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Distributed Prescribed-Time and Adaptive Synchronization of Complex Dynamical Networks under Directed Topologies

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***Abstract*— This paper addresses an adaptively distributed prescribed-time synchronization problem of complex dynamical networks (CDNs) via distributed pinning control schemes using neighboring information over the digraph. The novel distributed prescribed-time synchronization pinning control algorithms with static and dynamic coupling laws are developed to achieve global synchronization in the specified time, where each node can adjust its strategy on its procurable synchronization error. Based on the time transformation method and Lyapunov analysis theory, it is proved that global synchronization can be guaranteed in a pre-defined time and moreover, this synchronization can be preserved after the time, and further the control inputs are kept uniformly bounded. Lastly, the numerical simulation results are further presented to illustrate the effectiveness of the developed synchronization control methods.**

I. Introduction

Distributed synchronization control for complex networks has become a hot research topic over the past few decades, which has exhibited significant benefits from practical applications, e.g., the Internet networks, epidemic spreading networks, social networks, power systems, wireless sensor networks, mobile multi-robot systems (e.g., see [1], [2]). The pinning control strategies have been proven to be powerful and effective techniques that solve synchronization problems of CDNs, where a subset of nodes is pinned to achieve global synchronization, e.g., [3]–[7]. The synchronization in these works was achieved in the infinite time horizon in terms of either an asymptotic convergence [6], or a uniformly ultimately bounded convergence [4], [7], or an exponential convergence [5]. To enhance a convergence rate of synchronization, the study of finite-/fixed-time synchronization has been investigated in [8]–[14] with conservative settling-time estimations. The signum function plays a key role in enabling finite-time synchronization (see [8], [9]) with settling-time estimations that rely on the initial conditions of each node, leading to unfixed settling time associated with various initial conditions. The signum function based design is nonsmooth and indispensable for controllers. In order to remove above limitations, a fixed-time synchronization was studied in [10]. Fixed-time group synchronization was further considered in [11] over undirected graphs, where the distributed controller is nonsmooth with the conservative settling-time estimation. Moreover, this design was extended in [12] to obtain robust fixed-time synchronization with stochastic disturbances. The fractional power based feedback was adopted in [13] to have fixed-time synchronization with the semi-global convergence that relies on each node's initial conditions.

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As observed, one design limitation in [8]–[14] to achieve finite-/fixed-time synchronization is that settling time is estimated with a conservative upper bound that relies on design parameters and communication structures. Then, prescribed-/appointed-time synchronization is introduced in [15]–[18], where the settling time is user-defined according to task requirements. This specified-time synchronization or consensus is difficult to achieve in theory, yet preferable for practical applications. A prescribed-time consensus was given in [15] over undirected graphs via a state scaling parameter design. It was extended in [16] to yield an appointed-time consensus via a two-hop communication. The authors in [17] addressed appointed-time synchronization problems via a motion planning method using neighboring information over a directed yet balanced graph. The prescribed-time synchronization was considered in [18] via the discontinuous controller such that the tracking error was uniformly bounded.

Another shortcoming in [8]–[18] is that explicit quantification of control gains needs to rely on information on the communication structure, the Lipschitz constant of nonlinear functions, and the number of all nodes. To remove global information, the idea is to adjust control strengths adaptively. The most related works on adaptive synchronization were reported in [19] and [20]. The strategies in [20] resulted in the synchronization to a small bound by adaptively updating coupling strengths to each node on its overall procurable synchronization error. This bound relies on the initial conditions, the Lipschitz constant of nonlinear functions as well as the network topologies. More importantly, the adaptive pinning synchronization design in [20] over directed graphs requires to construct a graph-dependent Lyapunov function with the asymmetric information exchange matrix, which may not be reasonable. In addition, the developed algorithm heavily depends on the explicit solution to linear matrix inequalities, whose existence may not be easily ensured.

In light of above discussions, this paper addresses the distributed prescribed-time and adaptive synchronization problems of CDNs under the directed topology, where the settling time is pre-specified by task requirements and the controller does not require this known global information. The feasible and easy-to-implement distributed pinning control schemes are proposed to guarantee synchronization in a pre-specified convergence time by using the time transformation technique that transforms the prescribed-time synchronization into an infinite-time issue. Further, the Laypunov analysis method is adopted to obtain the convergence of systems. The zero-error convergence is thus achieved, where each node can adjust its strategy on its procurable synchronization error dynamically.

