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Disturbance attenuation in the Euler–Bernoulli beam with viscous and Kelvin–Voigt damping via piezoelectric actuators

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Abstract— **We design a state-feedback controller, applied via piezoelectric actuators, that suppresses the effect of a distributed disturbance in the Euler–Bernoulli beam with viscous and Kelvin–Voigt damping. The controller is designed to improve performance on a finite number of modes. Its effect on the remaining (infinitely many) modes is analysed by constructing an appropriate Lyapunov functional, whose properties are guaranteed by the feasibility of linear matrix inequalities (LMIs). The LMIs allow us to design suitable controller gain and estimate the induced L^2 gain. A numerical example demonstrates how this modal decomposition approach leads to a controller that significantly reduces the L^2 gain.**

I. Introduction

A natural first step in designing a controller for a system described by a partial differential equation (PDE) is to approximate it with ordinary differential equations (ODEs), which can be analyzed using standard control techniques. An efficient method of obtaining such ODEs, especially for linear systems, is modal decomposition, also called eigenfunction expansion, Galerkin's method, or model reduction [1], [2]. Its idea is to project the PDE state on a finite-dimensional subspace (comprised of modes) and design a controller for the resulting reduced-order model [3]–[10]. The main problem of this approach is the "spillover" effect: a controller designed for the ODE approximation may have deteriorating effect on the unaccounted dynamics [11], [12].

Recently, significant progress has been made in analyzing and reducing the "spillover" effect for *parabolic* PDEs. In particular, modal decomposition was used to establish the input-to-state stability with respect to boundary disturbances [13], [14], which enabled the design of sampled-data *state*-feedback boundary control [15]. Later, it was combined with Lyapunov functionals to design *state*-feedback boundary control for semilinear parabolic PDEs [16]. Modal decomposition approach to finite-dimensional *output* feedback was developed in [17], which derived linear matrix inequalities (LMIs) feasible for a large enough number of modes. Then, [18] improved these LMIs so that their complexity does not grow when the order of the reduced system increases. The latter approach was subsequently extended to input/output delays [18]–[20] and the Kuramoto–Sivashinsky equation [21].

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This paper develops a modal decomposition approach for the Euler–Bernoulli beam equation with viscous and Kelvin–Voigt friction. Though the eigenvalues of this PDE have negative real parts, they have a finite limit, which complicates the disturbance attenuation problem compared to parabolic PDEs, where the eigenvalues go to $-\infty$. A compensator avoiding the "spillover" phenomenon in Riesz-spectral systems with bounded input and output operators was proposed in [10]. This compensator was used in a flexible beam with Kelvin–Voigt damping and "point-shaped" actuators. Here, we consider more practical and challenging piezoelectric actuators. The controllability problem for the beam with piezoelectric actuators was studied in [22]. We address the distributed disturbance attenuation problem. Boundary disturbances and actuators were studied in [23], [24]. An experimental study of the disturbance attenuation with piezoelectric sensors and actuators (without a "spillover" analysis) was reported in [25], [26].

Our results extend the constructive modal decomposition approach of [17], where the heat equation was studied. The main difference is that the state of a beam comprises the displacement and its velocity, which lie in different functional spaces. This makes the Lyapunov-based analysis difficult since the Lyapunov functional has to contain the products of the Fourier coefficients corresponding to functions from different spaces. By carefully choosing the weights in the Lyapunov functional (Lemma 1), we ensure that the truncated dynamics are accounted for and do not deteriorate the stability. This allows us to derive linear matrix inequalities (LMIs) to identify how many modes to consider and what controller gain to take to avoid spillover and improve the induced L^2 gain in two different norms. The results of this paper have been improved and published in [27].

Notations: $|\cdot|$ is the Euclidean norm, $\|\cdot\|$ is the L^2 norm, $\langle\cdot,\cdot\rangle$ is the scalar product in L^2, I is the identity matrix, $\operatorname{diag}\{\lambda_1,\dots,\lambda_N\}$ is the diagonal matrix with diagonal elements λ_i, $i=1,\dots,N$, H^4 is the Sobolev space, δ is the Dirac delta function. For a matrix P, the notation $P<0$ implies that P is square, symmetric, and negative-definite with the symmetric elements sometimes marked as $*$. Partial derivatives are denoted by indices, e.g., $z_t=\partial z/\partial t$.

II. Problem statement

Consider the following Euler–Bernoulli beam model:

$$\begin{aligned} &z_{tt}+\mu z_t+\nu z_{txxxx}+\alpha z_{xxxx}\\ &\qquad=\left[\delta'(x-x_L)-\delta'(x-x_R)\right]u(t)+w(x,t), \qquad (1)\\ &z(0,t)=z(1,t)=z_{xx}(0,t)=z_{xx}(1,t)=0, \end{aligned}$$

where the state $z\colon [0,1]\times[0,\infty)\to\mathbb{R}$ represents the displacement of the beam with respect to the position at rest, the control $u\colon[0,\infty)\to\mathbb{R}$ is a moment on the beam applied via a piezoelectric actuator with ends at x_L and x_R, $w\colon[0,1]\times[0,\infty)\to\mathbb{R}$ is the unknown distributed disturbance, $\mu>0$ is the viscous damping coefficient, $\nu>0$ is the Kelvin–Voigt damping coefficient, and $\alpha>0$ depends on the beam's elastic modulus, second moment of area, and density. The boundary conditions represent hinged ends.

