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Tube-based MPC for Two-Timescale Discrete-Time Nonlinear Processes with Robust Control Contraction Metrics*

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Abstract— **A computationally efficient model predictive control scheme is proposed for constrained nonlinear processes with inherent slow and fast dynamics. Specifically, the nonlinear process is approximated by a surrogate model, whose slow state is sampled with a larger time interval than the fast one. This reduces optimization variables and, on the other hand, introduces prediction errors and henceforth may induce constraint violation without further treatments. To mitigate these issues, we tighten constraints by leveraging Robust Control Contraction Metrics. Furthermore, a back-up strategy is employed to ensure feasibility over time. The numerical efficiency of this proposed scheme is illustrated with a crops growth process, e.g., in indoor farming.**

I. INTRODUCTION

Model Predictive Control (MPC) solves an optimal control problem in a receding-horizon fashion. It is nowadays a prevalent approach to compute optimal control actions accounting for input and state constraints. The accuracy of models used for MPC plays a crucial role in closed-loop performance and constraint guarantees. However, high-fidelity models emerging from real applications, e.g. greenhouse cultivation [1] and chemical engineering [2], can be very complex, which renders real-time implementation challenging. In response to this challenge, tremendous research progress have been made in past decades, e.g. non-uniformly spaced horizon [3], move blocking [4], just to name a few.

Specially for multi-timescale systems, hierarchical MPC [5] appears promising and found numerous successful applications, e.g. [6], [7]. In this method, the system is approximately decomposed into slow and fast timescale reduced order models and used by higher and lower level MPCs operating in different timescale. However, the design methods tailored on specific practical problems lack control-theoretic guarantees. Indeed, as for state constraints, model errors arsing from model decomposition are not considered in [6], [7]. Another promising method is the decentralized approach proposed in [8], where multiple controllers operates with different timescale and reduced order models in decentralized fashion. Although stability is guaranteed, state constraints are not considered. Apart from decentralized and hierarchical control approaches, a centralized control framework with slow reduced order model in [9] emerges as an appealing approach, which rigorously guarantees constraints satisfaction, albeit limited to linear systems.

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In the presence of model errors, constraint satisfaction is typically ensured by means of tubes around arbitrary nominal trajectories. Among existing approaches for continuous-time systems, Control Contraction Metrics (CCM) [10] originating from contraction theory [11] is particularly befitting for this task, as it studies convergence between pairs of *arbitrary* trajectories and boils design problem down to *convex* semi-definite programming, see e.g. [12], [13]. In [14], CCM was extended to Robust CCM (RCCM) to consider disturbance rejection property into feedback controller design, and then used in [16] for rigid tubes construction. Recently, CCM was conceptualized for discrete-time system in [15]. However, results on RCCM are still lacking for discrete-time system.

Accounting for the aforementioned limitations, for a nonlinear discrete-time process with multi-timescale behavior, we propose a tube-based MPC using a coarse surrogate model, which is similar in spirit to [9], but employing other approach instead of model order reduction to reduce model complexity. Specifically, the two-timescale nonlinear process is approximated by a surrogate model, whereby the slow state updates with larger time interval than the fast one. To enforce state and input constraints, we leverage RCCM, as inspired by [16] for continuous-time systems. This permits jointly synthesizing homothetic tubes [17], i.e., *non-rigid* tube scaled depending on prediction steps, and feedback controllers with property of disturbance rejection against model approximation errors. Problems arsing from recursive infeasibility are circumvented by the auxiliary back-up strategy motivated by [18] for stochastic MPC.

The paper is organized as follows. We first clarify the problem setting in Section II. In Section III, the proposed MPC formulation with the back-up strategy is presented following preliminary results in terms of RCCM. The virtue of the proposed method is validated on a crop growth process in the Section IV, followed by conclusive remarks in Section V.

Notation: Let the set of integers in the interval $[a,b] \subset \mathbb{R}$ be denoted by $[a:b]$ and the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$. We use $\|\cdot\|$ to denote the 2-norm of a vector or matrix. The notation $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest integer less (larger) than or equal to x. For $r > 0$, the set $\{x \in \mathbb{R}^n : \|x\|^2 < r\}$ is denoted by $\mathcal{B}^n(r)$. For any matrix valued function $F(\cdot)$, we term $F_{ij}(\cdot)$ as the function positioned at i-th row and j-th column. The Minkowski(Pontryagin) difference is represented by $\ominus$. Finally, I and O denote the identity and zero matrices of appropriate sizes respectively. Given two functions $g : \mathbb{X} \to \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{Y}$, the composition $f \circ g$ assigns to each $p \in \mathbb{X}$ the value $f(g(p)) \in \mathbb{Y}$.

II. Problem Statement

We consider a discrete-time nonlinear system

$$x^f((i+1)\Delta_t) = f\left(x^f(i\Delta_t), x^s(i\Delta_t), u(i\Delta_t), w(i\Delta_t); \Delta_t\right) \quad (1)$$

$$x^s((i+1)\Delta_t) = g\left(x^f(i\Delta_t), x^s(i\Delta_t), u(i\Delta_t), w(i\Delta_t); \Delta_t\right) \quad (2)$$

with the time index $i \in \mathbb{N}_0$, $\Delta_t \in \mathbb{R}_{\geq 0}$ as sample time, online measurable states $x^f(i) \in \mathbb{R}^n$ and $x^s(i) \in \mathbb{R}^m$ governed by the fast and slow dynamics respectively, as well as the control inputs $u(i) \in \mathbb{R}^p$. The exogenous input $w(i)$ known from current time $i\Delta_t$ till $(i+N)\Delta_t$ lies in a compact set $\mathbb{W} \subset \mathbb{R}^{\widehat{p}}$. We assume that, for each fixed Δ_t, $f(\cdot;\Delta_t)$ and $g(\cdot;\Delta_t)$, defined on some open subsets of Euclidean space, are continuously differentiable and continuous respectively. For simplicity, we assume w.l.o.g. $\Delta_t = 1$ and drop Δ_t in the dynamics whenever sample time is 1. To streamline the presentation, in what follows, we assume that control inputs u appear *only* in the fast dynamic f.

