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Stabilization of a Parabolic-Elliptic System via Backstepping

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***Abstract*— Stabilization of a parabolic partial differential equation coupled with an elliptic partial differential equation is considered. Even in the situation when these equations are exponentially stable when uncoupled, the coupled system may be unstable. A backstepping approach is used to design a boundary control that stabilizes the coupled system. The result is illustrated with simulations.**

I. INTRODUCTION

Parabolic-elliptic systems result from the coupling of parabolic partial differential equations with elliptic partial differential equations. Examples include the mathematical modelling of lithium-ion and electrolytic cells [1], [2], biological transport networks [3], chemotaxis phenomena [4], [5] and the thermistor [6].

Parabolic-elliptic systems are an important class of partial differential-algebraic equations (PDAEs). The well-posedness of the latter has been addressed in [7], [8], [9], [10]. In [9] the authors gave conditions for well-posedness using a concept called (E, p)-radiality. This is an extension of the Hille-Yosida type conditions for PDEs to PDAEs. Later, [10] provided similar results with weaker conditions by restricting the state-space to a reflexive Banach space and also presented an extension of the classical Lumer-Phillips Theorem to partial differential-algebraic equations.

Stabilization through boundary control for coupled linear parabolic partial differential equations has been investigated in the literature. In [11], the backstepping approach was used to stabilize the dynamics of a linear coupled reaction-diffusion systems with constant coefficients. An extension of this work to systems with variable coefficients was presented in [12]. In Koga et al. [13] the authors described boundary control of the one phase Stefan problem, modeled by a diffusion equation coupled with an ordinary differential equation. Feedback stabilization of a PDE-ODE system was also studied in [14]. The backstepping method stands out as one of the rare strategies that yields an explicit control law for PDEs, without first approximating the PDE. This is achieved by mapping the unstable original system into an exponentially stable target system. The transformation is intended to send the destabilizing terms within the original system to the boundary, where they can be eliminated by the control input.

If the parabolic equation is stable, it would be expected that coupling between the parabolic and the elliptic equations would not lead to instability. However, this was displayed to be untrue in [15]. Krstic and Smyshlyaev [16] considered the boundary stabilization of several coupled parabolic-elliptic systems, linearized Kuramoto–Sivashinsky and Korteweg–de Vries equations, using two control inputs. Later [15], stabilization using Dirichlet boundary control of an unstable parabolic-elliptic system with input delay was shown.

In this paper, we consider boundary stabilization of a class of parabolic-elliptic systems with single Neumann control. The result is a feedback control law that exponentially stabilizes the dynamics of the system. The control input is directly designed using the system of partial differential equations, without approximation by finite-dimensional systems. Explicit calculation of the eigenfunctions is not required. This is done by using a backstepping approach [16]. The conventional backstepping methodology involves looking for an invertible state transformation that maps the unstable original system into an exponentially stable target system. However, our algorithm takes a slightly different approach. We use a backstepping transformation that has been previously used for parabolic equations [16]. This leads to an unusual target system. The next step is to establish stability of the obtained target system. The final result is an explicit expression for a single boundary control that stabilizes the coupled system.

In section II, the class of parabolic-elliptic equations under study is described and shown to be well-posed. Stability analysis for the system is also provided. The main result, in section III, is the use of a backstepping transformation that leads to an explicit expression for a boundary controller for the coupled system . In section IV, the theoretical results are illustrated with numerical simulations.

II. PROBLEM STATEMENT

The aim is stabilization of the following class of parabolic-elliptic systems

$$w_t(x,t) = w_{xx}(x,t) - \rho w(x,t) + \alpha v(x,t), \tag{1}$$
$$0 = v_{xx}(x,t) - \gamma v(x,t) + \beta w(x,t), \tag{2}$$

$$w_x(0,t) = 0, \quad w_x(1,t) = u(t), \tag{3}$$
$$v_x(0,t) = 0, \quad v_x(1,t) = 0, \tag{4}$$

where $x \in [0,1]$ and $t \geq 0$. The parameters ρ, α, β, γ are all real, with α, β both nonzero. Also $\gamma \neq -(n\pi)^2$ so the operator $\gamma I - \partial_{xx}$ is invertible. The given restriction on γ ensures the well-posedness of system (1)- (4); see [10]. The parabolic equation is controlled at $x = 1$ by Neumann boundary control $u(t)$.

