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A forwarding-based approach for the stabilization of linear systems in the presence of delayed nonlinear actuators

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Abstract— **In this paper, we revisit the forwarding approach for the input-to-states stabilization of linear systems subject to external perturbations, input nonlinearities (e.g. saturation and backlash functions), and different input delays. For this problem, we propose a Lyapunov functional analysis in the original coordinates certifying input-to-states stability of the origin with any desired type of convergence (asymptotic, finite-time, fixed-time, etc). At the end, we present some numerical simulations to show the effectiveness of our method and to show that in the linear case we recover the well-known backstepping methodology for delay compensation.**

I. Introduction

In control engineering, processes modeled by linear systems may include actuation delays in their formulation. This type of delays is critical and must be taken into account in the control design, as it may degrade the performance or induce instabilities of the closed-loop system [2], [6]. Due to the infinite-dimensional nature of linear systems with input delays, control design still continue to be challenging. One of the most effective methods to deal with input delays is the predictor feedback technique. This technique was first introduced, in [25], to overcome the dead time delay in open-loop stable systems. Later on, it was extended to handle stabilization problem for general unstable LTI systems with input-delay, introducing novel methodologies such as *finite spectrum assignment* [12], [15], [18], [27] and *model reduction* [1], [20], [24], [31]. These methods were generalized to more general classes of finite-dimensional systems subject to input delay with the *PDE backstepping* technique [3], [4], [10]. This technique is distinct because it accounts for the infinite dimensionality of the input and may allow possible extensions to more complex infinite dimensional systems with different types of input delays (e.g. distributed, stochastic, state-dependent, input-dependent). The idea of the PDE backstepping is to use an invertible Volterra like transformation, coupled with a state change of variables, if needed, to transform the unstable PDE into an easy-to-analyze system, called the target system, chosen to satisfy the desired stability property. While the backstepping technique offers significant advantages, it involves the use of some functions called *kernels*. For ODE-PDE cascade systems, the backstepping transformation change and the kernels become state dependent (predictor state) and they satisfy some PDEs that change when changing the studied cascade system and might be difficult in some cases to compute numerically. Moreover, when dealing with more demanding stability properties that requires the convergence towards the equilibrium to be achieved in a finite time, the backstepping transformation needs to become nonlinear [29] or time-varying [21], [28] even in the case of linear time-invariant (LTI) systems.

In this paper, we address the problem of achieving input-to-states stabilization, with any desired type of convergence (asymptotic, finite-time, fixed-time, or prescribed-time), of LTI systems subject to multiple input delays, input nonlinearities, and external disturbances. To solve this problem, we propose a novel delay compensation approach for LTI systems with input delays, inspired by the forwarding technique [16], [17], [22], rather than the existing backstepping-based techniques given in [10], [28], [29]. To our knowledge this approach has never been applied to time delay systems except in [9]. It is important to note that our approach significantly differs from the one presented therein. The main idea of our approach is to use a linear Sylvester-based change of coordinates on the ODE-PDE cascade representation of the studied delay system. A similar approach was also analyzed in the context of linear systems in [22]. Unlike the existing backstepping-based transformations - which is always applied on the transport PDE part of the ODE-PDE system and change when dealing with a different type of stability - our transformation is only applied on the ODE part and is independent of the desired closed-loop stability properties. This distinction allows us to ensure different types of stability (which depends only on the properties of the ODE) without changing the transformation which is instead fixed *a priori*. Moreover, the resulting target system given in our approach features the nonlinear non-delayed control input as the only coupling term of the cascade system. This property, alongside the **ISS** properties of transport PDEs w.r.t. to their boundary inputs, allows us to design a robust stabilizing controller that can be adapted to ensure any desired type of stability for the ODE-PDE cascade system by just stabilizing the ODE part with a non-delayed controller based on the existing results. Finally, by inverse transformation, the stability property and the desired convergence is transferred back to the original closed-loop system.

The rest of this article is organized as follows. In Section II, we present some relevant preliminary definitions on **ISS** notions. In Section III, we introduce the class of delay systems that we are interested in. In Section IV, we revisit the forwarding technique for stabilization of ODE-PDE cascade systems. In particular, we propose a general expression for

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the control design. In Section V, we give some application of our control design. In Section VI, we consider a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section VII.