II. Problem Formulation

A. Graph Theory

Define $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ as a directed graph, where $\mathcal{V} \in \{1, 2, \cdots, N\}$ denotes a set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, respectively, and the neighboring set of node i is given by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. A directed graph $\mathcal{G}$ is said to have a spanning tree, if there exists a directed path from the root node to every other nodes, and this root node has no incoming edges. Let $A = [a_{ij}]$ be the adjacency matrix of a graph $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. Let $\mathcal{L}$ be the Laplacian matrix of $\mathcal{G}$.

For a leader-follower network, we label the leader node as 0 and then, $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ can be used to define an augmented graph, where $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$ with $\bar{\mathcal{V}} = \{0\} \cup \{\mathcal{V}\}$. Further, we denote $\bar{\mathcal{L}} = [0_{1\times1}, 0_{1\times N}; -\mathcal{A}_0 1_N, H]$ as the Laplacian matrix of $\bar{\mathcal{G}}$, where $\mathcal{A}_0 = \mathrm{diag}\{a_{i0}\}$ is a diagonal matrix with $a_{i0} > 0$ if $(0, i) \in \bar{\mathcal{E}}$ and $a_{i0} = 0$, otherwise, and the matrix $H = \mathcal{L} + \mathcal{A}_0$ is allowed to be asymmetric for a digraph.

B. Network Model of CDNs

Consider CDNs consisting of a team of nonlinear subsystems, which is specified by (c.f., [9]–[14])

$$\dot{x}_i(t) = f(x_i(t), t) + u_i(t), \; x_i(0) = x_{i0}, \; i \in \mathcal{V}, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ denotes the node i's state vector, $f(x_i(t)) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function that describes the node i's intrinsic dynamics, and $u_i(t)$ is its control input.

It aims to synchronize states of (1) to a desired state $z_0(t) \in \mathbb{R}^n$ in a pre-defined time with $x_i(t) = x_j(t) = z_0(t)$, and the dynamics of $z_0(t)$ is described by (c.f., [9]–[14])

$$\dot{z}_0(t) = f(z_0(t), t), \quad (2)$$

where $z_0(t)$ is allowed to be an equilibrium point, a periodic orbit, a chaotic orbit, etc. Many existing distributed pinning controllers for synchronization are often designed as

$$u_i(t) = c \left[\sum\nolimits_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + a_{i0}(z_0(t) - x_i(t))\right],$$

where $j \in \mathcal{N}_i$ and $c > 0$ denotes the coupling strength that relies on some known global information as discussed above, to ensure asymptotic synchronization (c.f., [3], [5], [7]). In addition, as shown in [8]–[14], the developed pinning controllers that require known global information, ensure finite-/fixed-time synchronization with conservative estimation of settling time that relies on each node's initial states, control parameters and communication structures.

On the contrary, in this paper, the adaptively distributed prescribed-time synchronization problem is defined below.

Problem 1: Consider CDNs in (1)-(2). Under the digraph $\bar{\mathcal{G}}$, we aim to design a distributed pinning controller *(independent of known global information on the directed graph, Lipschitz constants of nonlinear functions, and the number of all nodes)* such that for any initial conditions, the prescribed-time synchronization is achieved in the sense that

$$\lim_{t\to T} ||x_i(t) - z_0(t)|| = 0 \text{ and } ||x_i(t) - z_0(t)|| \equiv 0 \text{ for } t \geq T,$$

where $T > 0$ *(independent of nodes' initial states, communication structures and design parameters)* is a pre-specified time defined by the user based on task requirements.

The standard assumptions are listed to solve this issue.

Assumption 1: The digraph $\bar{\mathcal{G}}$ has a directed spanning tree with the leading node as the root.

Assumption 2: For $\forall \xi, \vartheta \in \mathbb{R}^n$, there exists a scalar $l_f > 0$ to satisfy: $(\xi - \vartheta)^T (f(\xi, t) - f(\vartheta, t)) \leq l_f (\xi - \vartheta)^T (\xi - \vartheta)$.