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System (1)- (4) can be written as

$$w_t(x,t) = w_{xx}(x,t) - \rho w(x,t) + \alpha\beta(\gamma I - \partial_{xx})^{-1} w(x,t). \quad (5)$$

Defining

$$A = \partial_{xx} - \rho I + \alpha\beta(\gamma I - \partial_{xx})^{-1}, \qquad (6)$$
$$D(A) = \{w \in H^2(0,1),\ w'(0) = w'(1) = 0\},$$

equation (5) can be further written as

$$\frac{d}{dt} w(t) = Aw(t), \quad w(0) = w_0. \qquad (7)$$

Theorem 1. *The eigenvalues of the uncontrolled system (1)- (4) are*

$$\lambda_n = -\rho + \frac{\alpha\beta}{\gamma + (n\pi)^2} - (n\pi)^2, \ n = 0,\ 1,\ \ldots \qquad (8)$$

where $\gamma \neq -(n\pi)^2$.

Proof. The analysis is standard but given for completeness. Let $\{\phi_j\}_{j\geq 0} \subset \mathscr{C}^4(0,1)$ be the eigenfunctions of the operator A corresponding to the eigenvalues λ_j, then

$$\lambda_j \phi_j = \phi_j'' - \rho\phi_j + \alpha\beta(\gamma I - \partial_{xx})^{-1}\phi_j,$$
$$\phi_j'(0) = \phi_j'(1) = 0. \qquad (9)$$

Setting

$$(\gamma I - \partial_{xx})^{-1}\phi_j = e_j, \qquad (10)$$

then $\gamma e_j - e_j'' = \phi_j$. Substituting for e_j from (10) into (9),

$$e_j = \frac{\rho + \lambda_j}{\alpha\beta}\phi_j - \frac{1}{\alpha\beta}\phi_j''.$$

We obtain the fourth-order differential equation

$$\phi_j'''' - (\lambda_j + \rho + \gamma)\phi_j'' + (\gamma(\lambda_j + \rho) - \alpha\beta)\phi_j = 0, \qquad (11)$$

with the boundary conditions

$$\phi_j'(0) = \phi_j'(1) = \phi_j'''(0) = \phi_j'''(1) = 0. \qquad (12)$$

Solving system (11)- (12) for ϕ_j yields that $\phi_j = \cos(j\pi x)$ for $j = 0,\ 1, \ldots$. Subbing ϕ_j in (11) leads to (8). □

Theorem 2. *System (1)- (4) is exponentially stable if*

$$\rho > \frac{\alpha\beta}{\gamma}, \qquad (13)$$

and the decay rate in that case is bounded by the maximum eigenvalue

$$\rho - \frac{\alpha\beta}{\gamma}. \qquad (14)$$

Proof. Since A is a self-adjoint operator with a compact inverse, it follows from [17, section 3] that A is a spectral operator. Also, A generates a C_0-semigroup with growth determined by the eigenvalues. □

Thus, even in the case when the parabolic equation is exponentially stable, coupling with the elliptic system can cause the uncontrolled system to be unstable.

III. STABILIZATION VIA BOUNDARY CONTROL

In this section, we design a boundary control that stabilizes the dynamics of the coupled system by using a backstepping approach. These transformations are generally formulated as a Volterra operator, which guarantees under weak conditions invertibility of the transformation. One possible approach to stabilization is to convert system (1)- (4) into one equation in terms of the state $w(x,t)$. However, this will result in the presence of a Fredholm operator $\alpha\beta\int_0^1 g(x;y)w(y,t)dy$ where $g(x;y)$ is the Green's function of $(\gamma I - \partial_{xx})^{-1}$. This term makes it difficult to establish a suitable Volterra transformation. Another approach would be a vector-valued transformation for both $w(x,t)$ and $v(x,t)$. This is quite complex.

The simplest approach is to apply state transformation only on the parabolic state. This leads to a target system that is also a PDAE, but only one transformation is needed and there is now a wide literature on such transformations; see [16]. We use the transformation

$$\tilde{w}(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)dy. \qquad (15)$$

while the elliptic state $v(x,t)$ is unchanged, and the kernel $k(X,y)$ is given by the following lemma from [16, chap. 4].