II. Preliminaries

A. Notations

$\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. $\|\cdot\|$ denotes the Euclidean vector norm of $\mathbb{R}^n$. $L^2(0, D)$ denotes the set $\{f : [0, D] \to \mathbb{R}^n : \int_0^D \|f(x)\|^2 \mathrm{d}x < \infty\}$ and $\|f\|_{L^2} := (\int_0^D \|f(x)\|^2 \mathrm{d}x)^{\frac{1}{2}}$ its associated norm. $f_t(t, x)$ (resp. $f_x(t, x)$) denotes the partial derivative of a function f w.r.t. the variable t (resp. x). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be zero-at-zero if $f(0) = 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class-$\mathcal{K}$ function if it is continuous, zero-at-zero, and strictly increasing. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class-$\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$, and $\beta(r, \cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(r, t) = 0$ for each fixed $r \in \mathbb{R}_+$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *generalized class-$\mathcal{KL}$ function* ($\mathcal{GKL}-$function) if $r \mapsto \beta(r, 0)$ is a class-$\mathcal{K}$ function and for each fixed $r \geq 0$ $t \mapsto \beta(r, t)$ is continuous, decreases to zero and there exists some $T(r) \in [0, +\infty)$ such that $\beta(r, t) = 0$ for all $t \geq T(r)$.

B. Highlights on input-to-state stability (ISS) definitions

In this section, we present some relevant preliminary definitions on stability notions of finite-dimensional systems. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{0\}$, and $\mathcal{D} \subset \mathbb{R}$ be two open connected sets containing the origin. Consider the following finite-dimensional system:

$$\dot{z}(t) = f(z(t), d(t)), \tag{1}$$

where $f : \Omega \times \mathcal{D} \to \mathbb{R}^n$ is a continuous function such that $f(0, 0) = 0$. Assume that f is such that (1) has the property of existence and uniqueness of solutions in forward time outside the origin. Then, the following definition holds:

Definition 1: The origin of (1) is said to be

- **input-to-states stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the following holds:
$$\|z(t)\| \leq \max\left\{\beta\left(\|z_0\|, t\right), \gamma(\|d\|_\infty)\right\}, \tag{2}$$
for all $t \geq 0$ and any $z_0 \in \Omega$ and $d(t) \in \mathcal{D}$ for all $t \geq 0$;
- **finite-time ISS** (**FT-ISS**) if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that the following holds:
$$\|z(t)\| \leq \max\left\{\beta\left(\|z_0\|, t\right), \gamma(\|d\|_\infty)\right\}, \tag{3}$$
for all $t \geq 0$ and any $z_0 \in \Omega$ and $d(t) \in \mathcal{D}$ for all $t \geq 0$;
- **fixed-time ISS (FxT-ISS)** if it is **FT-ISS** and moreover $\sup_{r\in\Omega} T(r) < +\infty$ (where $T(r)$ is the finite settling-time associated to the class-$\mathcal{GKL}$ function β).

We remark that if $d \equiv 0$ for all times, the previous definitions of **ISS** imply asymptotic stability, finite-time stability, and fixed-time stability, respectively. We refer for instance to [8, Chapter 10], [26] for more details on **ISS** and to [7], [14] for more details on **FT-ISS** and **FxT-ISS**.

Remark 1: Notice that Definition 1 can be extended to infinite-dimensional systems by replacing the Euclidean norm by a suitable infinite-dimensional norm (in our case, it will be the L^2 norm).

III. Problem statement

The goal of this paper is to propose a new methodology to study the stabilization of LTI systems in the presence of a nonlinear input subject to both distributed and pointwise input delays and to external disturbances. This class of systems can be represented as an ODE-PDE cascade system of the form

$$\dot{X}(t) = AX(t) + \sum_{k=0}^{l\text{-}1} B_k u(t, x_k) + \int_0^D G(x) u(t, x) \mathrm{d}x, \tag{4a}$$