The control task is to find $u(\cdot)$ to optimize some cost $J_\infty(x^f(0), x^s(0), u(\cdot)) = \sum_{i=1}^{\infty} l(x^s(i), x^f(i), u(i-1))$ with the stage cost subject to the system dynamic (1)-(2), input constraint $u \in \mathbb{U}$ and state constraints $x^f \in \mathbb{X}_f$ and $x^s \in \mathbb{X}_s$, where $\mathbb{U}$, $\mathbb{X}_f$ and $\mathbb{X}_s$ are convex compact sets. This intractable infinite-time optimal control problem with disturbance forecasts available over finite time period is approximately solved by the below optimization problem within each sample time,

$$\begin{aligned}
\min_{\substack{u(\cdot|i), x^f(\cdot|i);\\ x^s(\cdot|i)}} \quad & \sum_{k=1}^{N} l(x^s(k|i), x^f(k|i), u(k-1|i)), \\
s.t.\ & x^f(k+1|i) = f(x^f(k|i), x^s(k|i), u(k|i), w(k+i)), \\
& x^s(k+1|i) = g(x^f(k|i), x^s(k|i), w(k+i)), \\
& x^f(0|i) = x^f(i),\ x^s(0|i) = x^s(i), \\
& u(k|i) \in \mathbb{U},\ k \in [0:N-1], \\
& x^s(k|i) \in \mathbb{X}_s,\ x^f(k|i) \in \mathbb{X}_f,\ k \in [1:N-1], \\
& (x^f(N|i), x^s(N|i)) \in \mathbb{Z}_N,
\end{aligned} \quad (3)$$

with the terminal set $\mathbb{Z}_N \subset \mathbb{X}_f \times \mathbb{X}_s$ defined with a terminal controller κ_N as follows, $\forall (x^f, x^s) \in \mathbb{Z}_N$ and $\forall w \in \mathbb{W}$, $\left(f(x^f, x^s, \kappa_N(x^f, x^s), w), g(x^f, x^s, w)\right) \in \mathbb{Z}_N$ and $\kappa_N(x^f, x^s) \in \mathbb{U}$. Recursive feasibility of MPC is hence ensured by the terminal controller and terminal set, see e.g. [19].

In practice, finding optimal inputs amounts to solving a static nonlinear optimization problem with state and input variables as optimization variables. If the number of variables within the prediction horizon is large, the computation complexity can be very demanding. Considering the slow dynamic associated with x^s, it is reasonable to adopt a larger sample time only for x^s, which reduces the number of optimization variables while still yields small states prediction errors. Employing multiple sampling intervals for the system (1)-(2) motivates the following surrogate model

$$\widehat{x}^f(i+1) = f(\widehat{x}^f(i), \widehat{x}^s(\sigma_{\bar{n}}(i)), u(i), w(i)), \quad (4)$$

$$\widehat{x}^s(\sigma_{\bar{n}}(i)+\bar{n}) = g(\widehat{x}^f(\sigma_{\bar{n}}(i)), \widehat{x}^s(\sigma_{\bar{n}}(i)), w(\sigma_{\bar{n}}(i)); \bar{n}), \quad (5)$$

with the time scaling factor $\bar{n} \in \{\bar{n} \in [2:N] : N/\bar{n} \in \mathbb{N}\}$ and the operator $\sigma_{\bar{n}}(i) = \lfloor \frac{i}{\bar{n}} \rfloor \bar{n}$, which maps the time index i from the time scale with Δ_t to the slower time scale with $\bar{n}\Delta_t$.

Based on the surrogate model (4)-(5), we reformulate problem (3) in such a way that the true process (1)-(2) still satisfies input and state constraints, but at reduced computational costs.

III. Tube-based MPC with RCCM

In this section, we formulate the tube-based MPC scheme using surrogate models. To this end, we first present the preliminary result on RCCM for a general discrete-time process in Theorem 1. Then we derive homothetic tubes for slow states in Proposition 2. Based on these and Theorem 1, homothetic tubes for fast states are constructed in Corollary 3. In Lemma 4, we analyze the open-loop and closed-loop property of process controlled by the reformulated MPC with the backup strategy.

A. RCCM for discrete-time process

We first recall some definitions and results from geometry theory [20]. A Riemannian metric on $\mathbb{R}^n$ is a smooth symmetric valued function $M(x) \in \mathbb{R}^{n\times n}$, which is positive definite for all x, and defines a structure for any two tangent vectors through the inner product. A metric is called uniformly bounded if there exists scalars $a_2 \geq a_1 > 0$, such that $a_1 I \prec M(x) \prec a_2 I$ for all $x \in \mathbb{R}^n$. Let $\Gamma(a,b) := \left\{ c \in C^1([0,1], \mathbb{R}^n \mid c(0) = a, c(1) = b) \right\}$ be the set of smooth paths from a to b and $c_s := \frac{\partial c}{\partial s}$. The energy of a path $c \in \Gamma(a,b)$ for a uniformly bounded metric $M(x)$ is defined as $E(c) := \int_0^1 c_s(s)^\top M(c(s)) c_s(s) ds$. The minimal energy of a path joining a and b is defined as $E^\star(a,b) := \min_{c\in\Gamma(a,b)} E(c)$, and the path achieving the minimal energy is geodesic, denoted by $c^\star$, which exists following Hopf-Rinow theorem and uniform boundedness of metrics, cf. [10].

