by (23) is Hurwitz. Given small $\sigma_0 > 0$ in (3), let a small enough $\sigma > 0$ (independent of $\varepsilon \in (0, \varepsilon^*]$) be the upper bound on $\Delta\mathcal{A}(t)$ defined in (21) for all $t \geq 0$, i.e.

$$\|\Delta\mathcal{A}(t)\| \leq \sigma \quad \forall t \geq 0. \tag{25}$$

Indeed, this upper bound can be found by using (3), (21):

$$\|\Delta\mathcal{A}(t)\| \leq \sigma_0 \|\Phi^{-1}(\tfrac{t}{\varepsilon}, 0)\| \|\Phi(\tfrac{t}{\varepsilon}, 0)\| \leq \sigma,$$

If particularly $\Delta A(t) = \Delta a(t) I$ with $\Delta a(t) \in \mathbb{R}$ satisfying $|\Delta a(t)| \leq \sigma_0$ for all $t \geq 0$, then $\sigma = \sigma_0$. Moreover, from (15) and (21) we find that all entries $\mathcal{A}_{ij}(\frac{t}{\varepsilon})$ of $\mathcal{A}(\frac{t}{\varepsilon})$ are uniformly bounded for $t \geq 0$. Thus, $\mathcal{A}(\frac{t}{\varepsilon})$ can be presented as a convex combination of the constant matrices $\mathcal{A}_i$ for all $t \geq \varepsilon$:

$$\mathcal{A}(\tfrac{t}{\varepsilon}) = \textstyle\sum_{i=1}^{N} \rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i, \quad \rho_i(\tfrac{t}{\varepsilon}) \geq 0, \quad \sum_{i=1}^{N} \rho_i(\tfrac{t}{\varepsilon}) = 1 \tag{26}$$

with some integer $N \geq 2$.

Following [9], [11], we will apply the time-delay approach to periodic averaging of system (20). Namely, we integrate both sides of system (20) over $[t-\varepsilon, t]$ for $t \geq \varepsilon + \varepsilon^2 h$, i.e.

$$\begin{aligned} &\tfrac{\zeta(t)-\zeta(t-\varepsilon)}{\varepsilon} = \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} [\mathcal{A}(\tfrac{s}{\varepsilon}) + \Delta\mathcal{A}(s)]\zeta(s)ds \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \chi(s)[\delta_1(s) + \delta_2(s)]ds \\ &+ \tfrac{k}{\varepsilon^2}\textstyle\int_{t-\varepsilon}^{t} (1-\chi(s))\mathrm{sq}\left(\tfrac{s}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{s}{\varepsilon}, 0\right) Be(s)ds. \end{aligned} \tag{27}$$

We present the left-hand side of (27) as

$$\begin{aligned} &\tfrac{\zeta(t)-\zeta(t-\varepsilon)}{\varepsilon} = \tfrac{d}{dt}[\zeta(t) - G(t)] \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \Delta\mathcal{A}(s)\zeta(s)ds - \Delta\mathcal{A}(t)\zeta(t) \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \chi(s)[\delta_1(s) + \delta_2(s)]ds - \chi(t)[\delta_1(t) + \delta_2(t)] \\ &+ \tfrac{k}{\varepsilon^2}\textstyle\int_{t-\varepsilon}^{t} (1-\chi(s))\mathrm{sq}\left(\tfrac{s}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{s}{\varepsilon}, 0\right) Be(s)ds \\ &-(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) Be(t), \end{aligned} \tag{28}$$

where

$$G(t) = \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} (s - t + \varepsilon)\mathcal{A}(\tfrac{s}{\varepsilon})\zeta(s)ds. \tag{29}$$

The term $G(t)$ depends on the nominal part $\mathcal{A}(\frac{t}{\varepsilon})\zeta(t)$ only (that is the fast-varying term to be "averaged") and not on the whole right-hand part of (20) as in [9]. Then we obtain

$$\begin{aligned} &\tfrac{d}{dt}[\zeta(t) - G(t)] = \Delta\mathcal{A}(t)\zeta(t) + \chi(t)[\delta_1(t) + \delta_2(t)] \\ &+(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) Be(t) \\ &+\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \mathcal{A}(\tfrac{s}{\varepsilon})[\zeta(s) - \zeta(t) + \zeta(t)]ds, \quad t \geq \varepsilon + \varepsilon^2 h. \end{aligned} \tag{30}$$

We present

$$\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \mathcal{A}(\tfrac{s}{\varepsilon})[\zeta(s) - \zeta(t)]ds = -\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \mathcal{A}(\tfrac{s}{\varepsilon})\int_{s}^{t} \dot{\zeta}(\theta)d\theta ds.$$