$$u_t(t, x) = u_x(t, x), \tag{4b}$$

$$u(t, D) = \varphi(U(t)) + d(t), \tag{4c}$$

where $t \geq 0$ and $x \in [0, D]$ are the time and space variables, $X(t) = [X_1(t), \ldots, X_n(t)]^\top \in \mathbb{R}^n$ ($n \in \mathbb{N}\backslash\{0\}$) and $u(t, x) \in \mathbb{R}$ are the states, $U(t) \in \mathbb{R}$ is the control input, the constant $D > 0$ is the upper-bound of the distributed delay as well as the biggest pointwise input delay, $(x_k)_{k\in\{0,\ \ldots,\ l\}}$, $l \in \mathbb{N}\backslash\{0\}$ is a finite increasing sequence of known real delays in $[0, D]$ satisfying $x_0 := 0$ and $x_l := D$ (i.e. $x_0 = 0 < x_1 < \ldots < x_l = D$), $d(t)$ is a bounded disturbance, and φ is a zero-at-zero nonlinear bounded function (e.g. saturation function, backlash function). A, B_0, ..., B_{l-1}, and $G(x)$ are the system and input matrices of appropriate dimensions. The input vector $G(\cdot)$ is a (piece-wise) continuous real-valued vector function defined in $L^2(0, D)$. The initial condition (z_0, u_0) is taken in $\mathbb{R}^n \times L^2(0, D)$ (u_0 assumed to be bounded).

Remark 2: Notice that, by method of characteristics, the transport PDE state u is given by:

$$u(t, x) = \begin{cases} u_0(t + x), & t + x \leq D, \\ \varphi(U(t + x - D)) + d(t + x - D), & t + x \geq D. \end{cases} \tag{5}$$

Moreover, the transport PDE (4b)-(4c) is **FxT-ISS** w.r.t. both the inputs $\varphi(U(t))$ and $d(t)$ (see proof of Proposition 1) in the following sense: there exist $\beta_u \in \mathcal{GKL}$ and $\gamma_u \in \mathcal{K}$ such that:

$$\|u(t, \cdot)\|_{L^2} \leq \max\{\beta_u\left(\|u_0\|_{L^2}, t\right), \gamma_u(|\varphi(U(t))|), \gamma_u(\|d\|_\infty)\}, \tag{6}$$

for all $t \geq 0$, where $\beta_u\left(\|u_0\|_{L^2}, t\right) = 0$, for all $t \geq D$. In particular, in the absence of the inputs $\varphi(U(t))$ and $d(t)$, the origin of (4b)-(4c) is fixed-time stable where we replace the Euclidean norm by the L^2 norm (see [19] for more details on the **ISS** property for infinite-dimensional systems).

IV. A Forwarding-based approach for delay compensation

In this section, we revisit the forwarding technique (or Sylvester-based change of coordinates) for stabilization of ODE-PDE cascade systems of the form (4), see e.g. [16],

[17], [22]. To that end, let us consider the following forwarding-based invertible change of coordinates:

$$\xi(X,u)(t) := X(t) - \sum_{k=0}^{l\text{-}1} \int_{x_k}^{x_{k+1}} M_k(x)u(t,x)\mathrm{d}x, \quad (7)$$

where M_k, $\forall k \in \{0,\ldots,l-1\}$, are given respectively by

$$M_0(x) = \text{-}e^{\text{-}Ax}B_0 + \int_{x_0}^{x} e^{\text{-}A(x\text{-}s)}G(s)\mathrm{d}x,\ \forall x\in[x_0,x_1], \quad (8)$$

and

$$M_k(x) = \text{-}e^{\text{-}A(x-x_k)}(B_k \text{ - } M_{k\text{-}1}(x_k)) + \int_{x_k}^{x} e^{\text{-}A(x\text{-}s)}G(s)\mathrm{d}x,$$
$$\forall x \in [x_k, x_{k+1}],\ \forall k \in \{1,\ldots,l\text{ - }1\}. \quad (9)$$

For simplicity and with some abuse of notation, we will use the notation ξ instead of $\xi(X,u)$ in the rest of the paper.

Consequently, (4) is equivalent to

$$\dot{\xi}(t) = A\xi(t) \text{-} M_{l\text{-}1}(D)\big[\varphi(U(t)) + d(t)\big], \quad (10a)$$
$$u_t(t,x) = u_x(t,x), \quad (10b)$$
$$u(t,D) = \varphi(U(t)) + d(t). \quad (10c)$$

in terms of trajectories, stabilizability, and well-posedness as we explain next.