Consider a nonlinear discrete-time system

$$\begin{aligned} x(i+1) &= f(x(i), u(i), d(i)), \\ y(i) &= h(x(i), u(i)),\ i \in \mathbb{N}_0 \end{aligned} \quad (6)$$

with continuously differentiable f and h, where states $x(i)$, control inputs $u(i)$ and disturbances $d(i)$ lie in convex bounded sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{D}$ respectively. Similar to [14] and [15], we then define matrix functions $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $B_d = \frac{\partial f}{\partial d}$, $C = \frac{\partial h}{\partial x}$, $D = \frac{\partial h}{\partial u}$ and $D_d = \frac{\partial h}{\partial d}$ as partial derivatives of f and h along arbitrary trajectory (x, d, u) associated with (6).

In the presence of disturbance errors, we will design a feedback controller with disturbance rejection property and then analyze output deviations of controlled system. To this aim, let us consider a uniformly bounded metric W, a matrix-valued function L, and scalars $\alpha \in (0,1)$, ν and γ satisfying

$$\begin{pmatrix} W \circ f & AW + BL & B_d \\ (AW+BL)^\top & (1-\alpha)W & O \\ B_d^\top & O & \nu I \end{pmatrix} \succ 0, \quad (7)$$

$$\begin{pmatrix} \alpha W & O & (CW+DL)^\top \\ O & (\gamma-\nu)I & O \\ CW+DL & O & \gamma I \end{pmatrix} \succ 0 \quad (8)$$

for all $(x,d,u) \in \mathbb{X}\times\mathbb{D}\times\mathbb{U}$. Given any $(\widehat{x},\widehat{u},\widehat{d},\widehat{y})$ as a nominal trajectory of the system (6), we then design a feedback control law as $u(i) = \kappa(\widehat{u}(i),\widehat{x}(i),x(i)) = \upsilon(i,1)$, where

$$\upsilon(s;i) = \widehat{u}(i) + \int_0^s K(c^\star(s;i))\frac{\partial c^\star(s;i)}{\partial s}ds,\ K(\cdot) = L(\cdot)W(\cdot)^{-1},$$
$$c^\star(s;i) = \underset{c\in\Gamma(\widehat{x}(i),x(i))}{\arg\min}\ E(c),\ M(\cdot) = W(\cdot)^{-1}. \quad (9)$$

Theorem 1: Consider (6). Given $\alpha \in (0,1)$, and let W, L, ν and γ satisfy (7) and (8) for all $(x,d,u)\in\mathbb{X}\times\mathbb{D}\times\mathbb{U}$ and $(c^\star,\upsilon)$ be defined by (9). For each $i\in\mathbb{N}_0$, assume that $(c^\star(s;i),\upsilon(s;i))\in\mathbb{X}\times\mathbb{U}$ for all $s\in(0,1)$ if it holds at $s=0$ and $s=1$. Then, for any trajectory $(\widehat{x},\widehat{u},\widehat{d},\widehat{y})$ satisfying (6), the state and output trajectories x and y of the system (6) driven by $u=\kappa(\widehat{u},\widehat{x},x)$ and d are confined to

$$E^\star(\widehat{x}(i+1),x(i+1)) < (1-\alpha)E^\star(\widehat{x}(i),x(i)) + \nu\|d(i)-\widehat{d}(i)\|^2 \quad (10)$$

and

$$\|\widehat{y}(i)-y(i)\|^2 < \gamma\alpha E^\star(\widehat{x}(i),x(i)). \quad (11)$$

Proof: Let $\mathscr{A} := A+BLM$ and $\mathscr{C} := C+DLM$. By applying congruence transformation to (7) and (8) with $\mathrm{diag}(I,M,I)$ and $\mathrm{diag}(M,I,I)$ respectively, and then performing Schur complement, we get

$$\begin{pmatrix} \mathscr{A}^\top(M\circ f)\mathscr{A}-(1-\alpha)M & \mathscr{A}^\top MB_d \\ B_d^\top M\mathscr{A} & B_d^\top MB_d-\nu I \end{pmatrix} \prec 0 \quad (12)$$

and

$$\begin{pmatrix} \alpha M-\gamma^{-1}\mathscr{C}^\top\mathscr{C} & O \\ O & (\gamma-\nu)I \end{pmatrix} \succ 0, \quad (13)$$

with M defined in (9). In virtue of uniform boundedness of W, M is also uniformly bounded, which can be used as Riemannian metric to construct minimal energy $E(\widehat{x}(i),x(i))$ yielding a smooth unique geodesic $c^\star(s;i)$ for each fixed i with $s\in[0,1]$. For each i, similar to [16], we parameterize disturbance as $\omega(s;i) = (1-s)\widehat{d}(i)+sd(i)$, and then use geodesic $c^\star(i,s)$ to parameterize control inputs, outputs, and states at i and $i+1$, which are given by, $\upsilon(s;i)$ from (9), $\xi(s;i) := h(c^\star(s;i),\upsilon(s;i),\omega(s;i))$, $c^\star(s;i)$, $c(s;i+1) := f(c^\star(s;i),\upsilon(s;i),\omega(s;i))$ jointing $\widehat{x}(i+1)$ and $x(i+1)$ respectively. By assumption and due to convexity of $\mathbb{D}$, the conditions (12)-(13) parameterized by these paths still hold for $s\in(0,1)$. Differentiating these paths with respect to s yields

$$\begin{aligned} c_s(s;i+1) &= \mathscr{A}c_s{}^\star(s;i)+B_d\omega_s \\ \xi_s(s;i) &= \mathscr{C}c_s{}^\star(s;i)+D_d\omega_s \end{aligned} \quad (14)$$

