Two competing populations with a common environmental resource

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Abstract— Feedback-evolving games is a framework that models the co-evolution between payoff functions and an environmental state. It serves as a useful tool to analyze many social dilemmas such as natural resource consumption, behaviors in epidemics, and the evolution of biological populations. However, it has primarily focused on the dynamics of a single population of agents. In this paper, we consider the impact of two populations of agents that share a common environmental resource. We focus on a scenario where individuals in one population are governed by an environmentally "responsible" incentive policy, and individuals in the other population are environmentally "irresponsible". An analysis on the asymptotic stability of the coupled system is provided, and conditions for which the resource collapses are identified. We then derive consumption rates for the irresponsible population that optimally exploit the environmental resource, and analyze how incentives should be allocated to the responsible population that most effectively promote the environment via a sensitivity analysis.

I. INTRODUCTION

Many game-theoretic analyses describe strategic interactions between individuals provided that the incentives for their choices are static, i.e. do not change over time. However, choices often have an impact on a shared environment, which in turn affect their incentives for future choices. The utilization of common resources such as water, fishing grounds, or traffic networks best illustrates this – high individual utilization makes fewer resources available to others in the future [2].

The recent framework of *feedback-evolving games* addresses this interplay by incorporating a changing environmental state coupled with existing evolutionary game theoretic dynamics [3]–[5]. The core model considers payoff functions to individuals that depend on whether the environment is in an abundant or depleted state. It is primarily used to understand how the payoff functions should be structured in order to avoid a "tragedy of the commons", an outcome in which the environment ultimately becomes depleted. In other words, these payoff functions reflect environmental incentive policies that can be designed to manage the common resource [6]–[8].

Much of the existing research focuses on a *single population* of agents whose decisions impact the environment in isolation [8]–[14]. These models are not sufficient to describe multi-population interactions. For example, individuals in

neighboring countries follow different environmental policies, yet utilize resources from the same common source (e.g. fish in the ocean, water from rivers, clean air). Extensions of feedback-evolving games to multiple populations, and possibly multiple local environments, would enable a vastly richer set of scenarios for study. Recent works in the literature have shifted attention to these multi-population scenarios [15]–[17]. For instance, [17] studies epidemic spreading within and between two populations, where the behaviors in each population have externalities on the health state of the other. Overall, an interesting and understudied direction involves hierarchical decision-making, i.e. the strategic selection of local incentive policies given there are environmental externalities between the populations.

In this paper, we extend the framework in this direction by considering two populations that share the same local environmental resource. We focus on a particular setting where one population, labelled "responsible", implements a pro-environmental incentive policy such that the common resource can be sustained in the absence of other populations. The other population, labelled "irresponsible", does not restrain its consumption activity. Our study centers on the following question, stated informally as

To what extent can the irresponsible population exploit the environmental resource?

While too much consumption could cause the resource to collapse, too little consumption leaves missed opportunities. Our highlighted contributions are:

- A stability analysis for the dynamics of the two-population system.
- Identify conditions for which the environment collapses, and when it can be sustained.
- Derivation of the optimal consumption rates for the irresponsible population.
- A sensitivity analysis for the incentive policies chosen by the responsible population.

The last item above illustrates how incentives should be allocated in order to maximally promote the environmental resource. Interestingly, we find that incentivizing mutual cooperation (i.e. coordinated) is far more effective in promoting resource levels than incentivizing unilateral cooperation (anti-coordinated).

The paper is organized as follows. Section II provides relevant background on single population feedback-evolving games, before presenting the two-population model. Section III states our main result regarding the dynamics of the model. Section IV proposes and solves an optimization problem regarding the irresponsible population's choice of

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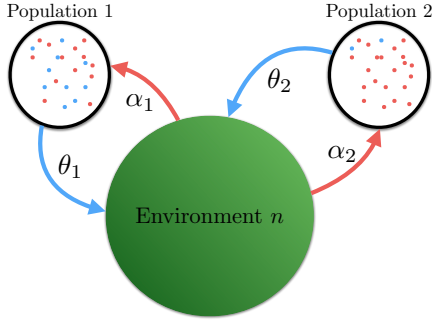The technical proofs of all results in this paper can be found in [1].

Fig. 1: Diagram of our two-population model. Activities from both populations impact the shared environmental state, n. The rates θ_i, α_i denote the restoration and degradation rates from the activities of population i, respectively.

consumption levels. Simulations and sensitivity analyses are given in Section V, followed by concluding remarks.

