Homogeneity-Based Finite-Time Stabilization of Linear Descriptor Systems

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Abstract—The homogeneity concept is extended to the area of descriptor systems. A homogeneity-based admissible control is designed for stabilization of linear descriptor systems in a finite time. The presented feedback homogenizes a linear system with a specified negative degree and stabilizes it in a finite time. The tuning procedure is formalized in LMI form. The theoretical results are supported by numerical simulations.

I. INTRODUCTION

The classical state-space representation of systems corresponds to a set of ordinary differential equations (ODEs) or difference equations with a minimum number of state variables required to represent a given system. In many cases such representation does not allow to choose physical variables as state variables in a natural way due to they may be related algebraically. This may result in a descriptor system model (also referred to as singular, differential-algebraic, or generalized state-space systems) since some of the relationships of the variables are dynamic while others are purely static [1], [2], [3], [4], [5]. Representation of systems in a descriptor form has found a number of applications, e.g., electrical circuits [1], [6], [7], [8] mechanical systems with phase constraints [9], [10], chemistry and biology [11], [12], [13], economics [14], etc.

Motivated by modern applications, the finite-time stabilization (ensuring the completion of all transients in a finite time) is continuing to be the subject of numerous research (see, for example, [15], [16], [17], [18], [19], [20], [21], [22], [23], etc.). The homogeneity property is widely used for finite-time control design: if a homogeneous system with negative degree is asymptotically stable, then it is finite-time stable (see, for example, [15], [18], [23], [24], etc.). Additionally, homogeneous systems have other very useful features, such as behaviour scalability (local property can be transformed to the whole state space), and robustness (e.g., Input-to-State Stability and delay robustness) [25], [26].

The finite-time control design problem for descriptor systems has received much less attention due to difficulty involved, and as a result, there are few works reported in literature (e.g., [27], [28], [29]). Note that in some cases with the use of proper decomposition it is possible to represent a system as two subsystems: differential and algebraic ones. However, a finite-time control cannot be easily generalized to the class of descriptor systems due to several reasons: solution of a descriptor system may contain impulsive modes [1]; a canonical form that allows to apply finite-time controller to differential part only may not exist; the representation in the canonical form may in some cases be accompanied by computational errors. To ensure that a closed-loop system is admissible, a controller must stabilize the system and make it impulse free. Thus, the problem of finite-time stabilization of descriptor system is a highly relevant problem.

It should be noted that the term finite-time stability, with a very different meaning than that considered here, has also appeared in the literature, e.g., [30], [31].

The present paper addresses two problems. First, the homogeneity concept is extended on the class of descriptor systems, and the necessary and sufficient conditions of linear descriptor systems to be homogeneous are obtained. Second, homogeneity-based finite-time stabilization control of linear descriptor systems is proposed. The presented feedback does not require special canonical forms or block decomposition, which can be accompanied by significant computational errors. The presented state-feedback controller contains two terms: a linear term that homogenizes the system with a specified negative degree and a nonlinear one, which provides convergence of the system solution to zero in a finite time. The tuning procedure is based on linear matrix inequalities and equations solution.

Notation: \( \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\} \), where \( \mathbb{R} \) is the field of real numbers; \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space with vector norm \( \| \cdot \| ; I_n \) denotes the identity matrix of order \( n \); \( \mathbb{C} \) is the field of complex numbers; \( \lambda(E, A) \) is a finite eigenvalues of matrix pencil \( (sE - A) \), i.e., \( \lambda(E, A) = \{s \in \mathbb{C}, s \) is finite, \( \det(sE - A) = 0\} \) for \( E, A \in \mathbb{R}^{n \times n} \), and \( \lambda(A) = \lambda(I_n, A) \) is a spectrum of the matrix \( A \); the inequality \( P > 0 \) \( (P \geq 0) \) means that a symmetric matrix \( P = P^T \in \mathbb{R}^{n \times n} \) is positive definite (positive semi-definite); the eigenvalues of a matrix \( G \in \mathbb{R}^{n \times n} \) are denoted by \( \lambda_i(G), \) \( i = 1, \ldots, n \); \( \Re(\lambda) \) denotes the real part of a complex number \( \lambda \).