Thus, we transform system (20) to the following time-delay system:

$$\begin{aligned} &\dot{z}(t) = [\mathcal{A}_{av} + \Delta\mathcal{A}(t)]\zeta(t) - Y(t) + \chi(t)[\delta_1(t) + \delta_2(t)] \\ &\quad +(1-\chi(t))\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon}, 0\right) Be(t), \quad t \geq \varepsilon + \varepsilon^2 h, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &z(t) = \zeta(t) - G(t), \\ &Y(t) = \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t} \mathcal{A}(\tfrac{s}{\varepsilon})\int_{s}^{t} \dot{\zeta}(\theta)d\theta ds, \end{aligned} \tag{32}$$

with $G(t)$ from (29) and $\dot{\zeta}$ satisfying (20). Note that system (31) with notations (32) is a neutral type system.

B. LMI conditions: L-K method

Theorem 1: ($c_2 \neq 0$) Let $a_2 < 0$ and k satisfy (24) (resulting in Hurwitz $\mathcal{A}_{av}$ given by (23)). Assume that (25) and (26) hold. Given matrices $\mathcal{A}_i$ $(i = 1, \ldots, N,\ N \geq 2)$ and scalars $\sigma > 0$, $\alpha > 0$, $\varepsilon^* > 0$, $\varsigma > 0$, $h > 0$ and $\psi_i > 0$ $(i = 1, 2)$, let there exist $n \times n$ matrices $P > 0$, $R > 0$, and scalars $Q > 0$, $W_i > 0$ $(i = 1, 2)$, $\lambda > 0$ that satisfy the following LMIs for $i = 1, \ldots, N$, $j = 1, 2$

$$\Upsilon_{ij} = \left[\begin{array}{c|cc} \Pi_j & \begin{matrix} \mathcal{A}_i^T R \\ 0 \\ 0 \end{matrix} & \begin{matrix} \mathcal{A}_i^T \Theta_j \\ 0 \\ \Theta_j \mathbf{I}_{4-j} \end{matrix} \\ \hline \begin{matrix} * \\ * \end{matrix} & \begin{matrix} -\frac{R}{\varepsilon^*} \\ * \end{matrix} & \begin{matrix} 0 \\ -\Theta_j \end{matrix} \end{array}\right] < 0, \ \mathbf{I}_{4-j} = [\underbrace{I, \cdots, I}_{4-j}]^T, \tag{33}$$

where $\Pi_j = [\Pi_{ij}^i]$ are the symmetric matrices composed of

$$\begin{aligned} &\Pi_{11}^1 = PA_{av} + A_{av}^T P + 2\alpha P + \lambda\sigma^2 I \\ &\quad +(hk^2\psi_1)^2 W_2 |CB|^2 I, \\ &\Pi_{11}^2 = PA_{av} + A_{av}^T P + 2\alpha P + \lambda\sigma^2 I \\ &\quad +\varsigma k^2 \psi_1^2 \psi_2^2 I, \quad \Pi_{12}^i = -A_{av}^T P - 2\alpha P, \\ &\Pi_{13}^i = \Pi_{24}^i = \Pi_{25}^i = \Pi_{26}^1 = -P, \quad \Pi_{44}^i = -\lambda I, \\ &\Pi_{14}^i = \Pi_{15}^i = \Pi_{16}^1 = \Pi_{23}^i = P, \quad \Pi_{33}^i = -\tfrac{2Q}{\varepsilon^*}e^{-2\alpha\varepsilon^*}I, \\ &\Pi_{22}^i = -\tfrac{4}{\varepsilon^*}e^{-2\alpha\varepsilon^*}R + 2\alpha P, \quad \Pi_{55}^2 = -I, \\ &\Pi_{55}^1 = -\tfrac{\pi^2}{4\psi_2^2}e^{-2\alpha\varepsilon^2 h}W_1 I, \quad \Pi_{66}^1 = -\tfrac{\pi^2}{4\psi_2^2}e^{-2\alpha\varepsilon^2 h}W_2 I \end{aligned} \tag{34}$$

and other blocks are zero matrices, and

$$\Theta_1 = \Theta_2 + (\varepsilon^* hk\psi_1)^2 W_1, \ \Theta_2 = \varepsilon^* Q \textstyle\int_0^1 \mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau. \tag{35}$$