Lemma 3. *The hyperbolic partial differential equation*

$$-k_{yy}(x,y) + k_{xx}(x,y) + (\rho - c_1)k(x,y) = 0,$$
$$k_y(x,0) = 0, \quad k(x,x) = -\frac{1}{2}(c_1 - \rho)x, \qquad (16)$$

is well-posed. Furthermore, the solution of the system above is

$$k(x,y) = -(c_1 - \rho)x \frac{I_1\left(\sqrt{(c_1 - \rho)(x^2 - y^2)}\right)}{\sqrt{(c_1 - \rho)(x^2 - y^2)}}, \qquad (17)$$

where $I_1(\cdot)$ *is the modified Bessel function of first order defined as*

$$I_1(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+1}}{m!(m+1)!}.$$

The following lemma in [16, chap. 4] presents the inverse transformation of (15).

Lemma 4. *The inverse transformation of (15) is*

$$w(x,t) = \tilde{w}(x,t) + \int_0^x l(x,y)\tilde{w}(y,t)dy,$$

where $l(x,y)$ *is the solution of the system*

$$l_{xx}(x,y) - l_{yy}(x,y) - (\rho - c_1)l(x,y) = 0,$$
$$l_y(x,0) = 0, \quad l(x,x) = -\frac{1}{2}(c_1 - \rho)x,$$

with

$$l(x,y) = -(c_1 - \rho)x \frac{J_1\left(\sqrt{(c_1 - \rho)(x^2 - y^2)}\right)}{\sqrt{(c_1 - \rho)(x^2 - y^2)}}, \qquad (18)$$

where $J_1(\cdot)$ *is the Bessel function of first order defined as*

$$J_1(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{2m+1}}{m!(m+1)!}.$$

Theorem 5. *Transformation* (15) *maps the original system (1)- (4) into the target system*

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - c_1\tilde{w}(x,t) + \alpha v(x,t) - \alpha\int_0^x k(x,y)v(y,t)dy, \quad (19)$$

$$0 = v_{xx}(x,t) - \gamma v(x,t) + \beta\tilde{w}(x,t) + \beta\int_0^x l(x,y)\tilde{w}(y,t)dy, \quad (20)$$

$$\tilde{w}_x(0,t) = 0, \quad \tilde{w}_x(1,t) = 0, \quad (21)$$

$$v_x(0,t) = 0, \quad v_x(1,t) = 0, \quad (22)$$

where c_1 is a free parameter restricted to be chosen such that $c_1 > \rho$ and the partial differential equation governing the kernel function $k(x,y)$ is (16)*, provided that*

$$u(t) = \int_0^1 k_x(1,y)w(y,t)dy + k(1,1)w(1,t). \quad (23)$$

Proof. We rewrite (15) as

$$w(x,t) = \tilde{w}(x,t) + \int_0^x k(x,y)w(y,t)dy. \quad (24)$$

Differentiating (15) with respect to x twice gives

$$w_{xx}(x,t) = \tilde{w}_{xx}(x,t) + \int_0^x k_{xx}(x,y)w(y,t)dy + k_x(x,x)w(x,t) + \frac{d}{dx}k(x,x)w(x,t) + k(x,x)w_x(x,t). \quad (25)$$

and with respect to t,

$$\begin{aligned} w_t(x,t) =& \tilde{w}_t(x,t) + \int_0^x k(x,y)w_t(y,t)dy \\ =& \tilde{w}_t(x,t) + \int_0^x k(x,y)[w_{yy}(y,t) - \rho w(y,t) + \alpha v(y,t)]dy \\ =& \tilde{w}_t(x,t) + k(x,x)w_x(x,t) - \int_0^x k_y(x,y)w_y(y,t)dy - \rho\int_0^x k(x,y)w(y,t)dy + \alpha\int_0^x k(x,y)v(y,t)dy \\ =& \tilde{w}_t(x,t) + k(x,x)w_x(x,t) - k_y(x,x)w(x,t) + k_y(x,0)w(0,t) + \int_0^x k_{yy}(x,y)w(y,t)dy \\ &- \rho\int_0^x k(x,y)w(y,t)dy + \alpha\int_0^x k(x,y)v(y,t)dy. \end{aligned} \quad (26)$$

Here, $k_x(x,x) = \frac{\partial}{\partial x}k(x,y)|_{x=y}$, $k_y(x,x) = \frac{\partial}{\partial y}k(x,y)|_{x=y}$, $\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x)$. Substituting (25) and (26) in (1), and after some mathematical steps we arrive to

$$\begin{aligned} \tilde{w}_t(x,t) =& \tilde{w}_{xx}(x,t) - \rho w(x,t) + 2\frac{d}{dx}k(x,x)w(x,t) \\ &- k_y(x,0)w(0,t) - \alpha\int_0^x k(x,y)v(y,t)dy + \alpha v(x,t) \\ &+ \int_0^x [-k_{yy}(x,y) + k_{xx}(x,y) + \rho k(x,y)]w(y,t)dy. \end{aligned}$$