For given $\rho_x\ge 0$ and $\rho_u\ge 0$, we say that (1) has the L^2 gain not greater than $\gamma>0$ if, for $z(\cdot,0)\equiv 0$,

$$\begin{aligned}\textstyle\int_0^\infty\big(&\|z(\cdot,t)\|^2+\rho_x\|z_{xx}(\cdot,t)\|^2\\ &+\rho_u u^2(t)-\gamma^2\|w(\cdot,t)\|^2\big)dt\le 0.\end{aligned}\quad (2)$$

The smallest γ satisfying (2) is called the L^2 gain. The objective of this paper is to design a finite-dimensional state-feedback controller that decreases γ. We design this controller based on the first N modes and study its effect on the remaining modes via Lyapunov-based analysis.

III. Modal decomposition and control design

The eigenfunctions and eigenvalues of the linear operator

$$\begin{aligned}&\mathcal{A}\varphi=\varphi_{xxxx},\\ &D(\mathcal{A})=\{\varphi\in H^4\mid \varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=0\},\end{aligned}$$

given by

$$\varphi_n=\sqrt{2}\sin(\pi n x),\quad \lambda_n=(\pi n)^4,\quad n\in\mathbb{N},\qquad (3)$$

form an orthonormal basis of $L^2(0,1)$. Substituting

$$z(x,t)=\textstyle\sum_{n=1}^\infty z_n(t)\varphi_n(x)$$

into (1), we obtain

$$\begin{aligned}\textstyle\sum_{n=1}^\infty&\left[\ddot z_n\varphi_n+\mu\dot z_n\varphi_n+\nu\lambda_n\dot z_n\varphi_n+\alpha\lambda_n z_n\varphi_n\right]\\ &=\left[\delta'(x-x_L)-\delta'(x-x_R)\right]u+w.\end{aligned}$$

Projecting both sides onto φ_n, $n\in\mathbb{N}$, we find

$$\ddot z_n(t)+(\mu+\nu\lambda_n)\dot z_n(t)+\alpha\lambda_n z_n(t)=b_n u(t)+w_n(t),\quad (4)$$

where

$$\begin{aligned}&b_n:=\sqrt{2}\pi n\left[\cos(\pi n x_R)-\cos(\pi n x_L)\right],\\ &w_n(t):=\langle w(\cdot,t),\varphi_n\rangle.\end{aligned}$$

In the above, we used the definition of $\delta'(x)$:

$$\begin{aligned}\textstyle\int_0^1\delta'(x-x_L)\varphi_n(x)\,dx&=-\textstyle\int_0^1\delta(x-x_L)\varphi_n'(x)\,dx\\ &=-\varphi_n'(x_L)=-\sqrt{2}\pi n\cos(\pi n x_L).\end{aligned}$$

We design a controller based on the first N modes. Therefore, it is convenient to split (4) into two groups:

$$\dot z^N=Az^N+Bu+w^N,\qquad (5a)$$

$$\dot{\bar z}_n=A_n\bar z_n+B_n u+\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]w_n,\quad n>N,\qquad (5b)$$

where

$$A=\left[\begin{array}{c|c}0 & I\\ \hline -\alpha\Lambda & -(\mu I+\nu\Lambda)\end{array}\right],\quad \Lambda=\operatorname{diag}\{\lambda_1,\dots,\lambda_N\},$$

$$B=\left[\begin{array}{c}0\\ \vdots\\ 0\\ \hline b_1\\ \vdots\\ b_N\end{array}\right],\quad z^N=\left[\begin{array}{c}z_1\\ \vdots\\ z_N\\ \hline \dot z_1\\ \vdots\\ \dot z_N\end{array}\right],\quad w^N=\left[\begin{array}{c}0\\ \vdots\\ 0\\ \hline w_1\\ \vdots\\ w_N\end{array}\right],$$

$$A_n=\left[\begin{smallmatrix}0 & 1\\ -\alpha\lambda_n & -(\mu+\nu\lambda_n)\end{smallmatrix}\right],\quad B_n=\left[\begin{smallmatrix}0\\ b_n\end{smallmatrix}\right],\quad \bar z_n=\left[\begin{smallmatrix}z_n\\ \dot z_n\end{smallmatrix}\right].$$

Proposition 1: Given $N\in\mathbb{N}$, (A,B) is controllable if and only if $\mu\nu\ne\alpha$ and $b_n\ne 0$ for all $n=1,\dots,N$.

Proof: We use the Hautus lemma [28, Lemma 3.3.7]. If $A^\top v=\lambda v$ for some $\lambda\in\mathbb{C}$ and $v=\operatorname{col}\{v_1,v_2\}\ne 0$ with $v_1,v_2\in\mathbb{C}^N$, then

$$[\nu\Lambda+(\mu+\lambda)I]v_2=v_1,\qquad (6a)$$

$$[(\lambda\nu+\alpha)\Lambda+(\lambda^2+\lambda\mu)I]v_2=0.\qquad (6b)$$

The relation (6a) implies $v_2\ne 0$ since otherwise $v=0$.

Let $\mu\nu\ne\alpha$ and $b_n\ne 0$ for any $n=1,\dots,N$. Then $(\lambda\nu+\alpha)$ and $(\lambda^2+\lambda\mu)$ cannot be zero simultaneously. Then, since all the elements of Λ are different, (6b) implies that exactly one element of v_2 is non-zero. But then $B^\top v$ is this element times $b_n\ne 0$, which is not zero. Therefore, (A,B) is controllable.