Then system (12) with $c_2 \neq 0$ and t_ℓ given by (7) is exponentially stable with a decay rate α for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists $M > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and initial conditions $x(0)$ the solutions of (12) with $c_2 \neq 0$ and t_ℓ given by (7) satisfy the following inequality:

$$|x(t)|^2 \leq Me^{-2\alpha t}|x(0)|^2 \quad \forall t \geq 0 \tag{36}$$

Proof: We start with $t \in [t_\ell, t_\ell + \varepsilon^2 h)$, i.e. $\chi(t) = 1$. Choose

$$V_P(t) = z^T(t)Pz(t), \quad 0 < P \in \mathbb{R}^{n\times n}. \tag{37}$$

Differentiating $V_P(t)$ along (31) with $\chi(t) = 1$ we find

$$\begin{aligned} &\dot{V}_P(t) = 2[\zeta(t) - G(t)]^T P[(\mathcal{A}_{av} + \Delta\mathcal{A}(t))\zeta(t) \\ &\quad -Y(t) + \delta_1(t) + \delta_2(t)]. \end{aligned} \tag{38}$$

To compensate $G(t)$ and $Y(t)$ in (38), we employ [11], [22]

$$\begin{array}{l} V_R(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} e^{-2\alpha(t-s)}(s-t+\varepsilon)^2\zeta^T(s) \\ \quad\times\mathcal{A}^T\left(\frac{s}{\varepsilon}\right)R\mathcal{A}\left(\frac{s}{\varepsilon}\right)\zeta(s)ds, \quad 0<R\in\mathbb{R}^{n\times n}, \\ V_Q(t) = \frac{Q}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} e^{-2\alpha(t-\theta)}(s-t+\varepsilon) \\ \quad\times|\mathcal{A}\left(\frac{s}{\varepsilon}\right)\dot\zeta(\theta)|^2 d\theta ds, \quad Q\in\mathbb{R}^+. \end{array} \tag{39}$$

By using Lemma 1.1 in [11], we obtain

$$\begin{array}{l} \dot V_R(t) + 2\alpha V_R(t) \le -\frac{4}{\varepsilon}e^{-2\alpha\varepsilon}G^T(t)RG(t) \\ \quad+\varepsilon\zeta^T(t)\mathcal{A}^T\left(\frac{t}{\varepsilon}\right)R\mathcal{A}\left(\frac{t}{\varepsilon}\right)\zeta(t), \\ \dot V_Q(t) + 2\alpha V_Q(t) \le -\frac{2Q}{\varepsilon}e^{-2\alpha\varepsilon}|Y(t)|^2 \\ \quad+\frac{Q}{\varepsilon}\dot\zeta^T(t)\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\mathcal{A}^T\left(\frac{s}{\varepsilon}\right)\mathcal{A}\left(\frac{s}{\varepsilon}\right)ds\dot\zeta(t) \\ \le -\frac{2Q}{\varepsilon}e^{-2\alpha\varepsilon}|Y(t)|^2 + \varepsilon Q\dot\zeta^T(t)\int_0^1\mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau\dot\zeta(t), \end{array} \tag{40}$$

where in the last inequality we used

$$\begin{array}{l} \frac{1}{\varepsilon^2}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\mathcal{A}^T(\frac{s}{\varepsilon})\mathcal{A}(\frac{s}{\varepsilon})ds \\ \le \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\mathcal{A}^T(\frac{s}{\varepsilon})\mathcal{A}(\frac{s}{\varepsilon})ds = \int_0^1\mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau. \end{array}$$

Using (17) and (21) we obtain

$$\begin{array}{l} |\delta_1(t)| = |\frac{k}{\varepsilon}\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)BC\int_{t-\tau(t)}^{t}\Phi\left(\frac{s}{\varepsilon},0\right)\dot\zeta(s)ds| \\ \quad\le|\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)B||\tilde\delta_1(t)|\le\psi_2|\tilde\delta_1(t)|, \\ |\delta_2(t)| = |\frac{k^2}{\varepsilon^2}\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)(BC)^2 \\ \quad\times\int_{t-\tau(t)}^{t}\mathrm{sq}\left(\frac{s}{\varepsilon}\right)\Phi\left(\frac{s}{\varepsilon},0\right)\zeta(s)ds| \\ \quad\le|\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)B||\tilde\delta_2(t)|\le\psi_2|\tilde\delta_2(t)|, \end{array}$$

where

$$\begin{array}{l} \tilde\delta_1(t) = \frac{k}{\varepsilon}C\int_{t-\tau(t)}^{t}\Phi\left(\frac{s}{\varepsilon},0\right)\dot\zeta(s)ds, \\ \tilde\delta_2(t) = \frac{k^2}{\varepsilon^2}CBC\int_{t-\tau(t)}^{t}\mathrm{sq}\left(\frac{s}{\varepsilon}\right)\Phi\left(\frac{s}{\varepsilon},0\right)\zeta(s)ds. \end{array}$$