II. MODEL

Before describing our two-population model, preliminary background on single-population feedback-evolving games is provided.

A. Preliminary on Feedback-evolving games

A single-population feedback-evolving game describes a population of agents that have access to a degradable environmental resource. The relative abundance of the resource is denoted by $n \in [0, 1]$. At any given time, each agent is choosing whether to use a low consumption action (strategy $\mathcal{L}$), or a high consumption action (strategy $\mathcal{H}$). High consumption degrades n, and low consumption improves n. The immediate payoff experienced by an agent is described by the environment-dependent 2×2 payoff matrix,

$$A_n = n \begin{bmatrix} R_1 & S_1 \\ T_1 & P_1 \end{bmatrix} + (1-n) \begin{bmatrix} R_0 & S_0 \\ T_0 & P_0 \end{bmatrix} \quad (1)$$

Here, the first row and column corresponds to a low consumer, and the second row and column corresponds to a high consumer. Entry ij ($i, j \in \{\mathcal{L}, \mathcal{H}\}$) indicates the experienced payoff to an agent using strategy i when encountering an agent using strategy j. We denote $x \in [0, 1]$ as the fraction (or frequency) of agents in the population using strategy $\mathcal{L}$. The payoff experienced by each type of agent is then given by

$$\pi_{\mathcal{L}}(x, n) = [A_n [x, 1-x]^\top]_1, \quad \pi_{\mathcal{H}}(x, n) = [A_n [x, 1-x]^\top]_2 \quad (2)$$

The payoffs are determined by the parameters in the A_0 and A_1 matrices. The A_1 matrix is the payoff matrix when the environment is abundant. Following the literature on feedback-evolving games, we make the following assumption about the A_1 matrix.

Assumption 1. *High consumption is the dominant strategy in A_1, i.e. $\delta_{TR1} := T_1 - R_1 > 0$ and $\delta_{PS1} := P_1 - S_1 > 0$*

On the other hand, the A_0 matrix describes payoffs when the environment is depleted. We interpret A_0 to be an "*envi-*

ronmental policy" that the population follows. For example, the low consumption strategy may be more incentivized when the environment is bad (e.g. subsidies for using electric vehicles). The payoff structure of the A_0 matrix is completely determined from the parameters $\delta_{SP0} := S_0 - P_0$ and $\delta_{RT0} := R_0 - T_0$.

We will use the replicator equation to describe how agents revise their decisions over time. Moreover, the environment n evolves over time, as it is influenced by the decisions in the population. The overall system dynamics is given by two coupled ODEs:

$$\begin{aligned} \dot{x} &= x(1-x)g(x, n) \\ \dot{n} &= \epsilon n(1-n)(\theta x - \alpha(1-x)) \end{aligned} \quad (3)$$

where

$$g(x, n) := \pi_{\mathcal{L}}(x, n) - \pi_{\mathcal{H}}(x, n) \quad (4)$$

is the payoff difference between low and high consumers, $\theta > 0$ is the restoration rate from low consumption activity, $\alpha > 0$ is the degradation rate from high consumption activity, and $\epsilon > 0$ is a time-scale separation constant. The form of the $\dot{n}$ equation is often referred to as the *tipping point dynamics*, since n is increasing only if there are sufficiently high fraction of low consumers.

We consider initial conditions in the interior $(x, n) \in (0, 1)^2$, which is forward-invariant under (3). The originating work [3] provided a full characterization of the asymptotic outcomes of system (3) for all possible environmental policy matrices A_0.

The result below summarizes the variety of behaviors that system (3) can exhibit.

Theorem 2.1 (adapted from [3]). *The environmental policy $(\delta_{SP0}, \delta_{RT0})$ determines the asymptotic properties of (3) as follows.*

1) Sustained resource: If $(\delta_{SP0}, \delta_{RT0}) \in \mathcal{V}$, where

$$\mathcal{V} := \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0 \text{ and } -\frac{\theta}{\alpha} y_1 < y_2 < \frac{\delta_{TR1}}{\delta_{PS1}} y_1\}, \quad (5)$$

then the fixed point $(x^*, n^*) = (\frac{\alpha}{\alpha + \theta}, \frac{g(x^*, 0)}{-\partial g/\partial n(x^*)}) \in (0, 1)^2$ is the only asymptotically stable fixed point in the system.