A. Analysis of the target systems

First, we show that the norm of the two systems:

$$\mathcal{N}(X,u) := \|X\|^2 + \|u\|^2_{L^2} \text{ and } \mathcal{N}(\xi,u) := \|\xi\|^2 + \|u\|^2_{L^2}$$

for any $X \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $u \in L^2(0,D)$, are equivalent in the following sense:

Lemma 1: There exists a positive constant $c > 0$ such that the following equivalence estimate holds:

$$\tfrac{1}{c}\mathcal{N}(\xi,u) \le \mathcal{N}(X,u) \le c\mathcal{N}(\xi,u). \quad (11)$$

Based on the previous result, we have the following "trivial" statement:

Proposition 1: System (4) *is well-posed if and only if* (10) *is well-posed.*

Next, we study the relations between the controllability properties of the target system (10) and the system in the original coordinates (4). In particular, we can show the following statement:

Proposition 2: System (4) *is stabilizable if and only if the pair* $(A, -M_{l-1}(D))$ *in* (10) *is stabilizable.*

Proof: The proof follows the same arguments used in the proof of [22, Theorem 3.4] for ODE-PDE cascade systems. The PDE part (4b)-(4c) (resp. (10b)-(10c)) being unchanged preserves its properties and the Hautus condition brought back to the ODE part (4a) (resp. (10a)) give the same stabilizability result by the linear change of coordinates (7). We also remark that the stabilizability test of the pair $(A, -M_{l-1}(D))$ corresponds to the condition proposed in [5, Theorem 5.2.12]. ■

Thanks to the equivalence between systems (4) and (10), proving the well-posedness and stability properties of either systems implies proving the same properties for the other system. We choose to do this on the later one (i.e. (10)) since unlike in (4), equations (10a) and (10b)-(10c) are only coupled through the input $U(t)$. This feature - alongside the properties of (10b)-(10c) in Remark 2 - makes the well-posedness and stabilization studies much easier. In fact, taking $U(t)$ equal to $\mathcal{F}(\xi(t))$ for some zero-at-zero function $\mathcal{F}(\cdot)$ guaranteeing the well-posedness and the **ISS** properties of the closed-loop system (10a), ensures the well-posedness and the **ISS** properties for the closed-loop cascade system (10) as well.

Remark 3: It is worth stressing that in view of Proposition 2, we can verify the stabilizability properties of the original system (4) by directly following the procedure at the beginning of the section, namely by computing the function M_k as in (8), (9), which are given in explicit closed form, and then directly check the stabilizability of the pair $(A, -M_{l-1}(D))$. This can be of particular interest in the presence of the distributed term G in (4a).

B. On the selection of the feedback controller $U(t)$

Notice that (10a) is a finite-dimensional system with a non-delayed nonlinear control input. This class of systems has been exhaustively studied in the literature. This means we have a variety of choices for the control input $U(t)$ depending on the nonlinear function φ and the desired convergence properties, e.g. asymptotic, finite-time, fixed-time, or prescribed-time. In this paper, we want to provide a general study taking into account all the possible choices of $U(t)$, that is why we assume the following:

Assumption 1: For a given continuous nonlinear zero-at-zero function $\varphi(\cdot)$, there exists a continuous zero-at-zero function $\mathcal{F}(\cdot)$ such that the origin of (10a) *with the feedback controller $U(t) = \mathcal{F}(\xi(t))$ is either **ISS**, **FT-ISS**, or **FxT-ISS** w.r.t. to external disturbances $d(t)$ in the sense of Definition 1. In particular, in the absence of d, there exists $T(\xi(0)) \in [0,+\infty]$ such that $\|\xi(t)\| = 0$ for all $t \ge T(\xi(0))$.*

When Assumption 1 is satisfied, the control input can be expressed, using (7), in terms of the original coordinates (X,u) as follows:

$$U(t) := \mathcal{F}\left(X(t) - \sum_{k=0}^{l\text{-}1}\int_{x_k}^{x_{k+1}} M_k(x)u(t,x)\mathrm{d}x\right). \quad (12)$$

Moreover, by continuity of $\varphi \circ \mathcal{F}$, the nonlinear input $\varphi(U(t))$ is similarly either **ISS**, **FT-ISS**, or **FxT-ISS** w.r.t. to external disturbances $d(t)$ in the sense of Definition 1. In particular, in the absence of d, $|\varphi(U(t))| = 0$ for all $t \ge T(\xi(0))$ where $T(\xi(0)) \in [0,+\infty]$ is given in Assumption 1.