with $\omega_s = d(i)-\widehat{d}(i)$ and $\upsilon_s = K(c^\star(s;i))c_s{}^\star(s;i)$, where subscript s denotes the partial derivative with respect to s. By multiplying (12) and (13) by $(c_s^{\star\top},\omega_s^\top)$, inserting (14) into the resulting inequalities and then integrating them over s from 0 to 1, we get

$$E(c(i+1,\cdot)) < (1-\alpha)E^\star(\widehat{x}(i),x(i)) + \nu\|d(i)-\widehat{d}(i)\|^2 \quad (15)$$

and (11) for all i. Since $E^\star(\widehat{x}(i+1),x(i+1)) \le E(c(i+1,\cdot))$ by definition of the minimal energy, the left side of (15) can be replaced by $E^\star(\widehat{x}(i+1),x(i+1))$, which gives (10). ■

Given bounds on errors of initial states and disturbances, the output error can be characterized point-wisely by invoking (10) together with (11) yielding homothetic tubes. Note that, conditions (7) and (8) are parameter dependent LMIs, which can be evaluated efficiently by the gridding technique or relaxation techniques, see [22] and the references therein.

B. Homothetic tubes for two-timescale processes

In what follows, we systematically derive tubes for states and inputs within horizon, i.e., $x^f(k)-\widehat{x}^f(k)$, $x^s(k)-\widehat{x}^s(\sigma_{\bar{n}}(k))$ and $u(k)-\widehat{u}(k)$ for $k\in[0:N]$ subject to the surrogate model (4)-(5) and the true process (1)-(2) controlled by the feedback control law κ with same exogenous inputs w and initial conditions. To this end, we treat slow and fast dynamics individually and first analyze the error of slow states, as presented in the following,

Proposition 2: Consider both slow submodels (2) and (5) with the initial approximation error

$$\bar{e}_0 := \|\widehat{x}^s(\sigma_{\bar{n}}(0))-x^s(0)\|^2 = 0. \quad (16)$$

Assume that $\widehat{x}^f(k), x^f(k)\in\mathbb{X}_f$, $x^s(k)\in\mathbb{X}_s$ for $k\in[0:i-1]$ with a fixed $i>0$. We further assume that $\widehat{x}^s(\sigma_{\bar{n}}(i-l\bar{n}))\in\mathbb{X}_s$ for $l\in[1:\lfloor i/\bar{n}\rfloor]$ in case of $i\ge\bar{n}$. Then

$$\|x^s(i)-\widehat{x}^s(\sigma_{\bar{n}}(i))\|^2 \le (i\varepsilon+\lfloor i/\bar{n}\rfloor\widehat{\varepsilon})^2 =: \bar{e}_i \quad (17)$$

for $i\in\mathbb{N}_0$, where ε and $\widehat{\varepsilon}$ are maximum of $\|g(x_1,x_2,x_3)-x_2\|$ and $\|g(x_1,x_2,x_3;\bar{n})-x_2\|$ respectively over $(x_1,x_2,x_3)\in\mathbb{X}_f\times\mathbb{X}_s\times\mathbb{W}$.

Proof: To alleviate notations, we use subscript i to indicate the time dependency of x^f, x^s as well as w and use $\widehat{x}_i^s$ and $\widehat{x}_i^f$ to denote $\widehat{x}^s(\sigma_{\bar{n}}(i))$ and $\widehat{x}^f(\sigma_{\bar{n}}(i))$ respectively. We also denote $g(\cdot;\bar{n})$ by $\widehat{g}(\cdot)$. For $j\in\mathbb{Z}$ and $i\in\mathbb{N}_0$ satisfying $i+j\bar{n}\ge 0$, we have

$$\widehat{x}^s(\sigma_{\bar{n}}(i+j\bar{n})) = \widehat{x}^s(\sigma_{\bar{n}}(i)+j\bar{n}), \quad (18)$$

since $\lfloor(i+j\bar{n})/\bar{n}\rfloor = \lfloor i/\bar{n}\rfloor + j$.

Let us consider the trajectory of (2), we have

$$\begin{aligned} x_i^s - x_{i-1}^s &= g(x_{i-1}^f,x_{i-1}^s,w_{i-1})-x_{i-1}^s \\ &\le \max_{x_1\in\mathbb{X}_f,x_2\in\mathbb{X}_s,x_e\in\mathbb{W}}\|g(x_1,x_2,x_3)-x_2\| =: \varepsilon, \end{aligned} \quad (19)$$

where the above maximum exists due to continuity of function g and compactness of $\mathbb{X}_f\times\mathbb{X}_s\times\mathbb{W}$.

Similarly, for the submodel (5) with $i\ge\bar{n}$, we have

$$\begin{aligned} \widehat{x}_i^s - \widehat{x}_{i-\bar{n}}^s &\overset{(18)}{=} \widehat{g}(\widehat{x}_{i-\bar{n}}^f,\widehat{x}_{i-\bar{n}}^s,w(\sigma_{\bar{n}}(i-\bar{n})))-\widehat{x}_{i-\bar{n}}^s \\ &\le \max_{x_1\in\mathbb{X}_f,x_2\in\mathbb{X}_s,x_e\in\mathbb{W}}\|\widehat{g}(x_1,x_2,x_3)-x_2\| =: \widehat{\varepsilon}. \end{aligned} \quad (20)$$