2) Oscillating Tragedy of the commons (OTOC): If $\delta_{SP0} > 0$ and $\frac{\delta_{TR1}}{\delta_{PS1}} \delta_{SP0} < \delta_{RT0}$, then system (3) exhibits a stable heteroclinic cycle between the four corner fixed points, $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

3) Tragedy of the commons (TOC): If $\delta_{SP0} \leq 0$, or $\delta_{SP0} > 0$ and $\delta_{RT0} < -\frac{\theta}{\alpha} \delta_{SP0}$, then the only asymptotically stable fixed point has $n = 0$.

The policies belonging to $\mathcal{V}$ are ones that can sustain a stable and non-zero resource level. In other words, they avert a TOC, which we refer to as a fixed point where the resource level is zero. The policies in $\mathcal{V}$ can be considered as more desirable operating conditions than the policies described in items 2 and 3 above. The OTOC is considered an undesirable outcome, as the system cycles between periods of nearly zero and fully abundant resource levels. Lastly, we note that the

time-scale separation ϵ does not affect the derived stability properties and fixed points of the system.

B. Model: Two-population feedback-evolving games

Now, we consider two populations that share the environmental resource. Figure 1 illustrates this scenario. Specifically, the consumption decisions of the members of both populations have effects on the environmental resource. Now, $x_i \in [0,1]$ denotes the fraction of low consumers in population $i \in \{1,2\}$. The behavior of population i is governed by the payoff matrices $A_0^{(i)}$ (parameters δ_{SP0}^i, δ_{RT0}^i) and $A_1^{(i)}$ (parameters δ_{TR1}^i, δ_{PS1}^i), restoration rate $\theta_i > 0$, degradation rate $\alpha_i > 0$. In line with Assumption 1, we maintain the parameters in the abundant state satisfy $\delta_{TR1}^i, \delta_{PS1}^i > 0$. The payoff differences for each population are given by

$$g_i(x_i, n) := \pi_{\mathcal{L}}^{(i)}(x_i, n) - \pi_{\mathcal{H}}^{(i)}(x_i, n) \qquad (6)$$

where $\pi_j^{(i)}$ is the experienced payoff to a j-strategist in population i. The system dynamics are now given by the set of three ODEs with state variable $\boldsymbol{z} = (x_1, x_2, n)$:

$$\begin{aligned}
\dot{x}_1 &= x_1(1-x_1)g_1(x_1, n) \\
\dot{x}_2 &= x_2(1-x_2)g_2(x_2, n) \\
\dot{n} &= \epsilon n(1-n)h(x_1, x_2)
\end{aligned} \qquad (7)$$

where $h(x_1, x_2) := \sum_{i=1}^{2}(\theta_i x_i - \alpha_i(1-x_i))$, and with initial condition $(x_1(0), x_2(0), n(0)) \in (0,1)^3$. By construction, $(0,1)^3$ is forward-invariant. We are interested in studying how the dynamics of the two-population environmental coupled system (7) behaves with respect to the population's defining parameters – especially how the environmental policies $A_0^{(1)}$, $A_0^{(2)}$ adopted by each population affects environmental resources. We will make the following assumptions on these environmental policies.

Assumption 2. *We assume that $\frac{\partial g_i}{\partial n}(x_i) < 0$ for all $x_i \in [0,1]$, $i = 1, 2$. This is equivalent to $\delta_{SP0}^i > -\delta_{PS1}^i$ and $\delta_{RT0}^i > -\delta_{TR1}^i$.*

Assumption 2 asserts that the relative payoff to low consumers in both populations monotonically decreases as the environmental state improves. The next and final assumption focuses our study on a scenario where one population 1 is environmentally *responsible*, and population 2 is *irresponsible*.

Assumption 3. *For population 1, we assume that $(\delta_{SP0}^1, \delta_{RT0}^1) \in \mathcal{V}$ (with $\theta = \theta_1$ and $\alpha = \alpha_1$). For population 2, we assume that $\delta_{SP0}^2, \delta_{RT0}^2 < 0$.*

The first part of Assumption 3 asserts that agents in population 1 are cooperative enough to sustain the resource (Theorem 2.1, item 1) in the absence of population 2. The second part asserts that agents in population 2 make no effort to conserve resources, always preferring the high consumption strategy. The set of all feasible policies specified by Assumptions 2 and 3 is visually depicted in Figure 2.
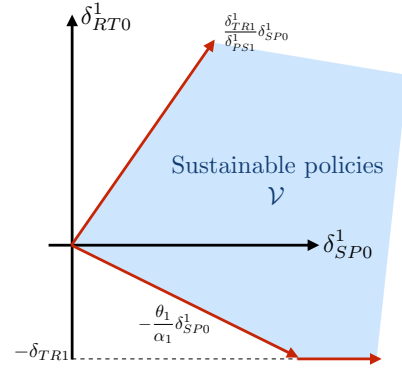

Fig. 2: The set of all sustainable policies for population 1 is shown as the blue region. A sustainable policy maintains a stable, nonzero resource in the absence of other populations (Theorem 2.1). This region is the focus of Assumptions 2 and 3.