Adding and subtracting the term $c_1 w(x,t)$ to the right-hand-side of the previous equation

$$\begin{aligned} \tilde{w}_t(x,t) =& \tilde{w}_{xx}(x,t) - c_1 w(x,t) + \alpha v(x,t) - \alpha\int_0^x k(x,y) \\ &\times v(y,t)dy + (c_1 - \rho + 2\frac{d}{dx}k(x,x))w(x,t) - k_y(x,0)w(0,t) \\ &+ \int_0^x [-k_{yy}(x,y) + k_{xx}(x,y) + (\rho - c_1)k(x,y)]w(y,t)dy = 0. \end{aligned} \quad (27)$$

It follows from Lemma 5 that equation (27) reduces to (19). Also, $\tilde{w}_x(0,t) = w_x(0,t) - k(0,0)w(0,t) = 0$. The other boundary condition on $w(x,t)$ holds by noting that $u(t)$ is given by (23). □

The boundary condition (23) defines the control signal for the original system.

Next, we provide conditions that ensure that the target system is exponentially stable. We need to establish some lemmas. The next lemma gives bounds on the induced L^2-norms of the kernel functions $k(x,y)$ and $l(x,y)$.

Lemma 6. *The L^2-norms of $k(x,y)$ and $l(x,y)$ are bounded by*

$$\|k\| \le \sqrt{\frac{(c_1-\rho)\pi}{8}}\left(erfi\left(\sqrt{\frac{(c_1-\rho)}{2}}\right) \times erf\left(\sqrt{\frac{(c_1-\rho)}{2}}\right)\right)^{\frac{1}{2}}, \quad (28)$$

$$\|l\| \le \sqrt{\frac{(c_1-\rho)\pi}{8}}\left(erfi\left(\sqrt{\frac{(c_1-\rho)}{2}}\right) \times erf\left(\sqrt{\frac{(c_1-\rho)}{2}}\right)\right)^{\frac{1}{2}}, \quad (29)$$

where $erfi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{\xi^2}d\xi$, *and* $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi$.

Proof. To prove relation (28), we recall the expression for the kernel $k(x,y)$ given in (17). We set $z = \sqrt{(c_1-\rho)(x^2-y^2)}$, then

$$\begin{aligned} k(x,y) =& \frac{-(c_1-\rho)}{z}x\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+1}\frac{1}{m!m+1!} \\ =& \frac{-(c_1-\rho)}{z}x\frac{z}{2}\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{1}{m!m+1!} \\ =& \frac{-(c_1-\rho)}{2}x\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!}\frac{1}{m+1!} \\ \le& \frac{-(c_1-\rho)}{2}x\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!} \end{aligned}$$

Thus the induced L_2- norm is bounded by

$$\begin{aligned} &\|k(x,y)\| \le \frac{(c_1-\rho)}{2}\|x\|\|e^{\frac{z^2}{4}}\| \\ &\le \frac{(c_1-\rho)}{2}\|x\|\|e^{\frac{(c_1-\rho)x^2}{4}}\|\|e^{\frac{-(c_1-\rho)y^2}{4}}\| \\ &\le \sqrt{\frac{(c_1-\rho)\pi}{8}}\left(erfi\left(\sqrt{\frac{(c_1-\rho)}{2}}\right)erf\left(\sqrt{\frac{(c_1-\rho)}{2}}\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Similarly, one can prove (29) by referring back to (18). $\square$

The following lemma will be needed to show stability of the target system.