Remark 1: Assumption 1 was widely employed in many existing works, which is supposed for a digraph. Assumption 2 is mild, e.g., all linear and piecewise linear functions satisfy this assumption. Further, if $\partial f_i / \partial x_k, i, k = 1, \cdots, n$ are bounded, the Lipschitz condition is satisfied.

Some lemmas are listed for convergence analysis later.

Lemma 1: [20] H is positive definite by Assumption 1, and there exists a matrix $\Omega = \mathrm{diag}\{\gamma_1, \cdots, \gamma_N\} > 0$ so that $\Xi = (\Omega H + H^T \Omega)/2 > 0$, where $\gamma = \mathrm{col}(\gamma_1, \cdots, \gamma_N) = (H^T)^{-1} 1_N$ is a vector and 1_N is a vector with all ones.

Lemma 2: [21] Consider a dynamical system $\dot{\zeta}(t) = g(\zeta(t), t)$ with $\zeta(0) = \zeta_0$, and let $\xi(t)$ be its solution. Then, there exist a time transformation function $t = \mu(s), s \in [0, \infty)$ and a scalar $T > 0$, with $\mu(s)$ satisfying conditions:

$$\mu(0) = 0, \; \mu'(0) = T \; -\text{(continuous differentiable)} \quad (3a)$$

$$s_1 > s_2 \geq 0 \Rightarrow \mu(s_1) > \mu(s_2) \; -\text{(strictly increasing)} \quad (3b)$$

$$\lim_{s\to\infty} \mu(s) = T, \; \lim_{s\to\infty} \mu'(s) = 0 \; -\text{(convergence in s)} \quad (3c)$$

so that we have that for $\psi(s) \triangleq \xi(t)$,

$$\psi'(s) = \mu'(s) g(\psi(s), \mu(s)), \; \psi(\mu^{-1}(0)) = \zeta_0, \quad (4)$$

where $\mu'(s) = d\mu(s)/ds$, $\psi'(s) = d\psi(s)/ds$ and $\lim_{s\to\infty} \psi(s) = \lim_{t\to T} \xi(t)$ for $\forall t \in [0, T)$ and $s \in [0, \infty)$ [21].

III. Main Results

In this work, the main goal is to develop the prescribed-time synchronization algorithm so that all nodes' states can reach synchronization and converge towards the state of the isolated node in a pre-defined time T, and this synchronization is kept maintained for $t \geq T$. To achieve this goal, the time transformation function that satisfies (3) is selected as

$$t = \mu(s) \triangleq T(1 - e^{-s}), \quad (5)$$

where $\mu(s)$ satisfies (3a) and (3b). Then, $t = \mu(s)$ converges towards T as $s \to \infty$ and $\mu'(s) = Te^{-s}$, which satisfies (3c). In light of (5), the general infinite-time interval $s \in [0, \infty)$ is transformed from the original time interval $t \in [0, T)$. As a consequence, the stability analysis is correspondingly studied over $s \in [0, \infty)$ using this time transformation technique.

A. Distributed Prescribed-Time Synchronization of CDNs over A Directed Communication Topology

Distributed prescribed-time synchronization design: we present a distributed prescribed-time synchronization pinning strategy such that the states of all nodes can not only reach synchronization, but also converge to the state of the isolated node in a pre-defined time T, and this synchronization is kept maintained for $t \geq T$. This design is developed as

$$\dot{x}_i(t) = f(x_i(t), t) + u_i(t), \; u_i(t) = (c + \varrho(t)) e_i(t), \quad (6a)$$

$$e_i(t) = \sum\nolimits_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + a_{i0}(z_0(t) - x_i(t)), \quad (6b)$$

where $c > 0$ is the static coupling strength, $\varrho(t)$ represents a time-varying scaling function: $\varrho(t) = \frac{m}{T-t}$, if $t \in [0, T)$ and 0, if $t \geq T$, $T > 0$ is the time scalar defined by the user and $m > 1$ is a scalar; $e_i(t)$ denotes an error collected from its neighboring information with a_{ij} as the (i, j)-th entry of A, and $a_{i0} > 0$ if $x_i(t)$ is pinned; $a_{i0} = 0$, otherwise.