III. STABILITY ANALYSIS OF TWO-POPULATION GAME

In this section, we characterize the dynamical behavior of the two-population coupled system by analyzing the local stability properties of all of its fixed points.

Theorem 3.1. *The asymptotic dynamics of the two-population system (7) are summarized below.*
1) Suppose $\alpha_2 > \theta_1$. Then a tragedy of the commons is asymptotically stable, i.e. $\lim_{t \to \infty} n(t) = 0$.

2) Suppose $\alpha_2 < \theta_1$.

(a) If $\frac{\alpha_2 - \theta_1}{\alpha_1 + \alpha_2}\delta_{SP0}^1 < \delta_{RT0}^1 < \frac{\delta_{TR1}}{\delta_{PS1}}\delta_{SP0}^1$, then the only asymptotically stable fixed point is of the form $(x_1^*, 0, n^*)$, where

$$x_1^* := \frac{\alpha_1 + \alpha_2}{\alpha_1 + \theta_1}, \quad n^* := -\frac{g_1(x_1^*, 0)}{\frac{\partial g_1}{\partial n}(x_1^*)}. \qquad (8)$$

(b) If $\delta_{RT0}^1 \leq \frac{\alpha_2 - \theta_1}{\alpha_1 + \alpha_2}\delta_{SP0}^1$, then a tragedy of the commons is the only stable outcome.

3) Suppose $\alpha_2 = \theta_1$. If $\delta_{RT0}^1 > 0$, then there is a locally stable line segment of fixed points given by

$$\left\{(1, 0, n) : n \in \left[0, \frac{\delta_{RT0}^1}{\delta_{RT0}^1 + \delta_{TR1}^1}\right)\right\} \qquad (9)$$

All other fixed points are isolated and unstable. If $\delta_{RT0}^1 \leq 0$, then a tragedy of the commons is asymptotically stable.

Several remarks are in order. In item 1 above, the irresponsible population induces a tragedy of the commons if its consumption rate is higher than the responsible population's restoration rate. Item 2a provides a region of sustainable policies for population 1. This region gets smaller as α_2 increases while remaining less than θ_1. Item 2b provides a region where the population 1 policy fails to sustain the resource even though $\alpha_2 < \theta_1$. Item 3 presents an edge case where there are an infinite number of fixed points, all with different resource levels. The fixed point on the line that the system converges to depends on the initial condition. The proof of this Theorem and all other results in this paper can be found in the online version [1].

Moreover, we observe the sustained resource level n^* in item 2a is a decreasing function in α_2. Taking the derivative with respect to α_2, we obtain

$$\frac{\partial n^*}{\partial \alpha_2} = \frac{g_1(x_1^*,0)\frac{\partial^2 g_1}{\partial n \partial x} - \frac{\partial g_1}{\partial x}(x_1^*,0)\frac{\partial g_1}{\partial n}(x_1^*)}{(\alpha_1+\theta_1)(\frac{\partial g_1}{\partial n}(x_1^*))^2}$$

It is negative since the numerator being negative is equivalent to the condition $\delta_{RT0}^1 \leq \frac{\delta_{TR1}}{\delta_{PS1}}\delta_{SP0}^1$.

IV. RESULTS: EXPLOITATION OF RESOURCES

In this section, we study the following hierarchical decision problem posed informally as: *how much consumption can the irresponsible population get away with?* To approach this question, we consider a local authority for population 2 that may set the consumption rate $\alpha_2 \geq 0$. This decision is representative of, for example, water usage or fishing regulations. The *utility* that the authority seeks to maximize is defined by

$$U(\alpha_2) := \alpha_2 \cdot R(\alpha_2). \qquad (10)$$

where $R(\cdot)$ summarizes the resource level that results in the asymptotic outcome of the dynamics. From Theorem 3.1, we define it as