Let $\mu\nu\ne\alpha$ and $b_n=0$ for some $n\in\{1,\dots,N\}$. Taking $\lambda\in\mathbb{C}$ such that $(\lambda\nu+\alpha)\lambda_n+(\lambda^2+\lambda\mu)=0$, v_2 with only the n-th component being non-zero, and v_1 as in (6a), we obtain $A^\top v=\lambda v$ and $B^\top v=0$ for $v\ne 0$. Therefore, (A,B) is not controllable.

If $\mu\nu=\alpha$, one can take $\lambda=-\mu$, any $v_2\ne 0$ such that $[b_1,\dots,b_N]v_2=0$, and $v_1=\nu\Lambda v_2$ to obtain $A^\top v=\lambda v$ and $B^\top v=0$. Therefore, (A,B) is not controllable. ∎

Remark 1 (Loss of controllability): Since

$$\begin{aligned}b_n&=\sqrt{2}\pi n\left[\cos(\pi n x_R)-\cos(\pi n x_L)\right]\\ &=-2\pi n\sqrt{2}\sin\left(\pi n\tfrac{x_R+x_L}{2}\right)\sin\left(\pi n\tfrac{x_R-x_L}{2}\right),\end{aligned}$$

the condition $b_n\ne 0$ holds if and only if

$$n\tfrac{x_R+x_L}{2}\notin\mathbb{Z}\quad\text{and}\quad n\tfrac{x_R-x_L}{2}\notin\mathbb{Z}.\qquad (7)$$

In particular, if the piezoelectric actuator is in the center and

$$x_L=\tfrac{1}{2}-\varepsilon,\quad x_R=\tfrac{1}{2}+\varepsilon,$$

then (A,B) is not controllable for $N=2$.

If $\mu\nu\ne\alpha$ and (7) is true, Proposition 1 implies that, for any δ_0, there is $K\in\mathbb{R}^{1\times 2N}$ such that

$$u(t)=-Kz^N\qquad (8)$$

exponentially stabilizes the finite-dimensional system (5a) with the decay rate $\delta_0>0$. In particular, there exists $P\in\mathbb{R}^{2N\times 2N}$ such that

$$P>0,\qquad P(A-BK)+(A-BK)^\top P\le -2\delta_0 P.$$

For

$$V_0(t)=(z^N(t))^\top P z^N(t),\qquad (9)$$

we obtain

$$\dot V_0(t)+|z^N(t)|^2-\gamma^2|w^N(t)|^2\le\left[\begin{smallmatrix}z^N(t)\\ w^N(t)\end{smallmatrix}\right]^\top Q\left[\begin{smallmatrix}z^N(t)\\ w^N(t)\end{smallmatrix}\right],$$

where

$$Q = \begin{bmatrix} I-2\delta_0 P & P \\ P & -\gamma^2 I \end{bmatrix}.$$

If $Q \leq 0$, we can integrate the above from 0 to ∞ and use the facts that $V_0(t) \geq 0$ and $V_0(0) = 0$ for $z^N(0) = 0$ to obtain

$$\int_0^\infty \left[|z^N(t)|^2 - \gamma^2 |w^N(t)|^2\right] dt \leq 0.$$

That is, the L^2 gain of the finite-dimensional system (5a) is not greater than γ (with $\rho_x = \rho_u = 0$). By the Schur complement lemma, $Q \leq 0$ is equivalent to

$$I + \gamma^{-2} P^2 \leq 2\delta_0 P.$$

Therefore, it is tempting to choose K that leads to a large δ_0 to obtain a small L^2 gain. However, the control signal (8) may have a destabilizing effect on (5b), which were neglected during the controller design. This is called the "spillover" effect. In the next two sections, we show how to quantify spillover and how to design a controller gain mitigating this behavior.

Substituting (8) into (5), we obtain the closed-loop system

$$\dot{z}^N = (A - BK)z^N + w^N, \tag{10a}$$
$$\dot{\bar{z}}_n = A_n \bar{z}_n - B_n K z^N + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] w_n, \quad n > N. \tag{10b}$$

The notations are given below (5). The eigenvalues of A_n, given by

$$s_n^\pm = -\tfrac{1}{2}\left[(\mu + \nu\lambda_n) \mp \sqrt{(\mu + \nu\lambda_n)^2 - 4\alpha\lambda_n}\right], \quad n \in \mathbb{N},$$

are such that

$$s_n^- \to -\infty \quad \text{and} \quad s_n^+ \to -\tfrac{\alpha}{\nu}$$

when $n \to \infty$. Moreover, $s_n^+ < -\alpha/\nu$ if $\nu\mu < \alpha$, that is, the decay rate of (10b) is at least α/ν if the terms with z^N and w_n are ignored. To quantify the effect of z^N and w_n on the stability of (10b), we will use Lypaunov functionals. The following lemma will be used to establish the required properties of these functionals.

Lemma 1: If $\nu\mu < \alpha$ and $\lambda_{N+1}\nu^2 \geq 2\alpha - \mu\nu$, then

$$P_n := \begin{bmatrix} 1 & \frac{1}{\nu\lambda_n} \\ \frac{1}{\nu\lambda_n} & \frac{2\alpha-\mu\nu}{\alpha(\nu\lambda_n)^2} \end{bmatrix} \tag{11}$$

satisfies

$$P_n \geq \begin{bmatrix} \frac{\alpha-\mu\nu}{2\alpha-\mu\nu} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_n A_n + A_n^\top P_n \leq -2\frac{\alpha}{\nu} P_n \tag{12}$$

for any $n > N$.