In the end, we can conclude that

$$\begin{aligned} \|x_i^s-\widehat{x}_i^s\| &\le \|x_i^s-x_0^s+\widehat{x}_0^s-\widehat{x}_i^s\|+\|x_0^s-\widehat{x}_0^s\| \\ &\overset{(18),(16)}{\le} \|\sum_{j=1}^{i}(x_j^s-x_{j-1}^s)+\sum_{j=1}^{\lfloor i/\bar{n}\rfloor}(\widehat{x}_{i-j\bar{n}}^s-\widehat{x}_{i-(j-1)\bar{n}}^s)\| \\ &\overset{(19),(20)}{\le} i\varepsilon+\lfloor i/\bar{n}\rfloor\widehat{\varepsilon},\ i\ge\bar{n}, \end{aligned}$$

together with

$$\|x_i^s - \widehat{x}_i^s\| = \|x_i^s - \widehat{x}_0^s\| \overset{(16)}{=} \|x_i^s - x_0^s\| \overset{(19)}{\leq} i\varepsilon,\ i \in [0:\bar{n}-1],$$

which lead to (17). ■

The error bound of slow states $\bar{e}_i$ obtained above can be treated as disturbance error bound for the fast submodel (4). With this in mind, we will derive homothetic tubes for fast states and control inputs in the following Corollary deduced from Theorem 1 together with stricter conditions on W and L. To this end, we first redefine matrix functions $A = \frac{\partial f}{\partial \widehat{x}^f}$, $B = \frac{\partial f}{\partial \widehat{u}}$, $B_d = \frac{\partial f}{\partial \widehat{x}^s}$ with f as the vector field of the fast submodel (4). Then, we consider a constant metric $W \succ 0$, a matrix L and scalars ν, $\gamma^x > \nu, \gamma^u > \nu$ satisfying (7),

$$\begin{pmatrix} \alpha W & W^\top \\ W & \gamma^x I \end{pmatrix} \succ 0, \quad \begin{pmatrix} \alpha W & L^\top \\ L & \gamma^u I \end{pmatrix} \succ 0, \tag{21}$$

for all $(\widehat{x}^f, \widehat{x}^s, \widehat{u}, w) \in \mathbb{X}_f \times \mathbb{X}_s \times \mathbb{U} \times \mathbb{W}$.

Corollary 3: Let the assumption in Proposition 2 hold. Consider the fast submodel (4). Given $\alpha \in (0,1)$, and assume that there exist a constant metric $W \succ 0$, a matrix L and scalars ν, $\gamma^x > \nu, \gamma^u > \nu$ satisfying (7) and (21) for all $(\widehat{x}^f, \widehat{x}^s, \widehat{u}, w) \in \mathbb{X}_f \times \mathbb{X}_s \times \mathbb{U} \times \mathbb{W}$. Then, for any trajectory $(\widehat{x}^f, \widehat{x}^s, \widehat{u}, w)$ of the model (4)-(5), the fast state x^f of the true process (1)-(2) driven by w and $u(i) = \kappa(\widehat{u}(i), \widehat{x}^f(i), x^f(i)) = \upsilon(i,1)$ with

$$\upsilon(s;i) = \widehat{u}(i) + sK(x^f - \widehat{x}^f),\ K = LW^{-1} \tag{22}$$

satisfies

$$E(\widehat{x}^f(i), x^f(i)) \leq \sum_{j=0}^{i-1} \nu(1-\alpha)^{i-j}\bar{e}_j =: e_i, \tag{23}$$

for $i \in \mathbb{N}$ with error bounds of slow states $\bar{e}_j$ defined in (17). Furthermore,

$$\|x^f(i) - \widehat{x}^f(i)\|^2 \leq \alpha\gamma^x e_i =: e_i^x, \tag{24}$$

$$\|u(i) - \widehat{u}(i)\|^2 \leq \alpha\gamma^u e_i =: e_i^u,\ i \in \mathbb{N}. \tag{25}$$

Proof: For a constant metric W, the geodesic $c^\star(i,x)$ in (9) is a straight line adjoining $\widehat{x}(i)$ and $x(i)$, see e.g. Section 16.1 in [21]. Hence, for constant L, (9) is reduced to (22), which is also a straight line. Consequently, the paths $\upsilon(s;i)$ and $c^\star(s;i)$ stay in convex sets $\mathbb{U}$ and $\mathbb{X}_f$ for $s \in (0,1)$, if both ends, i.e., $s \in \{0,1\}$, do. Since the assumption in Prop. 2 holds, the error of slow states is bounded by (17). Finally, (23)-(25) follow from the same line of argumentation as in the proof of Theorem 1. ■

C. MPC framework with backup strategy

We present the tube-based MPC framework using the surrogate model, which admits less computation costs than standard MPC (3). The optimization problem of MPC reads

$$\begin{aligned} \min_{\substack{\widehat{u}(\cdot|i);\\ \widehat{x}^f(\cdot|i), \widehat{x}^s(\cdot|i)}} \ & \sum_{k=1}^{N} l(\widehat{x}^s(\sigma_{\bar{n}}^k|i), \widehat{x}^f(k|i), \widehat{u}(k-1|i)) \\ s.t.\ & \widehat{x}^f(k+1|i) = f(\widehat{x}^f(k|i), \widehat{x}^s(\sigma_{\bar{n}}^k|i), \widehat{u}(k|i), w(i+k)), \\ & \widehat{x}^s(\sigma_{\bar{n}}^k + \bar{n}|i) = g(\widehat{x}^f(\sigma_{\bar{n}}^k|i), \widehat{x}^s(\sigma_{\bar{n}}^k|i), w(i+\sigma_{\bar{n}}^k); \bar{n}), \\ & \widehat{x}^s(\sigma_{\bar{n}}^0|i) = x^s(i),\ \widehat{x}^s(\sigma_{\bar{n}}^k|i) \in \mathbb{X}_s \ominus \mathcal{B}^m(\bar{e}_{\sigma_{\bar{n}}^k + \bar{n}-1}), \\ & \widehat{x}^f(0|i) = x^f(i),\ \widehat{x}^f(k|i) \in \mathbb{X}_f \ominus \mathcal{B}^n(e_k^x), \\ & \widehat{u}(k|i) \in \mathbb{U} \ominus \mathcal{B}^p(e_k^u),\ k \in [0:N-1], \\ & (\widehat{x}^f(N|i), \widehat{x}^s(\sigma_{\bar{n}}^N|i) \in \mathbb{Z}_N \ominus (\mathcal{B}(e_N^x) \times \mathcal{B}(\bar{e}_N)), \end{aligned} \tag{26}$$