Then

$$-|\tilde\delta_i(t)|^2 \le -\frac{1}{\psi_2^2}|\delta_i(t)|^2, \quad i=1,2. \tag{41}$$

Based on the latter bounding, we suggest

$$\begin{array}{l} V_{W_1}(t) = -\frac{\pi^2}{4}e^{-2\alpha h}W_1\int_{t-\tau(t)}^{t}e^{-2\alpha(t-s)}|\tilde\delta_1(s)|^2ds \\ \quad+\varepsilon^2h^2k^2W_1\int_{t-\tau(t)}^{t}e^{-2\alpha(t-s)}|C\Phi\left(\frac{s}{\varepsilon},0\right)\dot\zeta(s)|^2ds, \\ V_{W_2}(t) = -\frac{\pi^2}{4}e^{-2\alpha h}W_2\int_{t-\tau(t)}^{t}e^{-2\alpha(t-s)}|\tilde\delta_2(s)|^2ds \\ \quad+h^2k^4W_2\int_{t-\tau(t)}^{t}e^{-2\alpha(t-s)}|CBC\Phi\left(\frac{s}{\varepsilon},0\right)\zeta(s)|^2ds, \\ t\in[t_\ell,t_\ell+\varepsilon^2h), \quad \ell\in\mathbb{N}_0, \quad W_i\in\mathbb{R}^+, \quad i=1,2 \end{array} \tag{42}$$

to compensate the terms $\tilde\delta_1(t)$ and $\tilde\delta_2(t)$ (thus, $\delta_1(t)$ and $\delta_2(t)$ in (38)), respectively. Since $\tilde\delta_1(t_\ell)=\tilde\delta_2(t_\ell)=0$, $\frac{d}{dt}\tilde\delta_1(t) = \frac{k}{\varepsilon}C\Phi\left(\frac{t}{\varepsilon},0\right)\dot\zeta(t)$, $\frac{d}{dt}\tilde\delta_2(t) = \frac{k^2}{\varepsilon^2}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)CBC\Phi\left(\frac{t}{\varepsilon},0\right)\zeta(t)$ and

$$|\tfrac{k^2}{\varepsilon^2}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)CBC\Phi\left(\tfrac{t}{\varepsilon},0\right)\zeta(t)| \le \tfrac{k^2}{\varepsilon^2}|CBC\Phi\left(\tfrac{t}{\varepsilon},0\right)\zeta(t)|,$$

Lemma 1 of [23] implies $V_{W_1}(t)\ge0$ and $V_{W_2}(t)\ge0$ for $t\in[t_\ell,t_\ell+\varepsilon^2h)$, $\ell\in\mathbb{N}_0$. Using (41) we obtain

$$\begin{array}{l} \dot V_{W_1}(t) + 2\alpha V_{W_1}(t) = \varepsilon^2h^2k^2W_1|C\Phi\left(\frac{t}{\varepsilon},0\right)\dot\zeta(t)|^2 \\ \quad-\frac{\pi^2}{4}e^{-2\alpha\varepsilon^2h}W_1|\tilde\delta_1(t)|^2 \\ \le\varepsilon^2h^2k^2\psi_1^2W_1|\dot\zeta(t)|^2 - \frac{\pi^2}{4\psi_2^2}e^{-2\alpha\varepsilon^2h}W_1|\delta_1(t)|^2, \\ \dot V_{W_2}(t) + 2\alpha V_{W_2}(t) = -\frac{\pi^2}{4}e^{-2\alpha\varepsilon^2h}W_2|\tilde\delta_2(t)|^2 \\ \quad+h^2k^4W_2|CBC\Phi\left(\frac{t}{\varepsilon},0\right)\zeta(t)|^2 \\ \le h^2k^4\psi_1^2W_2|CB|^2|\zeta(t)|^2 - \frac{\pi^2}{4\psi_2^2}e^{-2\alpha\varepsilon^2h}W_2|\delta_2(t)|^2. \end{array} \tag{43}$$

We next consider $t\in[t_\ell+\varepsilon^2h, t_{\ell+1})$, i.e. $\chi(t)=0$. Choose $V_P(t)$ given by (37). Differentiating $V_P(t)$ along (31) with $\chi(t)=0$ leads to

$$\begin{array}{l} \dot V_P(t) = 2[\zeta(t)-G(t)]^TP[(\mathcal{A}_{av}+\Delta\mathcal{A}(t))\zeta(t) \\ \quad-Y(t)+\frac{k}{\varepsilon}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)Be(t)]. \end{array} \tag{44}$$