Proof: The first inequality follows from the Schur complement lemma. The second inequality holds since

$$P_n A_n + A_n^\top P_n + 2\frac{\alpha}{\nu} P_n = \begin{bmatrix} 0 & 0 \\ 0 & 2\frac{(\mu\nu - 2\alpha + \lambda_n\nu^2)(\mu\nu - \alpha)}{\alpha\nu^3\lambda_n^2} \end{bmatrix}.$$

■

IV. L^2-GAIN ANALYSIS WITH THE L^2 NORM

We start with the case when $\rho_x = 0$. Then (2) becomes

$$\int_0^\infty \left(\|z(\cdot,t)\|^2 + \rho_u u^2(t) - \gamma^2 \|w(\cdot,t)\|^2\right) dt \leq 0. \tag{13}$$

To study the L^2 gain in the sense of (13), we need to quantify the "spillover" affect that the control (8) has on the dynamics of (10b). To this end, we use the functional

$$V(t) = V_0(t) + q_1 V_1(t), \quad V_1 := \textstyle\sum_{n=N+1}^\infty \bar{z}_n^\top P_n \bar{z}_n \tag{14}$$

with V_0 from (9), P_n from (11), and $q_1 > 0$. This functional and Lemma 1, which enables its analysis, are the key ingredients allowing for the L^2 gain analysis in the sense of (13).

Theorem 1: Consider (1) with $\alpha > \mu\nu$ and subject to (7). For a given $\gamma \in [0, \infty)$, let $N \in \mathbb{N}$ be such that λ_{N+1}, defined in (3), satisfies

$$\lambda_{N+1}\nu^2 \geq 2\alpha - \mu\nu \text{ and } \lambda_{N+1}\gamma > \tfrac{2\alpha-\mu\nu}{\alpha\sqrt{\alpha(\alpha-\mu\nu)}}. \tag{15}$$

Given a controller gain, $K \in \mathbb{R}^{1\times 2N}$, if there exists a negative semidefinite $P \in \mathbb{R}^{2N\times 2N}$ such that

$$\Phi = \begin{bmatrix} \Phi_1 & P \\ P & -\gamma^2 I \end{bmatrix} \leq 0, \tag{16}$$

where

$$\Phi_1 = P(A - BK) + (A - BK)^\top P + \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] + [r_1 + \rho_u] K^\top K, \tag{17a}$$
$$r_1 = \tfrac{q_1}{\kappa_0} \tfrac{8(2\alpha-\mu\nu)}{\alpha\nu^2\pi^6}\left[\tfrac{\pi^6}{945} - \textstyle\sum_{n=1}^N \tfrac{1}{n^6}\right], \tag{17b}$$
$$q_1 = \tfrac{\nu(2\alpha-\mu\nu)}{\alpha(\alpha-\mu\nu)}, \tag{17c}$$
$$\kappa_0 = \tfrac{\alpha}{\nu} - \tfrac{(2\alpha-\mu\nu)^2}{\alpha^2\gamma^2\nu\lambda_{N+1}^2(\alpha-\mu\nu)}, \tag{17d}$$

then the control law (8) guarantees that the L^2 gain of (1) in the sense of (13) is at most γ.

Proof: The idea of the proof is to show that

$$\dot{V} + \|z(\cdot,t)\|^2 + \rho_u u^2 - \gamma^2 \|w(\cdot,t)\|^2 \leq 0 \tag{18}$$

for $V(t)$ from (14) and $q_1 > 0$ given in (17c). Then, since $V(t) \geq 0$ and $V(0) = 0$ for $z(0,t) = 0$, integrating (18) from 0 to ∞, we obtain (13).

To show that (18) holds under the conditions of the theorem, we first calculate the derivatives:

$$\dot{V}_0 \overset{(10a)}{=} \left(z^N\right)^\top \left[P(A-BK) + (A-BK)^\top P\right] z^N + 2\left(z^N\right)^\top P w^N,$$

$$\dot{V}_1 \overset{(10b)}{=} \textstyle\sum_{N+1}^\infty \bar{z}_n^\top [P_n A_n + A_n^\top P_n] \bar{z}_n - 2\sum_{N+1}^\infty \bar{z}_n^\top P_n B_n K z^N + 2\sum_{N+1}^\infty \bar{z}_n^\top P_n \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] w_n.$$

The second inequality in (15) guarantees $\kappa_0 > 0$ (see (17d)). Therefore, using Young's inequality, we obtain

$$-2\textstyle\sum_{N+1}^\infty \bar{z}_n^\top P_n B_n K z^N \leq \kappa_0 \sum_{N+1}^\infty \bar{z}_n^\top P_n \bar{z}_n + \kappa_0^{-1} (z^N)^\top K^\top \left[\sum_{N+1}^\infty B_n^\top P_n B_n\right] K z^N.$$