$$R(\alpha_2) := \begin{cases} n^*(\alpha_2), & \text{if } \alpha_2 \leq \theta_1 \text{ and} \\ & \frac{\alpha_2-\theta_1}{\alpha_1+\alpha_2}\delta_{SP0}^1 \leq \delta_{RT0}^1 \leq \frac{\delta_{TR1}}{\delta_{PS1}}\delta_{SP0}^1 \\ 0, & \text{else} \end{cases} \qquad (11)$$

where $n^*(\alpha_2) := -\frac{g_1(\frac{\alpha_1+\alpha_2}{\alpha_1+\theta_1},0)}{\frac{\partial g_1}{\partial n}(\frac{\alpha_1+\alpha_2}{\alpha_1+\theta_1})}$ is taken directly from (8). In the edge case $\alpha_2 = \theta_1$, Theorem 3.1 states there is a range $n \in [0, \frac{\delta_{RT0}^1}{\delta_{RT0}^1+\delta_{TR1}^1}]$ of possible asymptotic resource levels depending on the initial condition. In our definition of $R(\cdot)$, we have elected to assign the highest possible resource[1] value $R(\alpha_2) = \frac{\delta_{RT0}^1}{\delta_{RT0}^1+\delta_{TR1}^1}$.

The choice to increase consumption comes at the cost of worsening or even destroying the environmental resource. Thus, (10) captures the tension between resource consumption and the stability of the resource. The optimal consumption rate can be determined by solving the problem

$$\alpha_2^* := \arg\max_{\alpha_2 \geq 0} U(\alpha_2). \qquad \text{(OC)}$$

Note here that we are assuming a fixed environmental policy $(\delta_{SP0}^1, \delta_{RT0}^1)$ for population 1 that satisfies Assumption 3. Our main result below provides a full characterization of the optimal consumption rate and utility.

Theorem 4.1 (Optimal consumption rate). *The optimal solution and value of* (OC) *are given as follows.*

(a) If $C(\delta_{SP0}^1) \leq \delta_{RT0}^1 < \frac{\delta_{TR1}^1}{\delta_{PS1}^1}\delta_{SP0}^1$, then

$$\alpha_2^* = \theta_1 \qquad (12)$$

[1]This is done for two reasons. First, it makes the resource function well-defined and left-continuous at θ_1. Thus, R attains a maximum value in the interval $[0, \theta_1]$. Second, since no particular initial condition is prescribed in the model, we may view the irresponsible population to be "optimistic" regarding the best case among all possible outcomes.

and $R(\alpha_2^*) = \frac{\delta_{RT0}^1}{\delta_{RT0}^1+\delta_{TR1}^1}$.
(b) If $\max\{-\frac{\theta_1}{\alpha_1}\delta_{SP0}^1, -\delta_{TR1}^1\} \leq \delta_{RT0}^1 \leq C(\delta_{SP0}^1)$, then

$$\alpha_2^* = -\frac{(\alpha_1+\theta_1)}{a_1}\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)\left(1 - \sqrt{\frac{Y}{b_1\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}}\right) \qquad (13)$$

and $R(\alpha_2^*) = \frac{\sqrt{b_1\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}-\sqrt{Y}}{a_1\sqrt{b_1^{-1}\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}}$, where we have defined

$$C(\delta_{SP0}^1) := \frac{1}{2}\left[-((1-\bar{\alpha}_1)\delta_{PS1}^1 + \delta_{TR1}^1)+ \right.$$
$$\left. \sqrt{((1-\bar{\alpha}_1)\delta_{PS1}^1 + \delta_{TR1}^1)^2 + 4(1-\bar{\alpha}_1)\delta_{TR1}^1\delta_{SP0}^1}\right] \qquad (14)$$
$$a_1 := \delta_{SP0}^1 - \delta_{RT0}^1 + \delta_{PS1}^1 - \delta_{TR1}^1$$
$$b_1 := \delta_{RT0}^1 - \delta_{SP0}^1$$
$$\bar{\alpha}_1 := \frac{\alpha_1}{\alpha_1+\theta_1}$$

and

$$Y := \delta_{TR1}^1\delta_{SP0}^1 - \delta_{RT0}^1\delta_{PS1}^1 \qquad (15)$$

Item a) specifies the range of population 1 environmental policies where population 2 benefits most from the maximal consumption rate $\alpha_2^* = \theta_1$. Any higher consumption rate will cause the resource to collapse. In this range, the resource (and consequently, utility) is highly sensitive at the threshold, since $R(\theta_1) > 0$ and discontinuously drops to zero for $\alpha_2 > \theta_1$.