Lemma 7. *The states of the target system (19)-(22) satisfy the following inequality*

$$\|v(x,t)\| \leq \frac{|\beta|}{\gamma}(1+\|l\|)\|\tilde{w}\|. \tag{30}$$

Proof. Multiply equation (20) by $v(x,t)$ and integrate from 0 to 1,

$$\begin{aligned} 0 \;=\; & \int_0^1 v_{xx}(x,t)v(x,t)dx - \gamma\int_0^1 v^2(x,t)dx \\ & +\beta\int_0^1 \tilde{w}(x,t)v(x,t)dx \\ & +\beta\int_0^1 v(x,t)\int_0^x l(x,y)\tilde{w}(y,t)dydx, \end{aligned}$$

Thus,

$$\begin{aligned} \gamma\int_0^1 v^2(x,t)dx \leq& \beta\int_0^1 \tilde{w}(x,t)v(x,t)dx \\ & + \beta\int_0^1 v(x,t)\int_0^x l(x,y)\tilde{w}(y,t)dydx. \end{aligned} \tag{31}$$

Bounding the terms on the right-hand side of inequality (31) using Cauchy-Schwartz leads to (30). $\square$

Theorem 8. *The target system* $(19)-(22)$ *is exponentially stable if*

$$c_1 > \frac{|\alpha\beta|}{\gamma}(1+\|l\|)(1+\|k\|). \tag{32}$$

Proof. Define the Lyapunov function candidate,

$$V(t) = \frac{1}{2}\int_0^1 \tilde{w}^2(x,t)dx = \frac{1}{2}\|\tilde{w}(x,t)\|^2.$$

Taking the time derivative of $V(t)$,

$$\begin{aligned} \dot{V}(t) =& \int_0^1 \tilde{w}(x,t)\tilde{w}_t(x,t)dx \\ =& \int_0^1 \tilde{w}(x,t)[\tilde{w}_{xx}(x,t) - c_1\tilde{w}(x,t) \\ & + \alpha v(x,t) - \alpha\int_0^x k(x,y)v(y,t)dy]dx \\ \leq& -c_1\int_0^1 \tilde{w}^2(x,t)dx + \alpha\int_0^1 \tilde{w}(x,t)v(x,t)dx \\ & - \alpha\int_0^1 \tilde{w}(x,t)\int_0^x k(x,y)v(y,t)dydx. \end{aligned} \tag{33}$$

Using Cauchy-Schwartz inequality, we estimate the term of the right hand-side of inequality (33) as follows.

$$\begin{aligned} \alpha\int_0^1 \tilde{w}(x,t)v(x,t)dx \leq& |\alpha|\|\tilde{w}\|\|v\| \\ \leq& \frac{|\alpha||\beta|}{\gamma}(1+\|l\|)\|\tilde{w}\|^2, \end{aligned} \tag{34}$$

and

$$\begin{aligned} & -\alpha\int_0^1 \tilde{w}(x,t)\int_0^x k(x,y)v(y,t)dydx \\ & \leq |\alpha|\int_0^1 |\tilde{w}(x,t)|\int_0^1 |k(x,y)||v(y,t)|dydx \\ & \leq |\alpha|\|k\|\|\tilde{w}\|\|v\| \\ & \leq \frac{|\alpha||\beta|}{\gamma}\|k\|(1+\|l\|)\|\tilde{w}\|^2. \end{aligned} \tag{35}$$

Subbing (34) and (35) in (33),

$$\dot{V}(t) \leq -\left(c_1 - \frac{|\alpha||\beta|}{\gamma}(1+\|l\|)(1+\|k\|)\right)\|\tilde{w}\|^2. \tag{36}$$

Setting

$$c_2 = c_1 - \frac{|\alpha||\beta|}{\gamma}(1+\|l\|)(1+\|k\|),$$

then inequality (36) implies that,

$$V(t) \;\leq\; e^{-2c_2 t}V(0).$$

If the parameter c_1 is chosen such that (32) is satisfied, then $V(t)$ decays exponentially as $t \to \infty$, and so does $\|\tilde{w}(x,t)\|$. By means of lemma (7), the state $v(x,t)$ is asymptotically stable. Recalling (15), (2) and the fact that the operator $(\partial_{xx} - \gamma I)$ is bounded, the exponential stability of $v(x,t)$ follows from

$$\begin{aligned} \|v(x,t)\| \leq& \frac{|\beta|}{\gamma}(1+\|l\|)\|\tilde{w}\| \\ \leq& \frac{|\beta|}{\gamma}(1+\|l\|)\|\tilde{w}_0\|e^{-2c_2 t} \\ \leq& \frac{|\beta|}{\gamma}(1+\|l\|)(1+\|k\|)\|w_0\|e^{-2c_2 t} \\ \leq& \frac{1}{\gamma}(1+\|l\|)(1+\|k\|)\|\partial_{xx} - \gamma I\|\|v_0\|e^{-2c_2 t} \\ =& c_3\|v_0\|e^{-2c_2 t} \end{aligned}$$

where $c_3 = \frac{1}{\gamma}(1+\|l\|)(1+\|k\|)\|\partial_{xx} - \gamma I\|$. The conclusion of the theorem follows. $\square$

The decay rate of the target system is bounded by,

$$2c_2 = 2\left(c_1 - \frac{|\alpha||\beta|}{\gamma}(1+\|l\|)(1+\|k\|)\right). \tag{37}$$

The following corollary to Theorem 8 is now immediate.