C. Stability analysis

In this subsection, we provide a stability analysis for both (4) and (10). We start first by studying (10), then using Lemma 1 on norm equivalence, we extend the stability analysis to the original system (4).

Proposition 3: Under Assumption 1, the origin of the transport PDE (10b)-(10c) *is fixed-time **ISS** (**FxT-ISS**) w.r.t* *the two inputs $\varphi(U(t))$ and $d(t)$ in the sense of (6) in Remark 2. In particular, in the absence of the two inputs $\varphi(U(t))$ and $d(t)$, $\|u(t,\cdot)\|_{L^2} = 0$, for all $t \geq D$.*

Proof: Consider the following Lyapunov functional:

$$\mathcal{V}(t) = \int_0^D e^{\sigma x} u(t,x)^2 \mathrm{d}x,\ \sigma > 0. \tag{13}$$

In this case, computing the time derivative of $\mathcal{V}(t)$ along the trajectories of (10b)-(10c) and integrating by parts gives us

$$\begin{aligned}\dot{\mathcal{V}}(t) &= \int_0^D e^{\sigma x} \frac{\partial u(t,x)^2}{\partial t} \mathrm{d}x = \int_0^D e^{\sigma x} \frac{\partial u(t,x)^2}{\partial x} \mathrm{d}x,\\ &= -\sigma \int_0^D e^{\sigma x} u(t,x)^2 \mathrm{d}x + \left[e^{\sigma x} u(t,x)^2\right]_0^1,\\ &\leq -\sigma \mathcal{V}(t) + e^{\sigma D}[\varphi(U(t)) + d(t)]^2,\\ &\leq -\sigma \mathcal{V}(t) + 2e^{\sigma D}\varphi(U(t))^2 + 2e^{\sigma D} d(t)^2.\end{aligned}$$

From this last inequality, we can conclude that $\mathcal{V}(t)$ is an **ISS** Lyapunov functional for infinite-dimensional systems (see [19] for more details) and consequently the origin of (10b)-(10c) is **ISS** w.r.t to the two inputs $\varphi(U(t))$ and $d(t)$. Moreover, in the absence of the two inputs $\varphi(U(t))$ and $d(t)$, we can prove, from the transport PDE solution (5), that $u(t,x) = 0$, $\forall x \in [0,D]$, $\forall t \geq D$ and consequently $\mathcal{V}(t) = \|u(t,\cdot)\|_{L^2} = 0$ for all $t \geq D$. This means that the origin of (10b)-(10c) is not only **ISS** but it is **FxT-ISS** w.r.t to the two inputs $\varphi(U(t))$ and $d(t)$ in the sense of (6). ∎

Remark 4: Note that, in the absence of $d(t)$, the convergence of $\|u(t,\cdot)\|_{L^2}$ towards the origin is controlled by $\|\varphi \circ \mathcal{F}(\xi(t))\|$ which vanishes after $t = T(\xi(0)) \in [0,+\infty]$ and $\beta_u(\|u_0\|_{L^2}, t)$ which vanishes after $t = D$. This implies that $\|u(t,\cdot)\|_{L^2} = 0$, for all $t \geq T(\xi(0)) + D$.

In view of Assumption 1 and the above proposition, we have the following results which is a straightforward consequence of the cascade structure of system (10).

*Corollary 1: Under Assumption 1, the origin of the ODE-PDE cascade system (10) is either **ISS**, **FT-ISS**, or **FxT-ISS** w.r.t to $d(t)$ in the sense of Definition 1 and Proposition 3. In particular, $\mathcal{N}(\xi,u)(t) = 0$, $\forall t \geq T(\xi(0)) + D$, where $T(\xi(0)) \in [0,+\infty]$ is given in Assumption 1.*

Proof: The proof of this Corollary is a consequence of the **ISS** property of cascades of **ISS** systems combined with Assumption 1 and Proposition 3. ∎

*Theorem 1: Under Assumption 1, the origin of the ODE-PDE cascade system (4) is either **ISS**, **FT-ISS** or **FxT-ISS** in the sense of Definition 1 and Proposition 3. In particular, $\mathcal{N}(X,u)(t) = 0$, $\forall t \geq D + T(\xi(0)) \in [0,+\infty]$, where $T(\xi(0))$ is given in Assumption 1.*

Proof: The proof of this result is a direct application of Lemma 1 on norms equivalence and Corollary 1. ∎

V. Some characterizations of the feedback law

In this section, we provide some characterizations of the feedback law function $\mathcal{F}(\cdot)$ given in Assumption 1.