Note that

$$\sum_{N+1}^{\infty} B_n^\top P_n B_n = \frac{2\pi^2(2\alpha-\mu\nu)}{\alpha\nu^2}\times \sum_{N+1}^{\infty} \frac{n^2}{\lambda_n^2}(\cos(\pi n x_R) - \cos(\pi n x_L))^2$$
$$\leq \frac{8(2\alpha-\mu\nu)}{\alpha\nu^2\pi^6}\sum_{N+1}^{\infty}\frac{1}{n^6} = \frac{8(2\alpha-\mu\nu)}{\alpha\nu^2\pi^6}\left[\frac{\pi^6}{945} - \sum_{1}^{N}\frac{1}{n^6}\right],$$

where $\frac{\pi^6}{945} = \sum_{n=1}^{\infty}\frac{1}{n^6}$ is the value of the Riemann zeta function at 6. Young's inequality with $\kappa_1 > 0$ yields

$$2\sum_{N+1}^{\infty}\bar{z}_n^\top P_n \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] w_n \leq \kappa_1 \sum_{N+1}^{\infty}\bar{z}_n^\top P_n \bar{z}_n + \kappa_1^{-1}\sum_{N+1}^{\infty}\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right] P_n \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] w_n^2$$
$$\leq \kappa_1 \sum_{N+1}^{\infty}\bar{z}_n^\top P_n \bar{z}_n + \frac{2\alpha-\mu\nu}{\kappa_1\alpha(\nu\lambda_{N+1})^2}\sum_{N+1}^{\infty} w_n^2.$$

Combining the above, we obtain

$$\dot{V}_1 \leq \sum_{N+1}^{\infty}\bar{z}_n^\top[P_n A_n + A_n^\top P_n + (\kappa_0+\kappa_1)P_n]\bar{z}_n + \frac{8(2\alpha-\mu\nu)}{\kappa_0\alpha\nu^2\pi^6}\left[\frac{\pi^6}{945} - \sum_{1}^{N}\frac{1}{n^6}\right](z^N)^\top K^\top K z^N + \frac{2\alpha-\mu\nu}{\kappa_1\alpha(\nu\lambda_{N+1})^2}\sum_{N+1}^{\infty} w_n^2.$$

Note that

$$\begin{aligned}\|z(\cdot,t)\|^2 &= (z^N)^\top \left[\begin{smallmatrix}I & 0\\0 & 0\end{smallmatrix}\right] z^N + \sum_{N+1}^{\infty}\bar{z}_n^\top \left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right]\bar{z}_n,\\ \|w(\cdot,t)\|^2 &= (w^N)^\top w^N + \sum_{N+1}^{\infty} w_n^2.\end{aligned} \quad (19)$$

Using these representations, we arrive at

$$\dot{V} + \|z(\cdot,t)\|^2 + \rho_u u^2 - \gamma^2\|w(\cdot,t)\|^2 \leq \begin{bmatrix} z^N \\ w^N\end{bmatrix}^\top \Phi \begin{bmatrix} z^N \\ w^N\end{bmatrix} + \sum_{N+1}^{\infty}\bar{z}_n^\top\left[q_1\left(P_n A_n + A_n^\top P_n + (\kappa_0+\kappa_1)P_n\right) + \left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right]\right]\bar{z}_n + \left[\frac{q_1(2\alpha-\mu\nu)}{\kappa_1\alpha(\nu\lambda_{N+1})^2} - \gamma^2\right]\sum_{N+1}^{\infty} w_n^2$$

with Φ defined in (16). Therefore, (18) holds if

$$\Phi \leq 0, \quad (20a)$$
$$q_1\left(P_n A_n + A_n^\top P_n + (\kappa_0+\kappa_1)P_n\right) + \left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right] \leq 0, \quad (20b)$$
$$\frac{q_1(2\alpha-\mu\nu)}{\kappa_1\alpha(\nu\lambda_{N+1})^2} \leq \gamma^2. \quad (20c)$$

The smallest κ_1 satisfying (20c) is

$$\kappa_1 = \frac{q_1(2\alpha-\mu\nu)}{\gamma^2\alpha(\nu\lambda_{N+1})^2}.$$

Note that $\kappa_1 > 0$ since $\alpha > \mu\nu$ by the conditions of the theorem. The value of κ_0, defined in (17d), was chosen as the largest value satisfying (20b):

$$q_1(P_n A_n + A_n^\top P_n + (\kappa_0+\kappa_1)P_n) + \left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right] \overset{(12)}{\leq} -q_1\left[2\tfrac{\alpha}{\nu} - \kappa_0 - \kappa_1\right]P_n + \left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right] \overset{(12)}{\leq} \begin{bmatrix} -q_1\left(2\frac{\alpha}{\nu} - \kappa_0 - \kappa_1\right)\frac{\alpha-\mu\nu}{2\alpha-\mu\nu} + 1 & 0 \\ 0 & 0\end{bmatrix} = 0.$$

Here, we used Lemma 1, which holds if $\alpha > \mu\nu$ and the first inequality of (15) is true. The value of q_1, defined in (17c), was chosen to minimize q_1/κ_0, which appears in Φ_1. Summarizing, the conditions of the theorem imply (20), which guarantee (18). As discussed at the beginning of the proof, (18) implies the statement of the theorem. ■

Remark 2: Since $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$, (17b) implies $r_1 \to 0$ as $N \to \infty$. This constant characterizes the "spillover" effect of the neglected modes. It decreases when more modes are considered in the design.

It may be difficult to pick the right controller gain K. If K is too small, the finite-dimensional part may not be stable enough to compensate the residue from the tail. If K is too large, then the residue of the tail will be large. The following design LMIs resolve this issue.