Corollary 9. *The controlled system is exponentially stable if*

$$\begin{aligned} c_1 > \frac{|\alpha||\beta|}{\gamma}&\left[1+\sqrt{\frac{(c_1-\rho)\pi}{8}}\left(erfi(\sqrt{\frac{(c_1-\rho)}{2}})\right)^{\frac{1}{2}}\right. \\ & \left.\times\left(erf(\sqrt{\frac{(c_1-\rho)}{2}})\right)^{\frac{1}{2}}\right]^2. \end{aligned} \tag{38}$$

Thus, if c_1 satisfies (32), transformation (15) converts the original system (1)-(4) into the stable target system (19)-(22). From (23) an explicit definition of the control is signal is immediately obtained,

$$u(t) = \int_0^1 k_x(1,y)w(y,t)dy + k(1,1)w(1,t). \qquad (39)$$

IV. Numerical Simulations

The solutions of system (1)-(4), both controlled and uncontrolled, were simulated numerically using a finite-element approximation in COMSOL Multiphysic software. Linear splines, with 27 sub-intervals, were used to discretize the coupled system into a DAE. Time was discretized by a time-stepping algorithm, generalized alpha, with time-step=0.2.

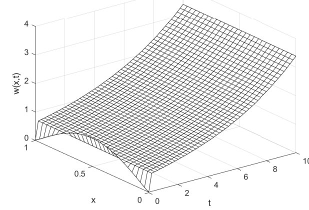
First, we considered the system with $\gamma = \frac{1}{4}$, $\rho = \frac{1}{3}$, $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$ and initial condition $w(0) = \sin(\pi x)$. For these coefficients, the system is unstable according to Theorem 2. Figure 1 presents the dynamics of the states $w(x,t)$ and $v(x,t)$ before and after applying the control (39). It can be seen that, in the absence of control, the solutions grow in magnitude t increases. With parameter $c_1 = 1.2$ inequality (38) is satisfied so the control law is stabilizing. Figure 1 shows the controlled system with the same initial condition. As predicted by the theory, the dynamics of the system decay to zero with time. A comparison between the L_2-norm of both states $w(x,t)$ and $v(x,t)$ before and after applying the control is shown in Figure 2. We also carried out the simulation for parameters $\gamma = 10$, $\rho = 9.5$, $\alpha = 10$ and $\beta = 10$, $c_1 = 15$. The open-loop and closed-loop dynamics, indicating the behaviour of the coupled system without and with control, respectively, were simulated (see Figure 4). Even though (38) is not satisfied by this set of parameters, the numerical simulations indicate that the control is stabilizing the system. This suggests that condition (38) on the control parameter c_1 is not necessary for the stability of the target system.

V. Conclusion

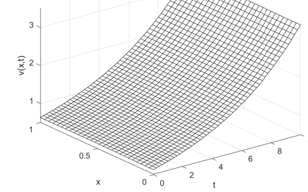
Boundary stabilization of a parabolic-elliptic systems is considered in this paper. Coupling between two stable parabolic and elliptic equations can result in an unstable coupled system. A boundary control law is designed for this system using a backstepping approach. The transformation is only applied to the parabolic part, simplifying the calculations. A sufficient condition for stability is obtained. Numerical simulations were conducted to illustrate the theoretical result. The state-feedback nature of the obtained control input (i.e. (39)) requires the knowledge of the state. Therefore, the observer design problem along with design of an output feedback controller is presented in [18]. Future work is aimed at weakening the sufficient condition on c_1 for stability for the target system. The design of a boundary control when some nonlinear terms are present in the system is also a point of interest. Furthermore, an extension of the work to the case when the coefficients are spatially-variant will be studied.

References

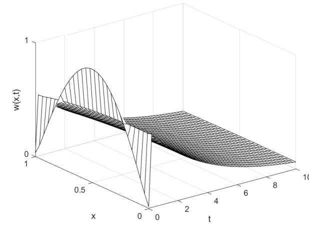
[1] L. Consiglieri, "Weak solutions for multiquasilinear elliptic–parabolic systems: application to thermoelectrochemical problems," *Boletín de la Sociedad Matemática Mexicana*, vol. 26, no. 2, pp. 535–562, 2020.