Note that the following holds:

$$|\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon},0\right)Be(t)| \le \tfrac{k\psi_2}{\varepsilon}|e(t)|.$$

Moreover, from (7) and (11) it follows that

$$|e(t)|^2 \le \varsigma\varepsilon^2|C\Phi\left(\tfrac{t}{\varepsilon},0\right)\zeta(t)|^2 = \varsigma\psi_1^2\varepsilon^2|\zeta(t)|^2.$$

Thus, we obtain

$$|\tfrac{k}{\varepsilon}\mathrm{sq}\left(\tfrac{t}{\varepsilon}\right)\Phi^{-1}\left(\tfrac{t}{\varepsilon},0\right)Be(t)|^2 \le \varsigma k^2\psi_1^2\psi_2^2|\zeta(t)|^2$$

leading to

$$\begin{array}{l} \dot V_P(t) \le 2[\zeta(t)-G(t)]^TP[(\mathcal{A}_{av}+\Delta\mathcal{A}(t))\zeta(t) \\ \quad-Y(t)+\frac{k}{\varepsilon}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)Be(t)] \\ \quad+\varsigma k^2\psi_1^2\psi_2^2|\zeta(t)|^2 - |\frac{k}{\varepsilon}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)Be(t)|^2. \end{array} \tag{45}$$

We use the terms $V_R(t)$ and $V_Q(t)$ in (39) to compensate $G(t)$ and $Y(t)$ in (45).

Define a Lyapunov functional as

$$\begin{array}{l} V(t) = V_P(t)+V_R(t)+V_Q(t)+\chi(t), \\ \quad\times[V_{W_1}+V_{W_2}], \quad t\ge\varepsilon+\varepsilon^2h, \end{array} \tag{46}$$

where $V_P(t)$ is from (37), $V_R(t)$ and $V_Q(t)$ are from (39), V_{W_1} and V_{W_2} are from (42), and $\chi(t)$ is given by (13). This functional is positive-definite for all $\varepsilon\in[0,\varepsilon^*]$:

$$\begin{array}{l} V(t)\ge V_P(t)+V_R(t) \\ \quad\ge\begin{bmatrix}\zeta(t)\\G(t)\end{bmatrix}^T\begin{bmatrix}P & -P\\ * & P+e^{-2\alpha\varepsilon^*}R\end{bmatrix}\begin{bmatrix}\zeta(t)\\G(t)\end{bmatrix}\ge\bar c_1|\zeta(t)|^2, \end{array}$$

by Jensen's inequality (see e.g. (3.87) in [24]) with $\bar c_1 = \lambda_{\min}(\begin{bmatrix}P & -P\\ * & P+e^{-2\alpha\varepsilon^*}R\end{bmatrix})$. To compensate $\Delta\mathcal{A}(t)\zeta(t)$ in (38) and (45), from (25) we have

$$\lambda[\sigma|\zeta(t)|^2 - |\Delta\mathcal{A}(t)\zeta(t)|^2] > 0 \tag{47}$$

with some $\lambda>0$. Applying S-procedure, where we add to $\dot V(t)$ the left-hand part of (47), and employing (38), (40), (43) and (45), we obtain for all $\varepsilon\in(0,\varepsilon^*]$

$$\begin{array}{l} \dot V(t)+2\alpha V(t)\le\chi(t)[\eta_1^T(t)\Pi_1\eta_1(t)+\dot\zeta^T(t)\Theta_1\dot\zeta(t)] \\ \quad+(1-\chi(t))[\eta_2^T(t)\Pi_2\eta_2(t)+\dot\zeta^T(t)\Theta_2\dot\zeta(t)] \\ \quad+\varepsilon^*\zeta^T(t)\mathcal{A}^T(\frac{t}{\varepsilon})R\mathcal{A}(\frac{t}{\varepsilon})\zeta(t), \quad t\ge\varepsilon+\varepsilon^2h. \end{array} \tag{48}$$

where Π_i $(i=1,2)$ are composed of (34) and

$$\begin{array}{l} \eta_1^T(t) = [\eta^T(t),\delta_1^T(t),\delta_2^T(t)], \\ \eta_2^T(t) = [\eta^T(t),(\frac{k}{\varepsilon}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)Be(t))^T], \\ \eta^T(t) = [\zeta^T(t),G^T(t),Y^T(t),\zeta^T(t)\Delta\mathcal{A}^T(t)]. \end{array}$$