The paper is organized in the following way. Section II recalls some basics on finite-time stability, descriptor systems and homogeneity. Section III presents the main result on homogeneous descriptor systems and finite-time feedback design for linear descriptor systems. Simulation results are shown in Section IV. Finally, concluding remarks are given.
in Section V.

II. PRELIMINARIES

A. Stability notions

Consider the system
\[ \dot{x}(t) = f(x(t)), \ x(0) = x_0, \ t \geq 0, \] (1)
where \( x(t) \in \mathbb{R}^n \) is the state vector, \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous, \( f(0) = 0 \).

**Definition 1** [16], [17] The origin of (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution \( \Phi_{x_0}(t) \) of the system (1) reaches the equilibrium point at some finite time moment, i.e. \( \Phi_{x_0}(t) = 0 \ \forall t \geq T(x_0) \) and \( \Phi_{x_0}(t) \neq 0 \ \forall t \in [0, T(x_0)), x_0 \neq 0 \), where \( T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}, T(0) = 0 \) is a settling-time function.

**Theorem 1** [16] Suppose there exist a positive definite \( C^1 \) function \( V \) defined on an open neighborhood of the origin \( D \subset \mathbb{R}^n \) and real numbers \( \beta \in \mathbb{R}_+ \) and \( \sigma \in (0, 1) \), such that
\[ \dot{V}(x) \leq -\beta V^\sigma(x), \ x \in D \setminus \{0\}. \]
Then the origin is finite-time stable and \( T(x_0) \leq \frac{1}{2(1-\sigma)}V_0^{1-\sigma} \), where \( V_0 = V(x_0) \). If \( D = \mathbb{R}^n \) and function \( V \) is radially unbounded, then the system (1) admits these properties globally.

B. Descriptor systems

Consider the following descriptor system
\[ E(x(t)) \dot{x}(t) = f(x(t), u(x(t))), \ x(0) = x_0, \ t \geq 0, \] (2)
where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is a control input, \( f : \mathbb{R}^n \to \mathbb{R}^n, E : \mathbb{R}^n \to \mathbb{R}^{n \times n} \), rank\(E(x(t))\) \leq n, \( f(0) = 0 \).

**Assumption 1** The unforced system (2) has a unique solution \( \Phi_{x_0}(t) \), i.e. an initial value \( x_0 \in \mathbb{R}^n \) is consistent.

**Definition 2** [32] A control law \( u(x) \) is called an admissible control law, if for any consistent initial condition, the resulting closed-loop descriptor system has no impulsive solution. If there exists such an admissible control law, then the original system is called impulse controllable.

Consider the linear descriptor systems in the form
\[ E \dot{x}(t) = Ax(t), \] (3)
where \( E, A \in \mathbb{R}^{n \times n} \), rank\(E = n_1 \leq n \). The solution behavior (regularity) of (3) depends on the properties of the matrix pair \((E, A)\).

**Definition 3** [1] A pair \((E, A)\) is called regular if \( \det(sE - A) \neq 0 \) for some \( s \in \mathbb{C} \).

The following lemma gives a necessary and sufficient condition of regularity for linear descriptor systems.

**Lemma 1** [1] For the descriptor system (3), the pair of matrices \((E, A)\) is regular (i.e. has a unique solution) if and only if one can choose two nonsingular matrices \( Q \in \mathbb{R}^{n \times n} \) and \( P \in \mathbb{R}^{n \times n} \) such that
\[ QEP = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \qquad QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \] (4)
where \( n_1 + n_2 = n, N \in \mathbb{R}^{n_2 \times n_2} \) is a nilpotent matrix, \( A_1 \in \mathbb{R}^{n_1 \times n_1} \).

C. Linear geometric homogeneity

Homogeneity is a certain invariance of a mathematical object with respect to a group of transformation called dilations (see, for example, [23, 24, 33]). The linear geometric homogeneity deals with the one-parameter group \( d(s) : \mathbb{R} \to \mathbb{R}^{n \times n} \) of linear dilations given by
\[ d(s) = e^{G_d s} = \sum_{i=0}^{+\infty} \frac{s^i G_d^i}{i!}, \quad s \in \mathbb{R}, \]
where \( G_d \in \mathbb{R}^{n \times n} \) is an anti-Hurwitz matrix (i.e. \( \Re(\lambda_i(G_d)) > 0, i = 1, \ldots, n \)) called the generator of the dilation [37].