Item b) gives the range of environmental policies where population 2 benefits most from a consumption rate that is not maximal, i.e. $\alpha_2^* < \theta_1$ (13). Any higher consumption causes the resource to degrade marginally faster, i.e. $U(\alpha_2)$ becomes a decreasing function for $\alpha_2 > \alpha_2^*$. Unlike in the policies from item a), the utility $U(\alpha_2)$ maintains continuity on $\alpha_2 \geq 0$, even as it becomes zero for all $\alpha_2 > \theta_1$.

Numerical computations of the optimal quantities are shown in Figure 3.

V. SIMULATIONS: SENSITIVITY OF COOPERATION INCENTIVES

From the perspective of population 2, the optimal consumption rate is chosen to always result in a non-zero resource level, $R(\alpha_2^*)$, given a *fixed* population 1 policy $(\delta_{SP0}^1, \delta_{RT0}^1)$. In this section, we seek to understand how the choice of environmental policy $(\delta_{SP0}^1, \delta_{RT0}^1)$ impacts the resulting resource level, under the optimal consumption rate for population 2. Intuitively, suppose a local authority for population 1 has the option to administer incentives to promote the low consumption strategy, given that the irresponsible population will optimally exploit the updated policy.

The updated policy becomes $(\delta_{SP0}^1+u_s, \delta_{RT0}^1+u_r)$, where $u_s, u_r \geq 0$ are the added incentives. The payoff matrix A_0^1 then reads as

$$\begin{bmatrix} R_0 + u_r & S_0 + u_s \\ T_0 & P_0 \end{bmatrix} \qquad (16)$$

The addition of the u_r term represents added incentives for *mutual cooperation* – individuals that practice low consump-
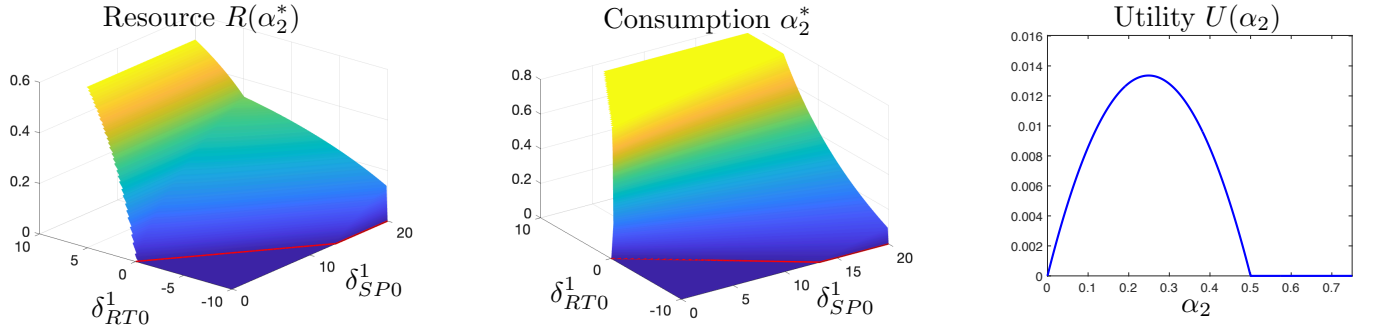
Fig. 3: (Left) This surface plot shows the resource level that results from the optimal consumption rate α_2^* detailed in Theorem 4.1. The red line indicates the bottom border of the set of feasible policies for population 1. We observe that $R(\alpha_2^*)$ is increasing in both δ_{RT0}^1 and δ_{SP0}^1. (Center) The optimal consumption rate detailed in Theorem 4.1. (Right) An example of the utility function $U_2(\alpha_2)$ under policy $(\delta_{SP0}^1, \delta_{RT0}^1) = (3, -0.5)$. For all simulations, the parameter values are: $\delta_{TR1} = 10$, $\delta_{PS1} = 6$, $\theta_1 = 0.75$, $\alpha_1 = 1$, $\epsilon = 0.1$.

tion are rewarded when many also practice the low consumption strategy. On the other hand, the addition of the u_s term represents added incentives for *unilateral cooperation* – individuals that practice low consumption are rewarded when many practice the high consumption strategy. How should the authority select u_s, u_r? Which type of incentive is more effective in promoting the resource level?