Moreover, from (20) and (26) it follows that

$$\begin{array}{l} \dot\zeta(t) = [\sum_{i=1}^N\rho_i(\frac{t}{\varepsilon})\mathcal{A}_i+\Delta\mathcal{A}(t)]\zeta(t)+\chi(t)[\delta_1(t) \\ \quad+\delta_2(t)]+(1-\chi(t))\frac{k}{\varepsilon}\mathrm{sq}\left(\frac{t}{\varepsilon}\right)\Phi^{-1}\left(\frac{t}{\varepsilon},0\right)Be(t). \end{array} \tag{49}$$

Substituting (49) into (48) and applying further Schur complement formula, we find that LMIs (33) guarantee that

$\dot{V}(t) + 2\alpha V(t) \leq 0$ for $t \in [t_\ell, t_\ell + \varepsilon^2 h)$ and $t \in [t_\ell + \varepsilon^2 h, t_{\ell+1})$, respectively. Taking into account the fact that $V(t)$ defined by (46) does not grow the switching instants t_ℓ and $t_\ell + \varepsilon^2 h$ and following arguments of [8], [9], we arrive at (36).

The feasibility of the strict LMIs (33) with $\alpha = 0$ implies the feasibility of (33) with the same decision variables and a small enough $\alpha = \alpha_0 > 0$, and thus guarantees exponential stability of system (9) with t_ℓ given by (7) with a small enough decay rate. Moreover, the feasibility of LMIs (33) is always guaranteed for small enough $\varepsilon^* > 0$, $\sigma > 0$, $\alpha > 0$, $h > 0$ and $\varsigma > 0$ provided $\mathcal{A}_{av}$ is Hurwitz. This completes the proof. □

Note that if $c_2 = 0$ in (2) we have $CB = 0$ (thus $\delta_2(t) = 0$ in (21)). Then, one employs the following ET (cf. (7))

$$t_{\ell+1} = \min_{t \geq t_\ell + \varepsilon h}\{|y(t) - y(t_\ell)|^2 \geq \varsigma\varepsilon^2|y(t)|^2\}, \quad (50)$$

where correspondingly the term $\delta_1(t)$ defined in (21) becomes of the order $O(h)$ (instead of $O(\varepsilon h)$ when using ET (7)). The latter leads to system (20) with $\delta_2(t) = 0$ and $\chi(t)$ define by (13) with $\varepsilon^2 h$ changed by εh. Moreover, the corresponding $\Phi\left(\frac{t}{\varepsilon}, 0\right)$ is obtained by using the limit of (15) as c_2 approaches zero:

$$\Phi\left(\tfrac{t}{\varepsilon}, 0\right) = \begin{bmatrix} 1 & 0 \\ kbc_1\rho(\frac{t}{\varepsilon}) & 1 \end{bmatrix}, \quad t \geq 0.$$

Then, by using the time-delay approach to periodic averaging [9], [11] we arrive at the time-delay system (31) with $\delta_2(t) = 0$ and $\chi(t)$ define by (13) with $\varepsilon^2 h$ changed by εh for $t \geq \varepsilon + \varepsilon h$. By arguments of Theorem 1, we arrive at the following:

Theorem 2: ($c_2 = 0$) Let $a_2 < 0$ and k satisfy (24) (resulting in Hurwitz $\mathcal{A}_{av}$ given by (23)). Assume that (25) and (26) hold. Given matrices $\mathcal{A}_i$ $(i = 1, \ldots, N,\ N \geq 2)$ and scalars $\sigma > 0$, $\alpha > 0$, $\varepsilon^* > 0$, $\varsigma > 0$, $h > 0$ and $\psi_i > 0$ $(i = 1, 2)$, let there exist $n \times n$ matrices $P > 0$, $R > 0$, and scalars $Q > 0$, $W_1 > 0$, $\lambda > 0$ that satisfy

$$\tilde{\Upsilon}_j < 0, \quad , i = 1, \ldots, N, \quad j = 1, 2.$$

Here $\tilde{\Upsilon}_1$ is obtained from Υ_1 in (33) by replacing Π^1_{11}, Π^1_{55}, Θ_1 with $PA_{av} + A^T_{av}P + 2\alpha P + \lambda\sigma^2 I$, $-\frac{\pi^2}{4|B|^2}e^{-2\alpha\varepsilon h}W_1 I$ and $\Theta_2 + h^2k^2W_1C^TC$ with Θ_2 defined in (35), respectively, and by taking away the 6th block-column and block-row whereas $\tilde{\Upsilon}_2$ is obtained from Υ_2 by replacing Π^2_{11} with $PA_{av} + A^T_{av}P + 2\alpha P + \lambda\sigma^2 I + \varsigma k^2|B|^2C^TC$. Then system (12) with $c_2 = 0$ and t_ℓ given by (50) is exponentially stable with a decay rate α for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists $M > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and initial conditions $x(0)$ the solutions of (12) with $c_2 = 0$ and t_ℓ given by (50) satisfy (36).