with $\sigma_{\bar{n}}^k$ denoting $\sigma_{\bar{n}}(k)$, e_k^x and e_k^u taken from (24)-(25) for $k \in \mathbb{N}$ and $e_0^x = e_0^u = 0$. The optimal input sequence $\widehat{u}^\star(\cdot|i)$ together with its associated state trajectory $\widehat{x}^f(\cdot|i)$ are used to construct an input sequence

$$u(\cdot|i) = \kappa_{[0,N-1]}(\widehat{u}^\star(\cdot|i), \widehat{x}^f(\cdot|t), x^f(\cdot|i)),$$

where

$$\begin{aligned} \kappa_{[i,j]}(\widehat{u}^\star(\cdot|i), \widehat{x}^f(\cdot|i), x^f(\cdot|i)) &:= (\kappa(\widehat{u}^\star(i|i), \widehat{x}^f(i|i), x^f(i|i)), \\ &\dots, \kappa(\widehat{u}^\star(j|i), \widehat{x}^f(j|t), x^f(j|i))),\ i,j \in [0:N-1],\ i<j, \end{aligned}$$

with κ specified in (9). The prediction $x^f(k|i)$ denoted by $x_{k|i}^f$, is subject to

$$\begin{aligned} x_{k+1|i}^f &= f(x_{k|i}^f, x_{k|i}^s, \kappa(\widehat{u}_{k|i}^\star, \widehat{x}_{k|i}^f, x_{k|i}^f), w(i+k)) \\ x_{k+1|i}^s &= g(x_{k|i}^f, x_{k|i}^s, w(i+k)),\ x_{0|i}^s = x^s(i),\ x_{0|i}^f = x^f(i). \end{aligned}$$

Since $\widehat{x}^f(0|i) = x^f(i)$, the applied closed-loop input $u_{0|i} = \kappa(\widehat{u}_{0|i}^\star, \widehat{x}_{0|i}^f, x_{0|i}^f)$ coincides with $\widehat{u}_{0|i}^\star$.

As inferred from (17) in Proposition 2, the error bound on the slow state grows in time, which precludes from ensuring recursive feasibility of (26) using terminal ingredients as done, e.g., in [19]. Inspired by [18], we devise a backup strategy to generate an input sequence for the problem (3) without enforcing the recursive feasibility of (26) via

$$u(\cdot|i) = \begin{cases} \kappa_{[0,N-1]}(\widehat{u}^\star(\cdot|i), \widehat{x}^f(\cdot|t), x^f(\cdot|i)), & \text{(26) is feasible} \\ (u_{[1,N-1]}(i-1), u_N(i)), & \text{otherwise} \end{cases} \tag{27}$$

with shorthand notations $u_{[l,j]}(i) := (u(l|i), \dots, u(j|i)), \forall l,j \in [0:N-1], l<j$, and $u_N(i) := \kappa_N(x^f(N-1|i), x^s(N-1|i), w(N+i-1))$. The states $x^f(N-1|i)$ and $x^s(N-1|i)$ are predictions of the true process (1)-(2) driven by input sequence $u_{[1,N-1]}(i-1)$ under initial conditions $x^s(i), x^f(i)$.

D. Open-loop and closed-loop constraints satisfaction

As indicated by the assumption related to Proposition 2 in Corollary 3, the validity of tubes for problem (26), which is crucial for ensuring constraint satisfaction of the true process (1)-(2), hinges on the assumption that the true process satisfies prescribed constraints. This in turn should be guaranteed by (26). Despite of such a seeming causality dilemma, we show that the proposed MPC scheme with the back-up strategy can still guarantee open-loop and closed-loop constraints satisfaction.

Lemma 4: Let problem (26) be feasible at $i=0$. The control input sequence $u(\cdot|0)=\kappa_{[0,N-1]}(\widehat{u}^{\star}(\cdot|0),\widehat{x}^{f}(\cdot|0),x^{f}(\cdot|0))$ constitutes a feasible solution of MPC problem (3) at $i=0$. Furthermore, the input sequence constructed recursively in accordance with the backup strategy (27) ensure open-loop and closed-loop constraints satisfaction over time.