**Definition 4** [23] A vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) (respectively a function \( h : \mathbb{R}^n \to \mathbb{R} \)) is said to be \( d \)-homogeneous of degree \( \nu \in \mathbb{R} \) if
\[ f(d(s)x) = e^{\nu s} f(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \forall s \in \mathbb{R} \]
(respectively \( h(d(s)x) = e^{\nu s} h(x), \forall x \in \mathbb{R}^n \setminus \{0\}, \forall s \in \mathbb{R} \)).

**Definition 5** A system \( \dot{x} = f(x) \) is said to be \( d \)-homogeneous if \( f \) is \( d \)-homogeneous.

A special case of homogeneous function is a homogeneous norm [23, 34]: a continuous positive definite \( d \)-homogeneous function of degree 1. The canonical homogeneous norm \( \| \cdot \|_d : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\} \) is defined as \( \|x\|_d = e^{s x} \) for \( x \neq 0 \), where \( s \in \mathbb{R} \) such that \( \|d(-s)x\|_d = 1 \) and, by continuity, we assign \( \|0\|_d = 0 \). Note that \( \|d(s)x\|_d = e^{s \|x\|_d} \) and
\[ \|d(-\ln s x)\|_d = 1. \] (5)

The following lemma gives the criterion for a linear system to be \( d \)-homogeneous with the generator \( G_d \).

**Lemma 2** [15] Let \( G_d \in \mathbb{R}^{n \times n} \) be a generator of the dilation \( d(s) = e^{s G} \), \( s \in \mathbb{R} \). Then the linear system \( \dot{x} = Ax, x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \) is \( d \)-homogeneous of degree \( \nu \in \mathbb{R} \) if and only if
\[ AG_d - G_d A = \nu A. \] (6)

Note that the equation (6) has a solution for \( \nu \neq 0 \) if and only if \( A \) is nilpotent [15].

The presented type of homogeneity is applicable for the descriptor systems (2) in the case of invertible \( E(x) \) (considering the system in the form \( \dot{x} = E(x)^{-1} f(x, u(x)) \), where \( E(x)^{-1} f(x, u(x)) \) is \( d \)-homogeneous). In this paper we extend the notion of the linear geometric homogeneity for the descriptor systems (2) for not necessary invertible \( E(x) \).
III. MAIN RESULT

A. Homogeneous descriptor systems

Let us formulate the definition of homogeneity for descriptor systems in the form (2).

**Definition 6** The pair \((E,f)\) is said to be \(d_E\)-homogeneous of degree \(\nu \in \mathbb{R}\), if for all \(x \in \mathbb{R}^n, s \in \mathbb{R}\)

\[
E(d_E(s)x)d_E(s) = W(x) \begin{bmatrix} e^{ns}I_n & 0 \\ 0 & e^{s}I_n \end{bmatrix} \Xi(x,s)E(x),
\]

\[
f(d_E(s)x) = W(x) \begin{bmatrix} e^{(\nu+s)n}I_n & 0 \\ 0 & e^{(\nu+s)I_n} \end{bmatrix} \Xi(x,s)f(x),
\]

where \(n, n_1, n_2, \nu_1, \nu_2 \in \mathbb{R}, d_E(s) = e^{Gds}, G_d \in \mathbb{R}^{n \times n}\) is anti-Hurwitz, \(\Xi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}\) and \(W : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) are invertible for all \(s \in \mathbb{R}, x \in \mathbb{R}^n\).

**Definition 7** The system (2) is said to be \(d_E\)-homogeneous if the pair \((E,f)\) is \(d_E\)-homogeneous.

**Remark 1** In the case \(E(x)\) is nonsingular for any \(x \in \mathbb{R}^n\) the system (2) is equivalent to \(d\)-homogeneous system \(\dot{x}(t) = E(x)^{-1}f(x)\) with the dilation \(d(s) = d_E(s)\). In order to avoid repetitions of results for ODEs, further it is assumed that \(E(x)\) is singular.

One of the most important properties of homogeneous systems is the scalability of the solutions [33], [34], [35], [36].