IV. Numerical Examples

For the cases of $c_2 \neq 0$ and $c_2 = 0$, we will present two examples in the presence of uncertainties, where the uncertainty is given by

$$\Delta A(t) = \sigma_0 \sin(t) I. \quad (51)$$

TABLE I

Maximum values of ε^* for different α, σ, h and ς (Example 1)

Method	α	σ	h	ς	ε^*
[8], Th. 1	10^{-6}	0	0	0	0.0097
	0.01	0.02	0	0	0.0079
Th. 1	10^{-6}	0	$8 \cdot 10^{-6}$	0	0.0061
	0.01	0.02	$8 \cdot 10^{-6}$	0	0.0044
	10^{-6}	0	$8 \cdot 10^{-6}$	$4 \cdot 10^{-9}$	0.0048
	0.01	0.02	$8 \cdot 10^{-6}$	$4 \cdot 10^{-9}$	0.0030

The latter satisfies (3). From (21), it follows that (25) with $\sigma = \sigma_0$.

Example 1: Consider system (1), (2) with [6], [8]

$$a_1 = b = c_1 = -c_2 = 1,\ a_2 = -\tfrac{1}{2}, \quad (52)$$

and (51) under a fast-varying output feedback controller (4). Note that this system with $\Delta A(t) = 0$ is not stabilizable by a static time-invariant output-feedback controller. Via the coordinate transformation (18), we obtain (20), where

$$\mathcal{A}(\tfrac{t}{\varepsilon}) = \begin{bmatrix} -e^{-k\rho(\frac{t}{\varepsilon})} + 1 & e^{-k\rho(\frac{t}{\varepsilon})} \\ -e^{-k\rho(\frac{t}{\varepsilon})} - \frac{1}{2}e^{k\rho(\frac{t}{\varepsilon})} + \frac{5}{2} & e^{-k\rho(\frac{t}{\varepsilon})} - \frac{3}{2} \end{bmatrix} \quad (53)$$

with $\rho(\cdot)$ given by (16) and $\Delta\mathcal{A}(t)$ in (21) satisfies (25) with $\sigma = \sigma_0$. Clearly, $\mathcal{A}(\frac{t}{\varepsilon})$ in (53) belongs to uncertain polytope with four vertices (that are omitted here) corresponding to $(-\rho, \rho) \in \{-\frac{1}{4}, \frac{1}{4}\} \times \{-\frac{1}{4}, \frac{1}{4}\}$. We obtain

$$\mathcal{A}_{av} = \begin{bmatrix} -\frac{4}{k}\sinh(\frac{k}{4}) + 1, & \frac{4}{k}\sinh(\frac{k}{4}) \\ -\frac{6}{k}\sinh(\frac{k}{4}) + \frac{5}{2} & \frac{4}{k}\sinh(\frac{k}{4}) - \frac{3}{2} \end{bmatrix}. \quad (54)$$

Let $k = 9$. From (17), (33) and (53) we obtain $\psi_1 = 13.4177$, $\psi_2 = 9.4877$ and

$$\int_0^1 \mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau = \begin{bmatrix} 10.9444 & -12.2628 \\ -12.2628 & 15.9964 \end{bmatrix}. \quad (55)$$

By verifying the feasibility of LMIs in Theorem 1 in the four vertices with different α, $\sigma = \sigma_0$, h and ρ, and using (54), (55), we find the upper bounds ε^* (see Table I) that guarantee the exponential stability of system (9), (52) with t_ℓ given by (7) and $k = 9$ for all $\varepsilon \in (0, \varepsilon^*]$.

Numerical simulation under the initial condition $x(0) = [-1, 1]^T$ shows that system (9), (52) with t_ℓ given by (7), $k = 9$ and $\Delta A(t) = 0$ $\forall t \geq 0$ is stable for a larger $\varepsilon = 2$, $h = 1 \cdot 10^{-3}$ and $\varsigma = 0.5 \cdot 10^{-3}$, see Fig. 1, where the simulation time is 50 seconds. Note that the amount of sent measurements under ET (7) is 3397, which is essentially smaller than 12500 under the periodic sampling of measurements with $t_\ell = \ell \cdot \varepsilon^2 h$ where $\varepsilon = 2$, $h = 1 \cdot 10^{-3}$ and $\ell \in \mathbb{N}$.