To achieve synchronization behaviors, we denote the error dynamics as $\tilde{x}_i(t) = x_i(t) - z_0(t)$. Let $\tilde{f}(x_i(t), z_0(t), t) = f(x_i(t), t) - f(z_0(t), t)$. Then, combining (2) and (6) gives

$$\begin{aligned}\dot{\tilde{x}}_i(t) &= f(x_i(t), t) - f(z_0(t), t) + (c + \varrho(t))e_i(t) \qquad (7)\\ &= \tilde{f}(x_i(t), z_0(t), t) - c[\sum\nolimits_{j=1}^{N} a_{ij}(\tilde{x}_i(t) - \tilde{x}_j(t))\\ &+ a_{i0}\tilde{x}_i(t)] - \varrho(t)[\sum\nolimits_{j=1}^{N} a_{ij}(\tilde{x}_i(t) - \tilde{x}_j(t)) + a_{i0}\tilde{x}_i(t)].\end{aligned}$$

Next, the goal is to provide the stability analysis of the developed distributed pinning control scheme for the synchronization problem, including the prescribed-time convergence property and the boundedness of each node's control input.

As can be observed, the selection of (5) gives $d\mu(s)/ds = T - t$ in (6) associated with the stretched time interval $s \in [0, \infty)$. Let $\xi_i(t)$ define a solution to (7) and denote $\psi_i(s) \triangleq \xi_i(t)$. Then, we obtain from (5) and Lemma 2 that

$$\begin{aligned}\psi_i^{'}(s) &= d\psi_i(s)/ds = \mu^{'}(s)[\tilde{f}(\psi_i(s), \mu(s)) + u_i(\mu(s))]\\ &= Te^{-s}[\tilde{f}(\psi_i(s), \mu(s)) + (c + \frac{m}{Te^{-s}})e_i(\mu(s))]\\ &= \theta(s)\tilde{f}(\psi_i(s), \mu(s)) - (c\theta(s) + m)[\sum\nolimits_{j=1}^{N} a_{ij}\\ &\quad \times (\psi_i(s) - \psi_j(s)) + a_{i0}\psi_i(s)], \qquad (8)\end{aligned}$$

where $\theta(s) = Te^{-s} \in (0, T]$ and $\theta^{'}(s) = -\theta(s)$.

Let $\psi(s) = \mathrm{col}(\psi_1(s), \cdots, \psi_N(s))$ and $\tilde{f}(s) = \mathrm{col}(\tilde{f}(\psi_1(s), \mu(s)), \cdots, \tilde{f}(s)(\psi_N(s), \mu(s))$ with $\tilde{f}(\psi_i(s)) = f(\psi_i(s), \mu(s)) - 1_N \otimes f(z_0(s), \mu(s))$. Then, (8) is rewritten as

$$\psi^{'}(s) = \theta(s)\tilde{f}(s) - (c\theta(s) + m)(H \otimes I_n)\psi(s). \qquad (9)$$

Theorem 1: Based on Assumptions 1 and 2 and for CDNs in (1)-(2), a digraph $\bar{\mathcal{G}}$ and any initial conditions $x_i(0)$, the developed distributed pinning controller in (6) ensures that we have: (a) prescribed-time synchronization can be achieved in a pre-defined time $T > 0$, i.e., $\lim_{t\to T} x_i(t) = z_0(t)$, $i \in \mathcal{V}$; (b) this synchronization is maintained (i.e., $x_i(t) = z_0(t)$ for all $t \geq T$); and (c) the control input $u_i(t)$ is uniformly bounded for all $t \in [0, \infty)$, provided that

$$c > 2l_f\lambda_{max}(\Omega)/\lambda_{min}(\Xi) \text{ for } \forall l_f > 0, \qquad (10)$$

where Ω, Ξ are from Lemma 1 and l_f is in Assumption 2.

Proof: (i) prove that synchronization can be obtained in a pre-defined time T.

Consider the non-negative Lyapunov function as

$$V(s) = \frac{1}{2}\sum_{i=1}^{N}\gamma_i\psi_i^T(s)\psi_i(s) = \frac{1}{2}\psi^T(s)(\Omega \otimes I_n)\psi(s), \qquad (11)$$

where γ_i and Ω have been defined in Lemma 1 and thus, $\frac{1}{2}\lambda_{min}(\Omega)||\psi(s)||^2 \leq V(\psi(s)) \leq \frac{1}{2}\lambda_{max}(\Omega)||\psi(s)||^2$.