A. An Asymptotic **ISS** controller for cone-bounded nonlinearities

Let us start by considering a linear feedback law $\mathcal{F}(\cdot)$ achieving **ISS** of the closed-loop system (10a) when φ is a cone-bounded nonlinearity, i.e.,

- when $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous zero-at-zero function such that: $\exists L > 0 : |\varphi(s)| \leq L|s|, \forall s \in \mathbb{R}$, and $(\varphi(s) - \varphi(z))(s-z) \geq 0$, $\forall (s,z) \in \mathbb{R}^2$.

For this case, we state the following assumption:

Assumption 2: There exists a positive definite symmetric matrix P satisfying

$$PA + A^\top P - 2PM_{l\text{-}1}(D)M_{l\text{-}1}(D)^\top P \prec 0, \tag{14}$$

$$PA + A^\top P \preceq 0. \tag{15}$$

Assumption 2 implies that the pair $(A, M_{l\text{-}1}(D))$ is stabilizable and that the open-loop ODE is not unstable, namely all the eigenvalues lie in the closed left-half complex plane and eigenvalues on the imaginary axes are simple. In turns, this is a necessary assumption (for $n > 2$) for the existence of a linear feedback guaranteeing global stabilization in the presence of nonlinearities φ such as saturation functions.

Based on the previous assumption, we define next the following feedback law:

$$\mathcal{F}(\xi) = \kappa M_{l\text{-}1}(D)^\top P\xi. \tag{16}$$

Consequently, we have the following result:

*Proposition 4: Suppose that Assumption 2 holds. Then, for any $\kappa > 0$, the origin of system (10a) in closed-loop with (16) is globally asymptotically stable when $d \equiv 0$, and locally **ISS** w.r.t. $d \in \mathcal{D}$ when $d \not\equiv 0$.*

Proof: Let us start first by assuming that $d \equiv 0$ and let us consider the Lyapunov function $W = \xi^\top P\xi$, whose time derivative along the solutions of (10a) satisfies

$$\begin{aligned}\dot{\mathcal{W}} &= \xi^\top\left[PA + A^\top P\right]\xi - 2\xi^\top PM_{l\text{-}1}(D)\varphi(U),\\ &\leq -\tfrac{2}{\kappa}\mathcal{F}(\xi)\varphi\big(\mathcal{F}(\xi)\big).\end{aligned}$$

From this last inequality, we conclude using the stabilizability properties in (14), and LaSalle's theorem that (10a) is globally **AS**. Standard **ISS** results allow to conclude local **ISS** properties in the presence of d, see, e.g. [26]. ∎

B. A non-asymptotic **ISS** controller in the absence of nonlinearities

For (10a), we can use homogeneity-based results, from [30] for instance, to characterize a new $\mathcal{F}(\cdot)$ from which we can subsequently design a nonlinear controller $U(t)$ achieving **FT-ISS** when $\varphi = \mathrm{Id}$. For this case, we state the following assumptions:

Assumption 3: There exist $L \in \mathbb{R}^{n\times n}$, $y_0 \in \mathbb{R}^{1\times n}$, and $\gamma \in \mathbb{R}$ satisfying

$$AL - LA - A - M_{l\text{-}1}(D)y_0 = 0, \tag{17}$$

$$(L - \gamma I_n)\, M_{l\text{-}1}(D) = 0, \tag{18}$$

$$L - (\gamma+1)I_n \prec 0. \tag{19}$$

Assumption 4: Under Assumption 3, there exist $X \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^{1\times n}$, $\delta \in \mathbb{R}_+$, *and* $\eta \in \mathbb{R}_+$ *satisfying*

$$(A - M_{l\text{-}1}(D)K_0)\,X + X(A - M_{l\text{-}1}(D)K_0)^\top \\ - M_{l\text{-}1}(D)y - y^\top M_{l\text{-}1}(D)^\top + \delta X \preceq 0, \quad (20)$$

$$X \succ 0,\ \eta X \succeq \nu L X + \nu X L^\top + 2\varepsilon X \succ 0, \quad (21)$$

for $K_0 = y_0\,[L - (\gamma+1)I_n]^{-1}$.