**Theorem 2** Let (2) be \(d_E\)-homogeneous of degree \(\nu \in \mathbb{R}\). If \(\Phi_{E,f} : [0,T) \rightarrow \mathbb{R}^n\) is a solution to (2), then \(\Phi_{d_E(s)x_0} : [0, e^{-\nu s}T) \rightarrow \mathbb{R}^n\) defined as

\[
\Phi_{d_E(s)x_0}(t) := d_E(s)\Phi_{x_0}(te^{\nu s})
\]

is a solution to (2) with the initial condition \(x(0) = d_E(s)x_0\) and \(s \in \mathbb{R}\).

**Remark 2** Note that if the system (10) is \(d_E\)-homogeneous with the generator \(G_d\), then the corresponding canonical form (11) is \(d_E\)-homogeneous with the generator \(G_d = P^{-1}G_dP\).

B. Finite-time control of linear descriptor systems

In this section we propose a finite-time control design method for linear descriptor systems based on \(d_E\)-homogeneity.

Firstly, let us define \(d_E\)-homogenization and \(d_E\)-homogeneous stabilization.

**Definition 8** A descriptor control system \(E(x)\dot{x} = f(x, u)\) is \(d_E\)-homogenizable with degree \(\nu \in \mathbb{R}\) if there exists a feedback \(u(x)\) such that the closed-loop system is \(d_E\)-homogeneous of degree \(\nu\).

**Remark 3** In the case \(E\) is nonsingular for any \(x \in \mathbb{R}^n\) the system (2) is equivalent to \(d\)-homogeneous system \(\dot{x}(t) = E(x)^{-1}f(x)\) with the dilation \(d(s) = d_E(s)\). In order to avoid repetitions of results for ODEs, further it is assumed that \(E(x)\) is singular.

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All proofs are skipped due to space limitations.

The scalability of the solutions implies a number of properties useful for qualitative analysis. For example, solutions scalability with local stability imply the global one; the existence of strictly invariant (in forward time) compact set implies asymptotic stability [33], [35].

Now consider the following linear regulator regular descriptor system

\[
E\dot{x}(t) = Ax(t),
\]

where \(x \in \mathbb{R}^n\) is the state vector, \(A, E \in \mathbb{R}^{n \times n}\) and \(\text{rank}E = n_1 < n\). The pair \((E,A)\) is regular, i.e., it admits the canonical form (see Lemma 1)

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = P^{-1}x(t), x_1(t) \in \mathbb{R}^{n_1}, x_2(t) \in \mathbb{R}^{n_2}.
\]

The following result gives the criteria for a linear descriptor system to be \(d_E\)-homogeneous of nonzero degree.

**Lemma 3** The next statements are equivalent.

1. The system (10) is \(d_E\)-homogeneous of degree \(\nu \neq 0\).
2. The matrix \(A_1 \in \mathbb{R}^{n_1 \times n_1}\) is nilpotent.
3. The condition \(\lambda(E,A) = 0\) is satisfied.

**Remark 2** Note that if the system (10) is \(d_E\)-homogeneous with the generator \(G_d\), then the corresponding canonical form (11) is \(d_E\)-homogeneous with the generator \(G_d = P^{-1}G_dP\).
\[(A + BK_E)M = (L + \nu In)(A + BK_E), \quad (16)\]
\[(L + (\nu - 1)In)B = 0, \quad (17)\]
\[M + MT + 2aIn > 0 \quad (18)\]
is feasible for some \(M, L \in \mathbb{R}^{n \times n}\). \(a \in \mathbb{R}\);
\(K \in \mathbb{R}^{m \times n}\), \(\beta \in \mathbb{R}_+\) are chosen such that
\[ER \geq 0, \quad (19)\]
\[
\begin{bmatrix}
E(M + aI_n)R + RT(M + aI_n)ET & RTET
\end{bmatrix} \geq 0, \quad (20)
\]
\[\Gamma > 0, \quad (21)\]
\[(A + BK_E)R + RT(A + BK_E)^T + BY + YTB^T < -\beta((L + aI_n)ER + RTET(L^T + aI_n)) \quad (22)\]
for some \(R, \Gamma \in \mathbb{R}^{n \times n}\), \(Y \in \mathbb{R}^{m \times n}\) with \(K = YX\),
\(X = R^{-1}\) and \(G_d = M + aI_n\).