Corollary 1: Consider (1) with $\alpha > \mu\nu$ and subject to (7). For a given $\gamma \in [0,\infty)$, let $N \in \mathbb{N}$ be such that λ_{N+1}, defined in (3), satisfies (15). If there exist $0 < \bar{P} \in \mathbb{R}^{2N\times 2N}$ and $Y \in \mathbb{R}^{1\times 2N}$ such that

$$\Psi = \begin{bmatrix} \Psi_1 & I & \bar{P}\left[\begin{smallmatrix}I\\0\end{smallmatrix}\right] & \sqrt{r_1+\rho_u}Y^\top \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -1\end{bmatrix} \leq 0,$$

where r_1 is from (17b) and

$$\Psi_1 = A\bar{P} + \bar{P}A^\top - BY - (BY)^\top,$$

then the control law (8) with $K = Y\bar{P}^{-1}$ guarantees that the L^2 gain of (1) in the sense of (13) is at most γ.

Proof: By the Schur complement lemma, $\Phi \leq 0$ is equivalent to

$$\begin{bmatrix} \Phi_1' & P & \left[\begin{smallmatrix}I\\0\end{smallmatrix}\right] & \sqrt{r_1+\rho_u}K^\top \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -1\end{bmatrix} \leq 0,$$

where $\Phi_1' = P(A-BK) + (A-BK)^\top P$. Multiplying this inequality by $\operatorname{diag}\{P^{-1}, I\}$ from left and right, we obtain that it is equivalent to $\Psi \leq 0$ with $\bar{P} = P^{-1}$ and $Y = KP^{-1}$. Then the corollary follows from Theorem 1. ■

V. L^2-GAIN ANALYSIS WITH THE H^2 NORM

Now we consider the general case when $\rho_x \geq 0$. In this case, the dynamics of (10b) is analyzed using

$$V_2 = \sum_{n=N+1}^{\infty}\lambda_n \bar{z}_n^\top P_n \bar{z}_n. \quad (21)$$

Theorem 2: Consider (1) with $\alpha > \mu\nu$ and subject to (7). For a given $\gamma \in [0,\infty)$, let $N \in \mathbb{N}$ be such that λ_{N+1}, defined in (3), satisfies $\lambda_{N+1}\nu^2 \geq 2\alpha - \mu\nu$ and

$$\lambda_{N+1}\gamma^2 > \frac{(2\alpha-\mu\nu)^2(\rho_x + \lambda_{N+1}^{-1})}{\alpha^3(\alpha-\mu\nu)}. \quad (22)$$

Given a controller gain, $K \in \mathbb{R}^{1\times 2N}$, if there exists a negative semidefinite $P \in \mathbb{R}^{2N\times 2N}$ such that

$$\Upsilon = \begin{bmatrix}\Upsilon_1 & P \\ P & -\gamma^2 I\end{bmatrix} \leq 0, \quad (23)$$

where

$$\Upsilon_1 = P(A-BK) + (A-BK)^\top P + \left[\begin{smallmatrix}I+\rho_x\Lambda & 0\\0 & 0\end{smallmatrix}\right] + [r_2 + \rho_u]K^\top K, \quad (24a)$$
$$r_2 = \frac{q_2}{\kappa_2}\frac{8(2\alpha-\mu\nu)}{\alpha\nu^2\pi^2}\left[\frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}\right], \quad (24b)$$
$$q_2 = \frac{\nu(2\alpha-\mu\nu)(\rho_x+\lambda_{N+1}^{-1})}{\alpha(\alpha-\mu\nu)}, \quad (24c)$$
$$\kappa_2 = \frac{\alpha}{\nu} - \frac{(2\alpha-\mu\nu)^2(\rho_x+\lambda_{N+1}^{-1})}{\alpha^2\gamma^2\nu\lambda_{N+1}(\alpha-\mu\nu)}, \quad (24d)$$

then the control law (8) guarantees that the L^2 gain of (1) in the sense of (2) is at most γ.

Proof: The proof is similar to that of the proof for Theorem 1. Namely, we show that

$$\dot{\bar{V}} + \|z(\cdot,t)\|^2 + \rho_x\|z_{xx}(\cdot,t)\|^2 + \rho_u u^2 - \gamma^2\|w(\cdot,t)\|^2 \le 0 \tag{25}$$

for $\bar{V}(t) = V_0(t) + q_2V_2(t)$ with V_0 from (9), V_2 from (21), and $q_2 > 0$ from (24c). Then, since $\bar{V}(t) \ge 0$ and $\bar{V}(0) = 0$ for $z(0,t) = 0$, integrating (25) from 0 to ∞, we obtain (2).