Thus, differentiating $V(\psi(s))$ associated with $s \in [0, \infty)$ along the system (9) gives rise to

$$\begin{aligned}V^{'} &= \psi^T(s)(\Omega \otimes I_n)[\theta(s)\tilde{f}(s) - (c\theta(s) + m)(H \otimes I_n)\psi(s)]\\ &= \theta(s)\psi^T(s)(\Omega \otimes I_n)\tilde{f}(s) - c\theta(s)\psi^T(s)(\Omega H \otimes I_n)\psi(s)\\ &\quad - m\psi^T(s)(\Omega H \otimes I_n)\psi(s), \qquad (12)\end{aligned}$$

where $V^{'}(\psi(s)) \triangleq \frac{dV(\psi(s))}{ds}$ and $\theta(s) \in (0, T]$ for $s \in [0, \infty)$.

In light of Assumption 2 where the term $\tilde{f}(s)$ satisfies the l_f-Lipschitz condition, we can obtain

$$\psi^T(s)(\Omega \otimes I_n)\tilde{f}(s) \leq l_f\lambda_{max}(\Omega)||\psi(s)||^2. \qquad (13)$$

Applying Lemma 1 gives rise to $-\psi^T(s)(\Omega H \otimes I_n)\psi(s) =$

$$-\frac{1}{2}\psi^T(s)(\Xi \otimes I_n)\psi(s) \leq -\frac{1}{2}\lambda_{min}(\Xi)||\psi(s)||^2. \qquad (14)$$

Next, it can be derived from (13) and (14) that

$$\begin{aligned}V^{'}(s) &\leq l_f\theta(s)\lambda_{max}(\Omega)||\psi(s)||^2 - \frac{1}{2}m\lambda_{min}(\Xi)||\psi(s)||^2\\ &\quad - \frac{1}{2}c\theta(s)\lambda_{min}(\Xi)||\psi(s)||^2\\ &\leq -\theta(s)\frac{c\lambda_{min}(\Xi) - 2l_f\lambda_{max}(\Omega)}{\lambda_{max}(\Pi)}V(\psi(s)) \qquad (15)\\ &\quad - \frac{m\lambda_{min}(\Xi)}{\lambda_{max}(\Pi)}V(\psi(s)) \leq -m\lambda_0 V(\psi(s)) \leq 0,\end{aligned}$$

provided that $c > 2l_f\lambda_{max}(\Omega)/\lambda_{min}(\Xi)$, $m > 1$, $\theta(s) = T e^{-s} \in (0, T]$ for $s \in [0, \infty)$ and $\lambda_0 = \lambda_{min}(\Xi)/\lambda_{max}(\Pi)$.

Based on the fact that $V(s)$ is non-negative for $s \in [0, \infty)$ and $V^{'}(s)$ is non-positive for $s \in [0, \infty)$, the obtained error system in (9) is globally stable. Moreover, the synchronization error $\psi(s)$ is bounded and converges towards zero exponentially based on the invariance-like theorem [22]. Thus, we have $\lim_{s\to\infty}\psi(s) = 0_{Nn}$. That is, $\lim_{s\to\infty}\psi_i(s) = 0_n$. In addition, notice that $t \to T$ when $s \to \infty$ using the time transformation function defined in (5) and also, $\psi_i(s) \triangleq \xi_i(t)$ where $\xi_i(t)$ is denoted as the solution to (7) by Lemma 2. Then, it can be obtained that $\lim_{t\to T}\tilde{x}_i(t) = 0_n$. That is, we have $\lim_{t\to T}(x_i(t) - z_0(t)) = 0_n$.

(ii) prove that this synchronization can be maintained for $t \in [T, \infty)$, and moreover, this control input $u_i(t)$ is always uniformly bounded over $t \in [0, \infty)$.

We first prove that $u_i(t)$ is bounded over $t \in [0, T)$.