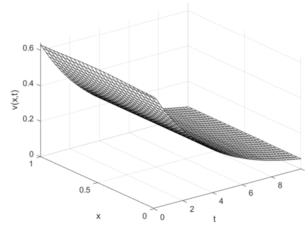
(a) uncontrolled $w(x,t)$



(b) uncontrolled $v(x,t)$

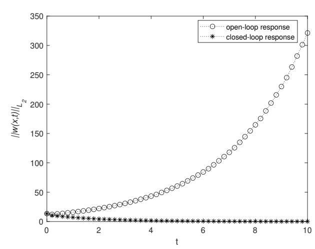


(c) controlled $w(x,t)$

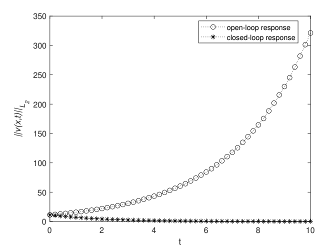


(d) controlled $v(x,t)$

Fig. 1. Trajectory of the coupled parabolic-elliptic system (1)-(4) with initial condition $w_0 = sin(\pi x)$ before and after applying the control (39). The system parameters $\gamma = \frac{1}{4}$, $\rho = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$ mean that without control, the system is unstable. The control is (39) with $c_1 = 1.2$, which satisfies the sufficient stability condition (38). As predicted by theory, the controlled system is stable.



(a) L_2-norm $w(x,t)$



(b) L_2-norm $v(x,t)$

Fig. 2. Comparison between the L_2-norm of the solutions $w(x,t)$ and $v(x,t)$ for the open and closed-loop systems;$\gamma = \frac{1}{4}$, $\rho = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$ and $c_1 = 1.2$, which satisfies the stability condition (38). The figure demonstrates the unstable behaviour of the solution without applying control. It also indicates that the control input forces the solutions of the coupled system to decay to the zero solution as t goes to ∞.

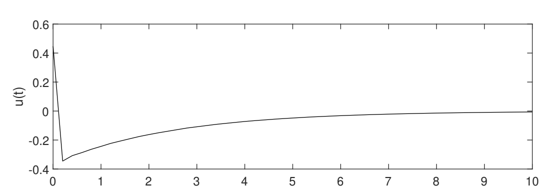
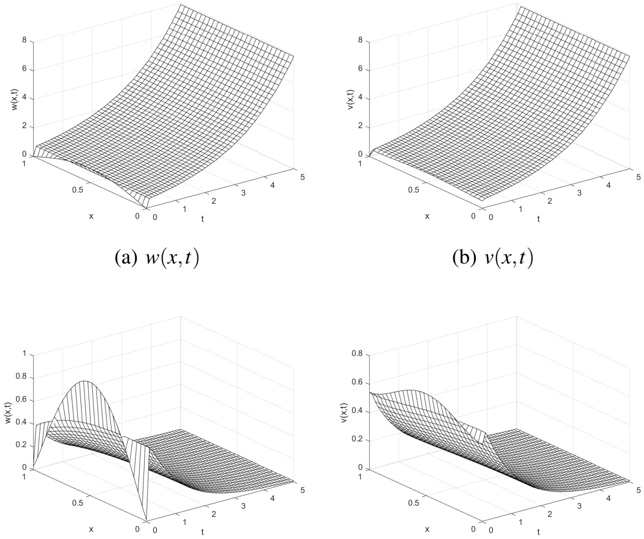


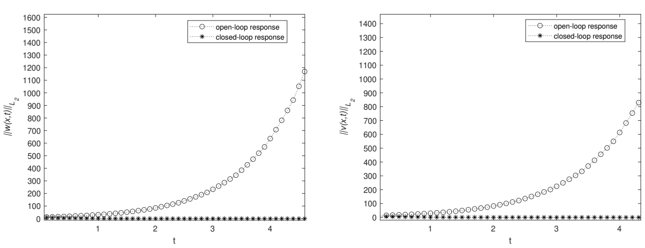
Fig. 3. The control gain signal (39); $\gamma = \frac{1}{4}$, $\rho = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$ and $c_1 = 1.2$.