Proof: Since problem (26) is feasible at $i=0$, we have $(\widehat{x}^{f}(k|0),\widehat{x}^{s}(\sigma_{\bar{n}}^{k}|0))\in\mathbb{X}_f\times\mathbb{X}_s$ for all $k\in[0:N-1]$ and $x^s(0|0)=x^s(0)=\widehat{x}^s(0|0)\in\mathbb{X}_s$, $x^f(0|0)=x^f(0)=\widehat{x}^f(0|0)\in\mathbb{X}_f$. As for an arbitrarily chosen but fixed $\bar{k}\in[0:N-2]$, if we assume $(x^f(k|0),x^s(k|0))\in\mathbb{X}_f\times\mathbb{X}_s$ for all $k\in[0:\bar{k}]$, then we have $\|x^s(k+1|0)-\widehat{x}^s(\sigma_{\bar{n}}^{k+1}|0)\|^2\le\bar{e}_{k+1}$ for all $k\in[0:\bar{k}]$ by applying Proposition 2. Due to monotonicity of $\bar{e}_k$, we have $\bar{e}_{k+1}\le\bar{e}_{\sigma_{\bar{n}}(k+1)+\bar{n}-1}$. By invoking Corollary 3, we further have $\|x^f(k+1|0)-\widehat{x}^f(k+1|0)\|^2\le e^x_{k+1}$ and $\|\widehat{u}^{\star}(k|0)-u(k|0)\|^2\le e^u_k$ with $u(k|0)=\kappa(\widehat{u}^{\star}(k|0),\widehat{x}^f(k|0),x^f(k|0))$ as the input of the true process (1)-(2) for all $k\in[0:\bar{k}]$. Henceforth, besides $u(k|i)\in\mathbb{U}$ for $k\in[0:\bar{k}]$, we can guarantee that $(x^f(k|0),x^s(k|0))\in\mathbb{X}_f\times\mathbb{X}_s$ for $k\in[0:\bar{k}+1]$. Proceeding similarly as above and assuming $(x^f(k|0),x^s(k|0))\in\mathbb{X}_f\times\mathbb{X}_s$ for all $k\in[0:N-1]$, we get $(x^f(N|0),x^s(N|0))\in\mathbb{Z}_N$. Finally, based on assume-guarantee reasoning, we can conclude that the input sequence $u(\cdot|0)$ is feasible for problem (3) at $i=0$ under the assumption in Lemma 4.

Since $u_{[0,N-1]}(0)=\kappa_{[0,N-1]}(\widehat{u}^{\star}(\cdot|0),\widehat{x}^f(\cdot|0),x^f(\cdot|0))$ is feasible for (3) at $i=0$, by shifting this input sequence and exploiting the terminal controller κ_N associated with the terminal set $\mathbb{Z}_N$, we can construct a feasible control input sequence $(u_{[1,N-1]}(0),u_N(1))$ according to the back-up strategy (27) for $i=1$. The statement then follows by induction. ∎

IV. Simulation Study

To illustrate the benefit of the proposed method, we apply standard MPC (sMPC) formulated in (3) and the proposed tube-based MPC (tbMPC) to a case study on a crop growth process adapted from [23]. Consider the discrete-time system

$$
\begin{aligned}
x^s_{d,j}(i+1) &= x^s_{d,j}(i)+\Delta_t\left(\alpha_{d1,j}\phi_{p,j}(i)-\alpha_{d2,j}\phi_{r,j}(i)\right),\\
x^f_C(i+1) &= x^f_C(i)+\Delta_t\left(-\alpha_{c1}\sum_{j=1}^{2}\phi_{p,j}(i)+\alpha_{c2}\sum_{j=1}^{2}\phi_{r,j}(i)\right)\\
&\quad+\Delta_t\left(\alpha_{c3}(d_C(i)-x^f_C(i))u_v(i)+\alpha_{c4}u_c(i)\right),\\
x^f_T(i+1) &= x^f_T(i)+\Delta_t\left(\alpha_{T1}u_h(i)-\alpha_{T2}\left(x^f_T(i)-d_T(i)\right)u_v(i)\right),
\end{aligned}
$$

with $j\in\{1,2\}$ where $\phi_{r,j}(i)=x^s_{d,j}(i)2^{0.1x^f_T(i)-2.5}$ and $\phi_{p,j}=(1-e^{-\xi_j x^s_{d,j}(i)})\phi(i)$ with

$$
\phi(i)=\frac{100\theta_0(x^f_C(i)-\theta_4)(-\theta_1 x^f_T(i)^2+\theta_2 x^f_T(i)-\theta_3)}{100\theta_0+(x^f_C(i)-\theta_4)(-\theta_1 x^f_T(i)^2+\theta_2 x^f_T(i)-\theta_3)}.
$$

The sample time $\Delta_t=60$, i.e.,, 1 min. The slow states $x^s_{d,j}$ represent dry weight of two different types of crop in kg·m^{-2}. The fast states x^f_C and x^f_T represent the inside CO2 concentration in kg·m^{-3} and the inside air temperature in °C respectively, see [23]. Physical limitations on the control input (u_v,u_h,u_c) is normalized to $\mathbb{U}=[0,1]^3\subset\mathbb{R}^3$. Outside temperature d_T and CO2 concentration d_C are assumed to be predicted based on weather forecasts. For simplicity, we assume $d_C(i)=6\times10^{-4}$, whereas $d_T\in\mathbb{W}=[2,10]$. The model parameters appearing also in [23], are chosen same as [23], while the other parameters are chosen as $\xi_2=55$, $\alpha_{d1,2}=0.57$ and $\alpha_{d2,2}=2.5\times10^{-7}$. The fast states (x^f_C,x^f_T) are subject to box constraints $\mathbb{X}_f=[0,0.00275]\times[5,35]$, whereas the slow states $(x^s_{d,1},x^s_{d,2})$ are unconstrained. However, for the parameterization of the tubes, we choose $\mathbb{X}_s=[0.08,0.15]^2$, which is sufficiently large in the sense that, under constrained inputs and fast states, the slow states of the true process and the surrogate model remain feasible.

In the simulation, three MPCs are implemented: standard MPC (sMPC) in (3), and the proposed tube-based MPC (tbMPC) in (26) with time scaling factors $\bar{n}=5$ and $\bar{n}=10$. The whole simulation is implemented on a PC with Intel i7-12700K and 32GB RAM. We use IPOPT [24] and CasADi [25] to implement MPC, and LMI solvers as well as optimization toolbox from Matlab to compute tube parameters.