Let $\tilde{x}(t) = \mathrm{col}(\tilde{x}_1(t), \cdots, \tilde{x}_N(t)) \in \mathbb{R}^{Nn}$ and $u(t) = \mathrm{col}(u_1(t), \cdots, u_N(t)) \in \mathbb{R}^{Nn}$. It follows from the distributed prescribed-time synchronization controller in (6) that

$$u(t) = -(c + \varrho(t))(H \otimes I_n)\tilde{x}(t). \qquad (16)$$

Differentiating (16) with respect to t gives rise to

$$\dot{u}(t) = -(c + \varrho(t))(H \otimes I_n)\dot{\tilde{x}}(t) - \dot{\varrho}(t)(H \otimes I_n)\tilde{x}(t). \quad (17)$$

Further, it obtains from (6) that

$$\begin{aligned}\dot{u}(t) &= -(c + \varrho(t))(H \otimes I_n)\dot{\tilde{x}}(t) + \frac{m}{(T-t)^2}(H \otimes I_n)\tilde{x}(t)\\ &= -(c + \varrho(t))(H \otimes I_n)(\tilde{f}(\tilde{x}(t), t) + u(t))\\ &\quad + \frac{c + \varrho(t)}{T-t}(H \otimes I_n)\tilde{x}(t) - \frac{c}{T-t}(H \otimes I_n)\tilde{x}(t)\\ &= -(c + \varrho(t))(H \otimes I_n)\tilde{f}(\tilde{x}(t), t) - \frac{c\varrho(t)}{m}(H \otimes I_n)\tilde{x}(t)\\ &\quad - [(c + \varrho(t))(H \otimes I_n) + \frac{\varrho(t)}{m}I_n]u(t). \qquad (18)\end{aligned}$$

Let $\zeta(t)$ denote the solution to (18). Since $t = \mu(s)$ in (5) for $s \in [0, \infty)$, we have $d\phi(s)/ds = \phi^{'}(s)$ with

$$\phi^{'}(s) = (c\theta(s) + m)(H \otimes I_n)\tilde{f}(s) - c(H \otimes I_n)\psi(s) - [(c\theta(s) + m)(H \otimes I_n) + I_{Nn}]\phi(s). \quad (19)$$

From Theorem 1, $V^{'}(\psi(s)) \leq -m\lambda_0 V(\psi(s))$ in (15) that for $s \in [0, \infty)$, $||\psi(s)|| \leq e^{-\frac{m\lambda_0}{2}s}||\psi(0)||$.

Denote $\tilde{\theta}(s) = (c\theta(s) + m) > m > 1$ and H is positive define by Assumption 1. From Assumption 2, we get

$$||\tilde{f}(s)|| \leq l_f||\psi(s)|| \leq l_f e^{-\frac{m\lambda_0}{2}s}||\psi(0)||. \quad (20)$$

Further, combining (19)-(20) gives rise to

$$\begin{aligned}\phi^{'}(s) &\leq -(\tilde{\theta}(s)\lambda_{min}(H) + 1)\phi(s) + l_f\tilde{\theta}(s)\lambda_{max}(H) \\ &\quad \times e^{-\frac{m\lambda_0}{2}s}||\psi(0)|| + c\lambda_{max}(H)e^{-\frac{m\lambda_0}{2}s}||\psi(0)|| \\ &\leq -h_0\phi(s) + (h_1 + h_2)e^{-\frac{m\lambda_0}{2}s} \\ &= -h_0\phi(s) + h_{12}h_3(s), \end{aligned} \quad (21)$$

where $h_0 = \lambda_{min}(H) + 1$, $h_1 = c\lambda_{max}(H)||\psi(0)||$, $h_2 = l_f(cT + m)\lambda_{max}(H)||\psi(0)||$, $h_{12} = h_1 + h_2$, and $h_3(s) = e^{-\frac{m\lambda_0}{2}s}$. Then, by using Comparison Lemma (Lemma 3.4 of [22]) and from (21), we have that for $s \in [0, \infty)$,

$$||\phi(s)|| \leq e^{-h_0}\phi(0) + h_{12}\int_0^s e^{-h_0(s-\tau)}h_3(\tau)d\tau. \quad (22)$$

Thus, $\phi(s)$ is a bounded solution to the dynamical system (19) over $s \in [0, \infty)$. Since $\phi(s) \triangleq \zeta(t)$, it can be concluded that $u(t)$ is bounded for $t \in [0, T)$. That is, the boundedness of the control signal $u_i(t)$ is obtained for $t \in [0, T)$.