A. Calculation of sensitivities

Our approach here is to evaluate the sensitivity of $R(\alpha_2^*)$ with respect to small changes in the policy. In other words, we calculate the gradient of the (re-defined) function $R^*(\delta_{SP0}^1, \delta_{RT0}^1) := R(\alpha_2^*)$ with respect to $(\delta_{SP0}^1, \delta_{RT0}^1)$.

The partial derivative of R^* with respect to δ_{SP0}^1, in the region $C(\delta_{SP0}^1) \le \delta_{RT0}^1 \le \frac{\delta_{TR1}^1}{\delta_{PS1}^1}\delta_{SP0}^1$ (part (a) of Theorem 4.1), is simply $\frac{\partial R^*}{\partial \delta_{SP0}^1} = 0$. In this case, the addition of any u_s incentive has *no effect* on the resource level. In the region $\max\{-\frac{\theta_1}{\alpha_1}\delta_{SP0}^1, -\delta_{TR1}^1\} \le \delta_{RT0}^1 \le C(\delta_{SP0}^1)$ (part (b) of Theorem 4.1), the partial derivative with respect to δ_{SP0}^1 is calculated to be

$$
\frac{\partial R^*}{\partial \delta_{SP0}^1} = \frac{\delta_{PS1}^1 - \delta_{TR1}^1}{a_1^2}(1-\phi) +
$$
$$
\frac{-b_1}{2a_1}\phi\left(\frac{1}{-b_1} - \frac{\delta_{TR1}^1}{Y} + \frac{1-\bar{\alpha}_1}{-\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}\right) \quad (17)
$$

where

$$
\phi := \sqrt{\frac{Y}{b_1 \frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}}, \quad (18)
$$

depends on the policy $(\delta_{SP0}^1, \delta_{RT0}^1)$, and a_1, b_1, and Y are variables that also depend on the policy $(\delta_{SP0}^1, \delta_{RT0}^1)$, and were defined in the statement of Theorem 4.1. The sign of $b_1 = \delta_{RT0}^1 - \delta_{SP0}^1$ is negative, since $C(\delta_{SP0}^1) \le \delta_{SP0}^1$ with equality if and only if $\delta_{SP0}^1 = 0$.

The partial derivative of R^* with respect to δ_{RT0}^1, in the region $C(\delta_{SP0}^1) \le \delta_{RT0}^1 \le \frac{\delta_{TR1}^1}{\delta_{PS1}^1}\delta_{SP0}^1$ (part (a) of Theorem 4.1), is calculated to be

$$
\frac{\partial R^*}{\partial \delta_{RT0}^1} = \frac{\delta_{TR1}^1}{(\delta_{RT0}^1 + \delta_{TR1}^1)^2} > 0.
$$

In the region $\max\{-\frac{\theta_1}{\alpha_1}\delta_{SP0}^1, -\delta_{TR1}^1\} \le \delta_{RT0}^1 \le C(\delta_{SP0}^1)$ (part (b) of Theorem 4.1), the partial derivative with respect to δ_{RT0}^1 is calculated to be

$$
\frac{\partial R^*}{\partial \delta_{RT0}^1} = \frac{\delta_{TR1}^1 - \delta_{PS1}^1}{a_1^2}(1-\phi) +
$$
$$
\frac{-b_1}{2a_1}\phi\left(\frac{1}{b_1} + \frac{\delta_{PS1}^1}{Y} + \frac{\bar{\alpha}_1}{-\frac{\partial g_1}{\partial n}(\bar{\alpha}_1)}\right)
$$

The following basic property holds for both sensitivities.

Proposition 5.1. *For any set of fixed parameter values δ_{TR1}^1, δ_{PS1}^1, $\bar{\alpha}_1$, and any feasible policy $(\delta_{SP0}^1, \delta_{RT0}^1)$, it holds that $\frac{\partial R^*}{\partial \delta_{SP0}^1}, \frac{\partial R^*}{\partial \delta_{RT0}^1} \ge 0$.*

Thus, increasing incentive for low consumption can never make the resource level worse under the optimal consumption rate of population 2.

B. Simulations: comparison of incentives

To address the question of which type of incentive, u_r or u_s, is more effective, we inspect the *sensitivity ratio*

$$
\rho(\delta_{SP0}^1, \delta_{RT0}^1) := \frac{\partial R^*/\partial \delta_{SP0}^1}{\partial R^*/\partial \delta_{RT0}^1} \quad (19)
$$

A ratio of $\rho < 1$ indicates that the incentive u_r is more effective than u_s, and $\rho > 1$ indicates the opposite. A numerical computation of the sensitivity ratio is provided in Figure 4.