In a manner similar to the proof of Theorem 1, we obtain

$$\begin{aligned}\dot{V}_2 \le\ & \textstyle\sum_{N+1}^{\infty} \lambda_n \bar{z}_n^\top[P_nA_n + A_n^\top P_n + (\kappa_2+\kappa_3)P_n]\bar{z}_n \\ &+ \tfrac{8(2\alpha-\mu\nu)}{\kappa_2\alpha\nu^2\pi^2}\left[\tfrac{\pi^2}{6} - \textstyle\sum_1^N \tfrac{1}{n^2}\right](z^N)^\top K^\top K z^N \\ &+ \tfrac{2\alpha-\mu\nu}{\kappa_3\alpha\nu^2\lambda_{N+1}}\textstyle\sum_{N+1}^{\infty} w_n^2\end{aligned}$$

with κ_2 given in (24d) and $\kappa_3 > 0$. Note that (22) guarantees $\kappa_2 > 0$. In addition to (19), we note that

$$\|z_{xx}(\cdot,t)\|^2 = (z^N)^\top \left[\begin{smallmatrix}\Lambda & 0\\ 0 & 0\end{smallmatrix}\right] z^N + \textstyle\sum_{N+1}^{\infty} \bar{z}_n^\top \left[\begin{smallmatrix}\lambda_n & 0\\ 0 & 0\end{smallmatrix}\right] \bar{z}_n.$$

Combining all the above, we obtain

$$\begin{aligned}&\dot{\bar{V}} + \|z(\cdot,t)\|^2 + \rho_x\|z_{xx}(\cdot,t)\|^2 + \rho_u u^2 - \gamma^2\|w(\cdot,t)\|^2 \\ &\le \left[\begin{smallmatrix}z^N\\ w^N\end{smallmatrix}\right]^\top \Upsilon \left[\begin{smallmatrix}z^N\\ w^N\end{smallmatrix}\right] \\ &+ \textstyle\sum_{N+1}^{\infty} \bar{z}_n^\top\Big[q_2\lambda_n\left(P_nA_n + A_n^\top P_n + (\kappa_2+\kappa_3)P_n\right) \\ &+ \left[\begin{smallmatrix}1+\rho_x\lambda_n & 0\\ 0 & 0\end{smallmatrix}\right]\Big]\bar{z}_n + \left[\tfrac{q_2(2\alpha-\mu\nu)}{\kappa_3\alpha\nu^2\lambda_{N+1}} - \gamma^2\right]\textstyle\sum_{N+1}^{\infty} w_n^2\end{aligned}$$

with Υ defined in (23). Therefore, (25) holds if

$$\Upsilon \le 0, \tag{26a}$$

$$q_2\lambda_n\left(P_nA_n + A_n^\top P_n + (\kappa_2+\kappa_3)P_n\right) + \left[\begin{smallmatrix}1+\rho_x\lambda_n & 0\\ 0 & 0\end{smallmatrix}\right] \le 0, \tag{26b}$$

$$\tfrac{q_2(2\alpha-\mu\nu)}{\kappa_3\alpha\nu^2\lambda_{N+1}} \le \gamma^2. \tag{26c}$$

The smallest κ_3 satisfying (26c) is

$$\kappa_3 = \tfrac{q(2\alpha-\mu\nu)}{\gamma^2\alpha\nu^2\lambda_{N+1}}.$$

Note that $\kappa_3 > 0$ since $\alpha > \mu\nu$ by the conditions of the theorem. The value of κ_2, defined in (24d), was chosen as the largest value satisfying (26b):

$$\begin{aligned}&q_2\lambda_n\left(P_nA_n + A_n^\top P_n + (\kappa_2+\kappa_3)P_n\right) + \left[\begin{smallmatrix}1+\rho_x\lambda_n & 0\\ 0 & 0\end{smallmatrix}\right] \\ &\overset{(12)}{\le} -q_2\lambda_n\left[2\tfrac{\alpha}{\nu} - \kappa_2 - \kappa_3\right]P_n + \left[\begin{smallmatrix}1+\rho_x\lambda_n & 0\\ 0 & 0\end{smallmatrix}\right] \\ &\overset{(12)}{\le} \begin{bmatrix}-q_2\lambda_{N+1}\left(2\tfrac{\alpha}{\nu} - \kappa_2 - \kappa_3\right)\tfrac{\alpha-\mu\nu}{2\alpha-\mu\nu} + \lambda_{N+1}\rho_x + 1 & 0\\ 0 & 0\end{bmatrix} \\ &= 0.\end{aligned}$$

The value of q_2, defined in (24c), was chosen to minimize q_2/κ_2, which appears in Υ_1. Summarizing, the conditions of the theorem guarantee (26), which imply (25) and, therefore, the statement of the theorem. ■

Remark 3: Similarly to Remark 2, $r_2 \to 0$ as $N \to \infty$. This indicates that the "spillover" effect of the neglected modes decreases when more modes are considered in the design.

Remark 4: Since the full energy of (1) involves z_t, another reasonable definition of the L^2 gain is the smallest γ such that

$$\begin{aligned}\textstyle\int_0^{\infty} \big(&\|z(\cdot,t)\|^2 + \rho_x\|z_{xx}(\cdot,t)\|^2 + \rho_t\|z_t(\cdot,t)\|^2 \\ &+\rho_u u^2(t) - \gamma^2\|w(\cdot,t)\|^2\big)\,dt \le 0.\end{aligned}$$

To establish this relation using the same Lyapunov-based approach as in the proof of Theorem 2, we need to modify P_n so that

$$\tilde{P}_n \ge \begin{bmatrix}\varepsilon_0 & 0\\ 0 & \frac{\varepsilon_1}{\lambda_n}\end{bmatrix}.$$

This requires the bottom right element of $\tilde{P}_n$ to decay at most as $\frac{1}{\lambda_n}$. This is not enough for the convergence of $\sum_{N+1}^{\infty} \lambda_n B_n^\top P_n B_n$, which characterizes the "spillover" effect.