Roughly speaking Assumption 3 implies that the pair $(A, -M_{l\text{-}1}(D))$ is stabilizable by a gain K_0 and that $A - M_{l\text{-}1}(D)K_0$ is homogeneous of degree ν. These properties are required in order to ensure finite-time stabilizability of (10a) via a homogeneous controller. Assumption 4 on the other hand is used to guarantee that the homogeneous controller is capable of stabilizing $(A - M_{l\text{-}1}(D)K_0, -M_{l\text{-}1}(D))$ in finite time. For more technical details we refer to [30].

Based on the previous assumption, we select a controller of the form

$$\mathcal{F}(\xi) = K_0\xi + \|\xi\|_{\mathbf{d}}^{\nu(1+\gamma)+\varepsilon} K\mathbf{d}\left(-\ln\|\xi\|_{\mathbf{d}}\right)\xi, \quad (22)$$

where $\mathbf{d}$ is the dilation defined by $\mathbf{d}(s) = e^{G_{\mathbf{d}}s}$, $\forall s \in \mathbb{R}$ with $G_{\mathbf{d}} = \nu L + \varepsilon I_n \in \mathbb{R}^{n\times n}$, and $\|\cdot\|_{\mathbf{d}}$ is its associated homogeneous norm (see [23] for more details on the homogeneous norm). The gains $K_0 \in \mathbb{R}^{1\times n}$, $K \in \mathbb{R}^{1\times n}$, and $\varepsilon \in \mathbb{R}_+$ are selected as $K_0 = y_0\,[L - (\gamma+1)I_n]^{-1}$, $K = yX^{-1}$.

Consequently, we have the following result

Proposition 5: The origin of system (10a) *in closed-loop with* (22), *with* $d \equiv 0$, *is globally* ***FTS*** *(in the sense of Definition 1) for* $\nu < 0$ *and the settling time is given by*

$$T(\xi_0) \le \frac{b}{a(-\nu)}\|\xi_0\|_{\boldsymbol{d}}^{-\nu}.$$

Moreover, using the induced homogeneity of (10a), *we conclude that is globally* ***FT-ISS*** *in the sense of Definition 1 for* $\nu < 0$.

Proof: see [30] for details. ■

VI. SIMULATION

In this section, we focus on (4) for $n = 3$ and $l = 1$, i.e.

$$\dot{X}(t) = AX(t) + B_0 u(t,0) + \int_0^D G(x)u(t,x)\mathrm{d}x, \quad (23a)$$

$$u_t(t,x) = u_x(t,x), \quad (23b)$$

$$u(t,D) = \varphi(U(t)), \quad (23c)$$

$$U(t) = \kappa M_0(D)^\top P\left[X(t) - \int_0^D M_0(x)u(t,x)\mathrm{d}x\right], \quad (23d)$$

where A, B_0, $G(\cdot)$, $\varphi(\cdot)$ are chosen respectively as

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ G(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}, \quad (24)$$