Then the closed-loop system (12), (14) is impulse controllable and finite-time stable with
\[T(x_0) < -\frac{1}{\beta \nu} \|x_0\|_{d_E}^{-\nu}. \quad (13)\]

**Remark 3** The presented control can be considered as an extension of the results in [15] for finite-time stabilization of linear ODEs. Indeed, in the case \(E\) is nonsingular (see Remark 1) the control (14) coincides with the given in [15].

**Remark 4** Note that \(V(x) := \|x\|_{d_E}\) is defined implicitly by (13). To realize the control (14) the bisection method may be utilized (see, e.g., [38]):

**Algorithm 1** [38]

**INITIALIZATION:** \(V_0 = 1; \ a = V_{\text{min}}; \ b = 1;\)

**STEP:**
1. If \(x_i^T d_E(-\ln b)^T X^T E d_E(-\ln b)x_i > 1\) then \(a = b; \ b = 2b;\)
2. Elseif \(x_i^T d_E(-\ln a)^T X^T E d_E(-\ln a)x_i < 1\) then \(b = a; \ a = \max\{a, V_{\text{min}}\};\)
3. Else \(c = \frac{a + b}{2}\)
4. If \(x_i^T d_E(-\ln c)^T X^T E d_E(-\ln c)x_i < 1\) then \(b = c;\)
5. Else \(a = \max\{V_{\text{min}}, c\};\)
6. **endif;**
7. **endif;**
8. \(V_i = b;\)

If **STEP** is applied recurrently many times to the same vector \(x_i\) then it allows to localize the unique positive root of the equation \(\|d_E(-\ln \|x_i\|_{d_E})x\|_{X^T E} = 1\).

**IV. EXAMPLE**

Consider the system (12) with
\[
E = \begin{bmatrix}
0.1 & 2 & 0 \\
1.6 & 3.1 & 0 \\
0.6 & 1 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
2.1 & 1.45 & -1.6 \\
4.7 & 6.75 & -0.7 \\
1.6 & 5.1 & 5.2
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0.1 & 2 \\
1.6 & 3.1 \\
0.6 & 1
\end{bmatrix}
\]

Define the finite-time control \(u\) in the form (14) with \(\nu = -0.5\), where the matrices \(P \in \mathbb{R}^{3 \times 3}, K_E, K \in \mathbb{R}^{2 \times 3}, G_d \in \mathbb{R}^{3 \times 3}\) are obtained from (15)-(22),
\[
P = \begin{bmatrix}
0.0642 & -0.0356 & 0 \\
-0.0356 & 0.0610 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad K_E = \begin{bmatrix}
-1 & -2 & 1 \\
-1 & -1 & 0
\end{bmatrix},
\]
\[
K = \begin{bmatrix}
-1.358 & 0.7544 & 0.2297 \\
0.5554 & -1.1535 & 0.9749
\end{bmatrix}, \quad G_d = 1.5I_3.
\]

The numerical simulation of the closed-loop system has been done for \(x_0 = (-2, 2, -1)^T\) by the Euler method with the fixed step size \(h = 10^{-3}\). To find values of \(\|x\|_{d_E}\) Algorithm 1 was used. The simulation results are shown in Fig. 1, 2. The results of simulation with using the logarithmic scale are shown in Fig. 3 in order to demonstrate finite-time convergence rate of \(\|x\|_{d_E}\).
V. Conclusions

The paper extends the homogeneity concept for descriptor systems in the form (2). The necessary and sufficient conditions of linear descriptor systems to be $d_k$-homogeneous are obtained, and it is shown that the linear descriptor system can be homogenized with any degree via linear feedback. The homogeneity-based finite-time stabilizing control for linear descriptor systems is presented. The settling time estimates are obtained. Tuning control parameters is presented in the form of linear matrix equations and inequalities.

The presented approach opens a lot of topics for future research. For example, robustness analysis, optimal tuning of the control parameters, development of homogeneity-based finite-time observers for descriptor systems, finite-time control design for nonlinear descriptor systems, etc.

REFERENCES