Corollary 2: Consider (1) with $\alpha > \mu\nu$ and subject to (7). For a given $\gamma \in [0,\infty)$, let $N \in \mathbb{N}$ be such that λ_{N+1}, defined in (3), satisfies $\lambda_{N+1}\nu^2 \ge 2\alpha - \mu\nu$ and (22). If there exist $0 < \bar{P} \in \mathbb{R}^{2N\times 2N}$ and $Y \in \mathbb{R}^{1\times 2N}$ such that

$$\Psi' = \begin{bmatrix}\Psi_1 & I & \bar{P}\left[\begin{smallmatrix}\rho_x\Lambda+I\\ 0\end{smallmatrix}\right] & \sqrt{r_2+\rho_u}Y^\top\\ * & -\gamma^2 I & 0 & 0\\ * & * & -(\rho_x\Lambda + I) & 0\\ * & * & * & -1\end{bmatrix} \le 0,$$

where r_2 is from (24b) and Ψ_1 is from Corollary 1, then the control law (8) with $K = Y\bar{P}^{-1}$ guarantees that the L^2 gain of (1) in the sense of (2) is at most γ.

The proof is similar to the proof of Corollary 1.

VI. Numerical simulations

Consider (1) with

$$\mu = \nu = 0.02, \quad \alpha = 1, \quad x_L = 0.3, \quad x_R = 0.5.$$

The condition $\mu\nu < \alpha$ is satisfied and the first relation in (15) holds for $N = 2$. Using Corollary 1, we find

$$K \approx \begin{bmatrix}-896.56 & 1.64 & -52.41 & 3.52\end{bmatrix} \times 10^2, \quad \gamma = 0.06.$$

Note that the LMIs of Theorem 1 with $K = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$ are feasible for a larger $\gamma = 0.53$. This shows that feedback (8) attenuates the affect of the external disturbance in (1).

The results of numerical simulations for the external disturbance $w(x,t) = \sin x \cos 10t$ and initial conditions $z(x,0) = 0$ are given in Figs. 1 and 2. Figure 1 shows the state without (top) and with (bottom) control (8). Figure 2 compares the resulting L^2 norms of the states. It is clear that feedback attenuates the affect of the external disturbance.

Table I shows the L^2 gains obtained by solving the derived LMIs. One can see that, though Corollary 2 can be applied to the case when $\rho_x = 0$, it gives a larger L^2 gain $\gamma = 0.08$ compared to $\gamma = 0.06$ given by Corollary 1. Table I also shows the L^2 gain found for $\rho_x = 0.1$. Corollary 2 with $\gamma = 1.09$ gives

$$K \approx \begin{bmatrix}-4.13 & 8 & -1.28 & 0.21\end{bmatrix}.$$

Without control input, Theorem 2 gives a larger L^2 gain $\gamma = 1.71$.

TABLE I
L^2 GAINS OBTAINED FROM LMIS

	Cor. 1	Cor. 2 ($\rho_x = 0$)	Cor. 2 ($\rho_x = 0.1$)
Without control	0.53	0.53	1.71
With control (8)	0.06	0.08	1.09

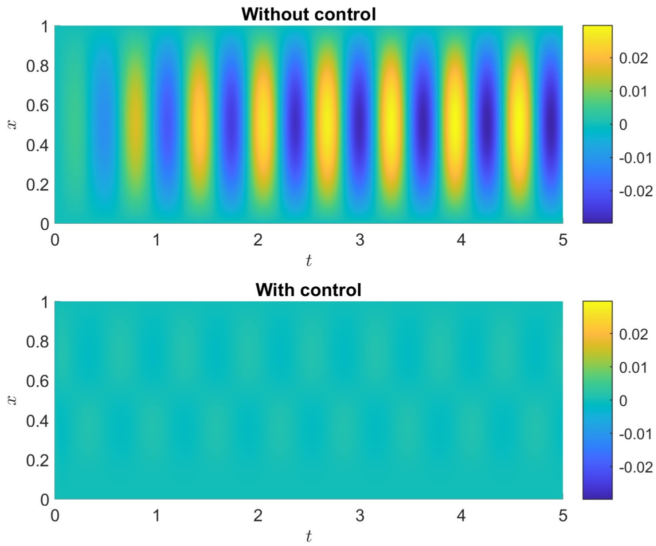

Fig. 1. State of (1) without (top) and with (bottom) control (8)

VII. CONCLUSIONS

We considered the Euler–Bernoulli beam with viscous and Kelvin–Voigt damping. We designed a finite-dimensional controller applied via piezoelectric actuators that attenuates the effect of a distributed disturbance. We derived linear matrix inequalities (LMIs) that allow one to assess how many modes should be considered and provide a bound on the L^2 gain. These LMIs also provide the controller gains that avoid the "spillover" effect.

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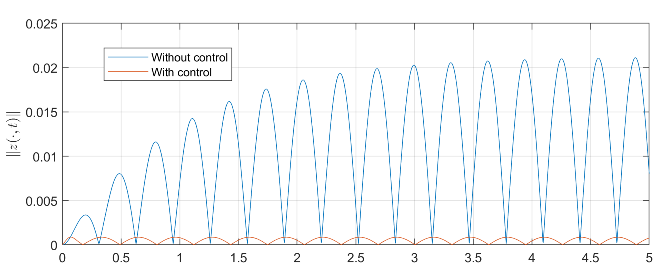

Fig. 2. The L^2 norm of the state without (blue) and with (red) control

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