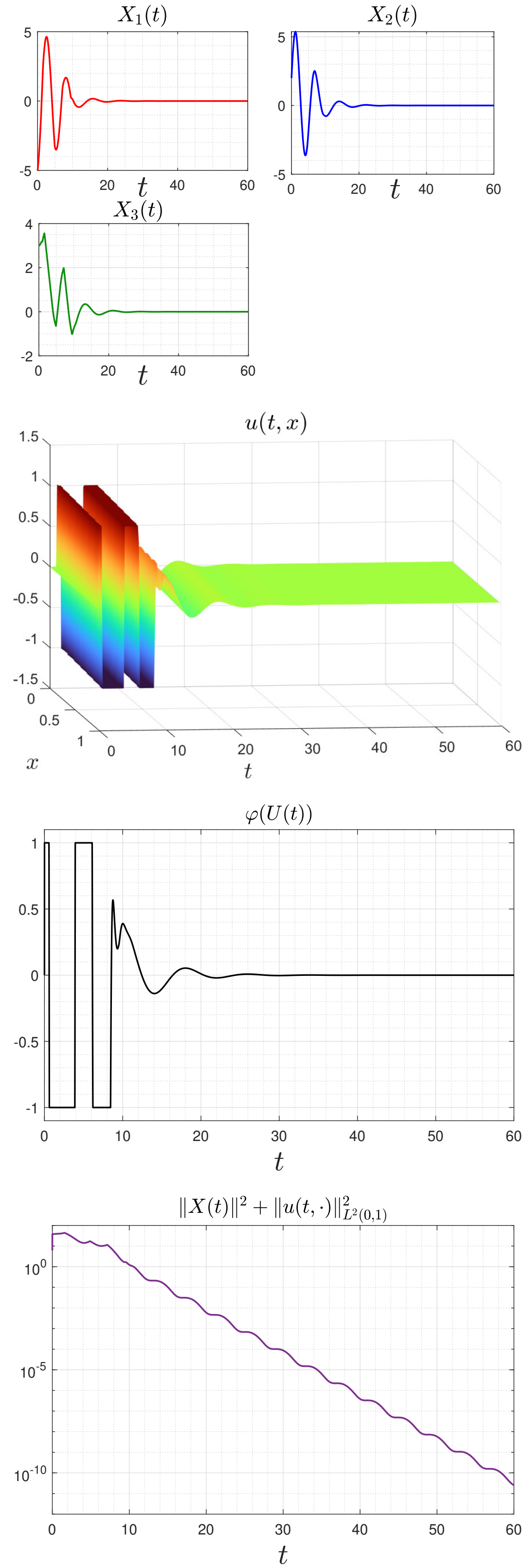

Fig. 1. On the top: the evolution of the states of (23a)-(23c) with $X_0 = (-5, 2, 3)$, $u_0 = 0$ and $D = 1$s.On the bottom left: the evolution of the used saturated controller $\varphi(U(t))$ given in (23d). On the bottom right: the evolution of norm of the closed-loop system (23a)-(23c).

$$\varphi(s) = \mathrm{Sat}_\Delta(s) := \begin{cases} -\Delta, & s \leq -\Delta, \\ s, & s \in [-\Delta, \Delta], \\ \Delta, & s \geq \Delta, \end{cases} \quad (25)$$

and $M_0(\cdot)$ is given in this scenario by

$$M_0(x) = -e^{-Ax}B_0 + \int_0^x e^{-A(x-s)}G(s)ds, \ \forall x \in [0, D]. \quad (26)$$

For the rest of the parameters, we take the saturation level $\Delta = 1$, the delay $D = 1$, the control gain $\kappa = 20$. The matrix P is computed as follows:

$$P = \begin{bmatrix} 4.380 & -1.971 & -0.538 \\ -1.971 & 5.015 & -5.891 \\ -0.538 & -5.891 & 10.336 \end{bmatrix}, \quad (27)$$

by solving (14) and (15) for $Q = \mathrm{Diag}([3, 4, 5])$, and $M_0(D)$ is computed as

$$M_0(D)^\top = \begin{bmatrix} -0.540 & -0.841 & -0.542 \end{bmatrix}. \quad (28)$$

For the numerical simulations, we approximate the closed-loop system (23a)-(23c) using the Lax-Wendroff scheme that can be set in Shampine's solver for Matlab as presented in [13] for hyperbolic PDEs ((23a) is treated as a hyperbolic PDE with a coefficient of convection equal to 0). The spatial and temporal discretization were done with steps $\Delta x = 0.002$ and $\Delta t = 0.006$. Notice that, the Courant-Friedrich-Levy (CFL) condition for the numerical stability holds. All the integral terms present in (23a)-(23d) are approximated using the trapezoidal rule. Figure 1 shows on the top the evolution of the states of (23a)-(23c) with $X_0 = (-5, 2, 3)$, $u_0 = 0$ and $D = 1$s. On the bottom left, it shows the evolution of the used saturated controller $\varphi(U(t))$ given in (23d). On the bottom right, it shows the evolution of norm of the closed-loop system (23a)-(23c).

VII. Conclusion

This paper dealt with the problem of input-to-states stabilization, with any desired type of convergence (asymptotic, finite-time, fixed-time, etc), of linear systems subject to external perturbations, input nonlinearities, and different input delays using a forwarding-based approach. Future works will extend this work to more general classes of interconnected systems namely cascades of different PDEs to handle different types of actuator models, see, e.g. [11], [22] and references therein. Extensions will also aim to design numerically safe approximations of the proposed control design, based on Legendre and Fourier polynomials.

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