(a) $w(x,t)$ (b) $v(x,t)$

(c) $w(x,t)$ (d) $v(x,t)$

Fig. 4. Trajectory of the coupled parabolic-elliptic system (1)-(4) with initial condition $w_0 = sin(\pi x)$ before and after applying the control (39). The system parameters $\gamma = 10$, $\rho = 9.5$, $\alpha = 10$ and $\beta = 10$ mean that without control, the system is unstable. The control is (39) with $c_1 = 15$, which does not satisfy the sufficient stability condition (38). However, the controlled system appears to be stable, indicating that the condition may be stronger than needed.



(a) L_2-norm $w(x,t)$ (b) L_2-norm $v(x,t)$

Fig. 5. Comparison between the L_2-norm of the solutions $w(x,t)$ and $v(x,t)$ for the open and closed-loop systems with $\gamma = 10$, $\rho = 9.5$, $\alpha = 10$, $\beta = 10$ and $c_1 = 15$, which does not satisfy the stability condition (38). The figure demonstrates the unstable behaviour of the solution without applying control. It also indicates that the control input forces the solutions of the coupled system to decay to the zero solution as t goes to ∞ although the stability condition (38) is not satisfied.

[2] J. Wu, J. Xu, and H. Zou, "On the well-posedness of a mathematical model for lithium-ion battery systems," *Methods and Applications of Analysis*, vol. 13, no. 3, pp. 275–298, 2006.

[3] B. Li, "Global existence and decay estimates of solutions of a parabolic–elliptic–parabolic system for ion transport networks," *Results in Mathematics*, vol. 75, no. 2, p. 45, 2020.

[4] J. Ahn and C. Yoon, "Global well-posedness and stability of constant equilibria in parabolic–elliptic chemotaxis systems without gradient sensing," *Nonlinearity*, vol. 32, no. 4, p. 1327, 2019.

[5] M. Negreanu, J. I. Tello, and A. M. Vargas, "On a parabolic-elliptic chemotaxis system with periodic asymptotic behavior," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 4, pp. 1210–1226, 2019.

[6] H. Meinlschmidt, C. Meyer, and J. Rehberg, "Optimal control of the thermistor problem in three spatial dimensions, part 1: existence of optimal solutions," *SIAM Journal on Control and Optimization*, vol. 55, no. 5, pp. 2876–2904, 2017.

[7] S. Trostorff, "Semigroups associated with differential-algebraic equations," in *Conference on Semigroups of Operators: Theory and Applications*, pp. 79–94, Springer, 2018.

[8] B. Thaller and S. Thaller, "Factorization of degenerate cauchy problems: The linear case," *Journal of Operator Theory*, pp. 121–146, 1996.

[9] G. A. Sviridyuk and V. E. Fedorov, *Linear Sobolev type equations and degenerate semigroups of operators*, vol. 42. Walter de Gruyter, 2012.

[10] B. Jacob and K. Morris, "On solvability of dissipative partial differential-algebraic equations," *IEEE Control Systems Letters*, vol. 6, pp. 3188–3193, 2022.

[11] A. Baccoli, A. Pisano, and Y. Orlov, "Boundary control of coupled reaction–diffusion processes with constant parameters," *Automatica*, vol. 54, pp. 80–90, 2015.

[12] R. Vazquez and M. Krstic, "Boundary control of coupled reaction-advection-diffusion systems with spatially-varying coefficients," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 2026–2033, 2016.

[13] S. Koga, M. Diagne, S. Tang, and M. Krstic, "Backstepping control of the one-phase stefan problem," in *2016 American Control Conference (ACC)*, pp. 2548–2553, IEEE, 2016.

[14] A. Hasan, O. M. Aamo, and M. Krstic, "Boundary observer design for hyperbolic pde–ode cascade systems," *Automatica*, vol. 68, pp. 75–86, 2016.

[15] H. Parada, E. Cerpa, and K. A. Morris, "Feedback control of an unstable parabolic-elliptic system with input delay," *preprint*, 2020.

[16] M. Krstic and A. Smyshlyaev, *Boundary control of PDEs: A course on backstepping designs*. SIAM, 2008.

[17] R. Curtain and H. Zwart, *Introduction to infinite-dimensional systems theory: a state-space approach*, vol. 71. Springer Nature, 2020.

[18] A. Alalabi and K. Morris, "Boundary control and observer design via backstepping for a coupled parabolic-elliptic system," *arXiv preprint arXiv:2309.00093*, 2023.