The metric W and associated parameters are computed by solving the optimization problem with the conditions (7) and (21) subject to state and control constraints. As (7) is affine in (u_v,u_h,u_c) and external inputs d_T,d_C, but non-affine in states $(x^s_{d,j},x^f_C,x^f_T)$, we leverage gridding technique to render optimization problem numerically tractable. Since x^f_C varies in a significantly smaller scale range than x^f_T, we treat them separately in optimization by replacing the first inequality in (21) with

$$
\begin{pmatrix}\alpha W_1 & W_1^\top\\ W_1 & \gamma^{x^f_C}I\end{pmatrix}\succ0,\ \begin{pmatrix}\alpha W_2 & W_2^\top\\ W_2 & \gamma^{x^f_T}I\end{pmatrix}\succ0,\ W=\begin{pmatrix}W_1 & 0\\ 0 & W_2\end{pmatrix},
$$

and modifying the cost to $\beta_C\gamma^{x^f_C}+\beta_T\gamma^{x^f_T}+\gamma^u$. Consequently, the tube for states (x^f_C,x^f_T) becomes $\mathcal{B}(e_k^{x^f_C})\times\mathcal{B}(e_k^{x^f_T})$ with $e_k^{x^f_C}=\gamma^{x^f_C}\alpha e_k$ and $e_k^{x^f_T}=\gamma^{x^f_T}\alpha e_k$. By solving the aforementioned optimization problem with $(\beta_C,\beta_T,\alpha)=(1.25\times10^5,0,0.9)$, we obtain $\nu=0.0056$, $(\gamma^{x^f_C},\gamma^{x^f_T},\gamma^u)=(0.0084,294.01,813.79)$. Resolving two optimization problems, (19) and (20) in Proposition 2, in terms of $\bar{n}=5$ and $\bar{n}=10$ yields $(\varepsilon,\widehat{\varepsilon})=(1.4\times10^{-5},7.2\times10^{-5})$ and $(\varepsilon,\widehat{\varepsilon})=(1.4\times10^{-5},1.4\times10^{-4})$, respectively.

For the MPC framework, the stage cost is chosen to be

$$
l(x^s(i),x^f(i),u(i-1))=-(x^s_{d,1}+x^s_{d,2})^2+u^\top Ru
$$

with $R=\mathrm{diag}(0.001,0.05,0.05)$. In view of slowly varying dry weights $x^s_{d,j}$, a long prediction horizon N is desired to maximize the growth and penalize control effort. Here, we choose $N=7200$, i.e., 120 min, for sMPC and tbMPC. Since the proposed tbMPC is already feasible during the simulation without terminal constraints, which, together with Lemma 4, indicates that the sMPC is meanwhile also feasible, we henceforth drop the terminal set $\mathbb{Z}_N$ for the implementation.

The following table lists the computation time for the online implementation and the closed-loop cost of each considered MPC simulated over 20 hours. It clearly shows the computational efficiency of the proposed approach, whereas the cost only marginally increases.

TABLE I: Summary of computation times and costs

	Computation Time (s)	Cost
sMPC	263.65	-71.71
tbMPC ($\bar{n} = 5$)	179 (-32%)	-71.66 ($+0.07\%$)
tbMPC ($\bar{n} = 10$)	174.64 (-34%)	-71.6 ($+0.15\%$)

The simulation results of the closed-loop system with sMPC and tbMPC with $\bar{n} = 10$ are visualized in Fig. 1. The result of tbMPC with $\bar{n} = 5$ is left out for better visibility. It can be seen that, all constraints are satisfied. The inputs of tbMPC are generally smaller than that of sMPC, as shown in Fig. 1 (d) and (e). This is mainly attributed to the implicitly changed weighting ration between control inputs and slow states resulted from model errors within each prediction horizon.

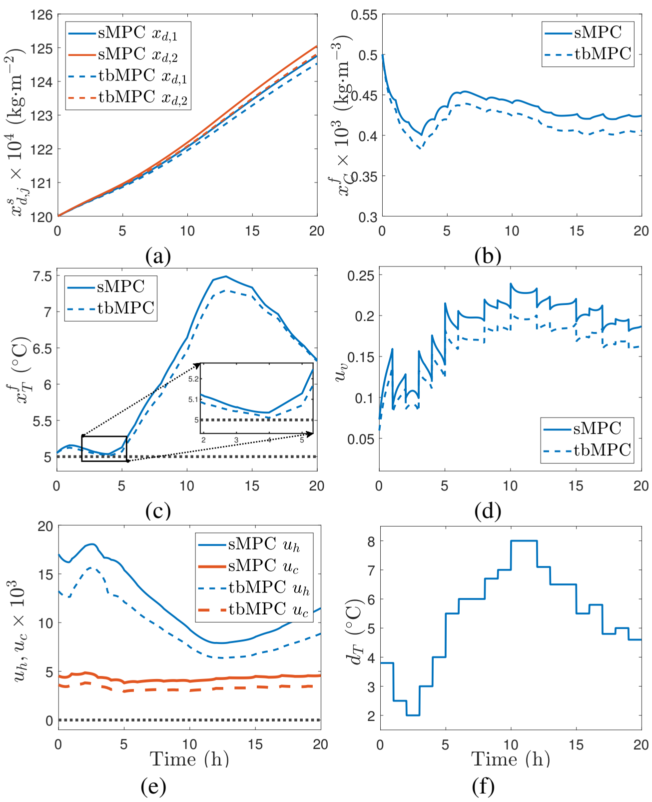

Fig. 1: Closed-loop trajectories by tbMPC with $\bar{n} = 10$ and sMPC. In (c) and (e), the dotted black lines represent the lower bound constraint for states and control inputs.

V. CONCLUSIONS

In this paper, we propose a novel computationally efficient MPC framework for two-timescale discrete-time nonlinear processes using surrogate models with lower complexity. By leveraging RCCM and constraints tightening, state and input constraints are guaranteed.

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