We next prove that the synchronization is maintained for $t \geq T$, and $u_i(t)$ is uniformly bounded for $t \in [T, \infty)$.

In light of (7), we can further obtain that for $t \in [T, \infty)$,

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), t) - c(H \otimes I_n)\tilde{x}(t), \ t \in [T, \infty), \quad (23)$$

which implies that for choosing a similar non-negative Lyapunov function to (11) with $V(\tilde{x}(t)) = \frac{1}{2}\tilde{x}^T(t)(\Omega \otimes I_n)\tilde{x}(t)$, its time derivative along (23) is expressed as

$$\dot{V}(\tilde{x}(t)) = \tilde{x}^T(t)(\Omega \otimes I_n)[\tilde{f}(\tilde{x}(t), t) - c(H \otimes I_n)\tilde{x}(t)]. \quad (24)$$

Similar to (12)-(15), we have that for $t \in [T, \infty)$

$$\dot{V}(\tilde{x}(t)) \leq \frac{2l_f\lambda_{max}(\Omega) - c\lambda_{min}(\Xi)}{\lambda_{max}(\Pi)}V(\tilde{x}(t)). \quad (25)$$

By the condition in (10), $\dot{V}(\tilde{x}(t)) \leq 0$ for $t \in [T, \infty)$. Due to the continuity of the system, $\tilde{x}(t)$ is continuous at $t = T$, and thus, $V(\tilde{x}(t))$ is continuous at $t = T$. Thus, we have

$$V(T) = \frac{1}{2}\lim_{t \to T^-}\tilde{x}^T(t)(\Omega \otimes I_n)\tilde{x}(t) = 0. \quad (26)$$

Combining (24) and (25), we have

$$0 \leq V(t) \leq V(T) = 0, \ t \in [T, \infty), \quad (27)$$

which obtains that $V(t) \equiv 0$ can hold for $[T, \infty)$ and then, $\tilde{x}(t) \equiv 0_{Nn}$ for $[T, \infty)$. That is, this synchronization can be maintained, i.e., $x_i(t) = z_0(t)$ for $[T, \infty)$. Further, it gets from (16) that $u_i(t) = 0_n$ for $[T, \infty)$. Thus, for $t \in [T, \infty)$, not only the synchronization can be maintained, but also the control signal $u_i(t)$ keeps zero.

Overall, it follows from Steps (i) and (ii) that synchronization is achieved for $t \in [0, T)$ and is maintained for $[T, \infty)$, and $u_i(t)$ is uniformly bounded for $t \in [0, \infty)$. ■

B. Distributed Prescribed-Time & Adaptive Synchronization of CDNs over A Directed Communication Topology

In this part, we develop another algorithm to remove the requirement that c in (10) requires known information on the digraph (e.g., $\lambda_{min}(\Omega)$ and $\lambda_{min}(\Xi)$), the Lipschitz constant of the nonlinear function (l_f), or the number of all nodes (N).

Distributed prescribed-time and adaptive synchronization algorithm: instead of using static control gains c and m in (6), we design a time-varying gain based distributed integral loop that tunes on-line the coupling strength with an adaptive law over a digraph. In light of (6), an adaptively distributed prescribed-time synchronization algorithm is developed as

$$u_i(t) = (c_i(t) + \varrho_i(t))\vartheta_i(r_i(t))e_i(t), \ i \in \mathcal{V}, \quad (28a)$$

$$\dot{m}_i(t) = \kappa_i(t)[r_i(t) - \sigma_i(m_i(t) - \hat{m}_i(t))], \quad (28b)$$

$$\dot{\hat{m}}_i(t) = \kappa_i(t)[\delta_i(m_i(t) - \hat{m}_i(t))], \quad (28c)$$

where $e_i(t) = \sum_{j=1}^N a_{ij}(x_j(t) - x_i(t)) + a_{i0}(z_0(t) - x_i(t))$ has been defined in (6), $\vartheta_i(r_i(t))$ are smooth and monotonically increasing functions which satisfies that $\vartheta_i(r_i(t)) \geq 1$ for $r_i(t) = e_i^T(t)e_i(t) \geq 0$, i.e., $\vartheta_i(r_i(t)) = (1 + r_i(t))^2$, $c_i(t) = c_{0i}m_i(t)$ with $c_{0i} > 0$ being a scalar, $\varrho_i(t) = \frac{m_i(t)}{T-t}$, if $t \in [0, T)$ and 0, if $t \geq T$, $\kappa_i(t) = k_i/(T - t)$, if $t \in [0, T)$ and 0, if $t \geq T$ and $k_i > 1$, $m_i(0) \geq \hat{m}_i(0) > 1$, $\sigma_i > 0$, $\delta_i > 0$.