While a formal analysis is not yet provided in this paper, the numerical computations strongly suggest that the u_r incentive is generally more effective than u_s. We observe that $\rho < 1$ for a large space of policies $(\delta_{SP0}^1, \delta_{RT0}^1)$, and $\rho > 1$ only when $(\delta_{SP0}^1, \delta_{RT0}^1)$ is close to the threshold curve $(\delta_{SP0}^1, C(\delta_{SP0}^1))$ and δ_{PS1}^1 is sufficiently small relative to δ_{TR1}^1. For larger values of δ_{PS1}^1, we observe that $\rho > 1$ for all policies $(\delta_{SP0}^1, \delta_{RT0}^1)$. We also reiterate that $\rho = 0$ for all policies above the threshold curve $C(\delta_{SP0}^1)$, i.e. only the u_r incentive can help improve the resource level.

VI. CONCLUSION AND FUTURE WORK

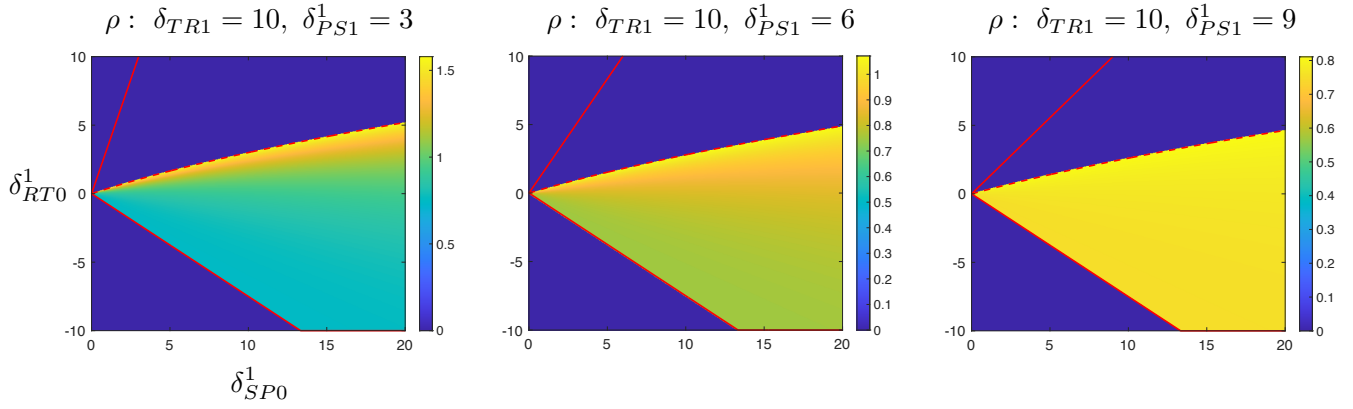In this paper, we studied a feedback-evolving game with two populations that share a common environmental re-

Fig. 4: *Sensitivity ratios*. This series of plots shows the sensitivity ratio ρ as the population 1 policy $(\delta^1_{SP0}, \delta^1_{RT0})$ varies. The red lines indicate the boundaries of the set of feasible policies for population 1. In all plots, we observe the ratio is highest near the curve $C(\delta^1_{SP0})$ (dashed red line), i.e. u_s becomes effective, and indeed can be more effective than u_r for lower values of δ^1_{PS1} (left and center plots). However, u_s is generally not as effective relative to u_r. For higher values of δ^1_{PS1}, u_s will never be as effective as u_r, i.e. $\rho < 1$ for all policies (right plot). For all simulations, the parameter values are: $\delta_{TR1} = 10$, $\theta_1 = 0.75$, $\alpha_1 = 1$, $\epsilon = 0.1$.

source. This marks initial steps in extending the framework to multi-population interactions and hierarchical decsion-making. We focused on a scenario where one population is "responsible" about using environmental resources, while the other population is "irresponsible". We evaluated to what extent the irresponsible population can take advantage of the resources by deriving optimal consumption rates. Lastly, a sensitivity analysis was provided that suggests incentivizing mutual cooperation is more effective than unilateral cooperation. Future work will analyze how multiple irresponsible populations interact with one another. Beyond the scope of the present study, one future direction is to investigate populations that each have local environmental resources. Another interesting consideration is to study evolutionary dynamics beyond the replicator equation.

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