Example 2: Consider system (1), (2) with [8]

$$a_1 = 52.973, \quad a_2 = -5, \quad b = c_1 = 1, \quad c_2 = 0 \quad (56)$$

and $\Delta A(t)$ given by (51) under a fast-varying output feedback controller (4). Via the coordinate transformation (18), we obtain (20), where

$$\mathcal{A}(\tfrac{t}{\varepsilon}) = \begin{bmatrix} k\rho(\frac{t}{\varepsilon}) & 1 \\ 52.973 - 5k\rho(\frac{t}{\varepsilon}) - k^2\rho^2(\frac{t}{\varepsilon}) & -5 - k\rho(\frac{t}{\varepsilon}) \end{bmatrix} \quad (57)$$

with $\rho(\cdot)$ given by (16), and $\Delta\mathcal{A}(t)$ in (21) satisfies (25) with $\sigma = \sigma_0$. Clearly, $\mathcal{A}(\frac{t}{\varepsilon})$ in (57) belongs to uncertain polytope

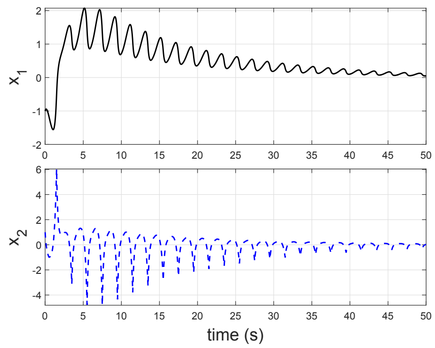

Fig. 1. State trajectory of (9), (52) with t_ℓ given by (7), $k = 9$ and $\Delta A(t) = 0\ \forall t \geq 0$ when $\varepsilon = 2$, $h = 1 \cdot 10^{-3}$ and $\varsigma = 0.1 \cdot 10^{-3}$.

TABLE II

MAXIMUM VALUES OF ε^* FOR DIFFERENT α, σ, h AND ς (EXAMPLE 2)

Method	α	σ	h	ς	ε^*
[8], Th. 2	10^{-6}	0	0	0	0.0018
	0.2	0.2	0	0	0.0013
Th. 2	10^{-6}	0	0.002	0	0.0010
	0.2	0.2	0.002	0	0.0005
	10^{-6}	0	0.002	0.0005	0.0009
	0.2	0.2	0.002	0.0005	0.0004

with four vertices (that are omitted here) corresponding to $\rho \in \{-\frac{1}{4}, \frac{1}{4}\}$ and $\rho^2 \in \{0, \frac{1}{16}\}$. We obtain

$$\mathcal{A}_{av} = \begin{bmatrix} 0 & 1 \\ 52.973 - \frac{k^2}{48} & -5 \end{bmatrix}. \quad (58)$$

Let $k = 57$. From (17), (33) and (57) we obtain

$$\int_0^1 \mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau = 10^3 \times \begin{bmatrix} 5.6417 & 0.412 \\ 0.412 & 0.09377 \end{bmatrix}. \quad (59)$$

By verifying the feasibility of LMIs in Theorem 2 in the four vertices with different α, $\sigma = \sigma_0$, h and ς, and using (58), (59), we find the upper bounds ε^* (see Table II) that guarantee the exponential stability of system (9), (56) with t_ℓ given by (50) and $k = 57$ for all $\varepsilon \in (0, \varepsilon^*]$.

Numerical simulation under the initial condition $x(0) = [-1, 1]^T$ shows that system (9), (56), with t_ℓ given by (50), $k = 57$ and $\Delta A(t) = 0\ \forall t \geq 0$ is stable for a larger $\varepsilon = 0.36$, $h = 0.01$ and $\varsigma = 0.8$, see Fig. 2, where the simulation time is 5 seconds. Note that the amount of sent measurements under ET (50) is 482, which is essentially smaller than 1388 under the periodic sampling of measurements with $t_\ell = \ell \cdot \varepsilon h$ where $\varepsilon = 0.8$, $h = 0.01$ and $\ell \in \mathbb{N}$.

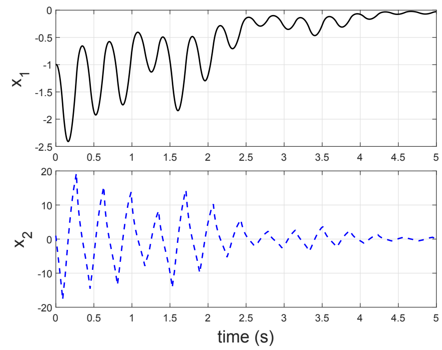

Fig. 2. State trajectory of (9), (56) with t_ℓ given by (50), $k = 57$ and $\Delta A(t) = 0\ \forall t \geq 0$ when $\varepsilon = 0.36$, $h = 0.01$ and $\varsigma = 0.8$.

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