Denote $\tilde{\vartheta}_i(t) = m_i(t)\vartheta_i(r_i(t))$. Then, based on $\tilde{x}_i(t) = x_i(t) - z_0(t)$ and in light of (28), we have

$$\begin{aligned}\dot{\tilde{x}}_i(t) &= \tilde{f}(x_i(t), z_0(t), t) + (c_i(t) + \varrho_i(t))\vartheta_i(r_i(t))e_i(t) \\ &= \tilde{f}(x_i(t), z_0(t), t) + (c_{0i} + \frac{1}{T - t})\tilde{\vartheta}_i(t)e_i(t), \end{aligned} \quad (29a)$$

$$\dot{m}_i(t) = \frac{k_i}{T - t}[r_i(t) - \sigma_i(m_i(t) - \hat{m}_i(t))], \quad (29b)$$

$$\dot{\hat{m}}_i(t) = \frac{k_i}{T - t}[\delta_i(m_i(t) - \hat{m}_i(t))]. \quad (29c)$$

Let $\varsigma_i(t)$, $\varepsilon_i(t)$ and $\hat{\varepsilon}_i(t)$ be solutions to dynamical systems of $\dot{e}_i(t)$, $\dot{m}_i(t)$ and $\dot{\hat{m}}_i(t)$, respectively, and $\chi_i(s) \triangleq \varsigma_i(t)$, $\omega_i(s) \triangleq \varepsilon_i(t)$, and $\hat{\omega}_i(s) \triangleq \hat{\varepsilon}_i(t)$, respectively. Furthermore, define $r_i(s) = \chi_i^T(s)\chi_i(s)$ and $\tilde{\vartheta}_i(s) = \omega_i(s)\vartheta_i(r_i(s))$ with respect to $s \in [0, \infty)$. In light of (5), $d\mu(s)/ds = T - t$ in (29). According to the fact that $t = \mu(s)$ in (5) and Lemma 2, it follows from (29a) that for $s \in [0, \infty)$, we have

$$\begin{aligned}\chi^{'}(s) &= -\mu^{'}(s)\tilde{H}[\tilde{f}(s) + ((C_0 + \frac{I_N}{\mu^{'}(s)})\tilde{\vartheta}(s) \otimes I_n)\chi(s)] \\ &= -\tilde{H}[\theta(s)\tilde{f}(s) + ((\theta(s)C_0 + I_N)\tilde{\vartheta}(s) \otimes I_n)\chi(s)], \end{aligned}$$

where $\tilde{H} = H \otimes I_n$, $C_0 = \text{diag}\{c_1, c_2, \cdots, c_N\}$, and further, $\chi(s) = \text{col}(\chi_1(s), \cdots, \chi_N(s))$ is defined with respect to $e(t) = \text{col}(e_1(t), \cdots, e_N(t))$ with $e_i(t) = \sum_{j=1}^N a_{ij}(x_j(t) - x_i(t)) + a_{i0}(z_0(t) - x_i(t))$, i.e., $e(t) = -(H \otimes I_n)\tilde{x}(t)$ and $\tilde{\vartheta}(s) = \text{diag}\{\tilde{\vartheta}_1(s), \cdots, \tilde{\vartheta}_N(s)\}$, $\theta(s) = Te^{-s}$.

Further, let $\omega(s) = \text{col}(\omega_1(s), \cdots, \omega_N(s))$ and $\hat{\omega}(s) = \text{col}(\hat{\omega}_1(s), \cdots, \hat{\omega}_N(s))$. Then, the closed-loop error system in (29) for $s \in [0, \infty)$, is expressed in a compact form as

$$\begin{aligned}\dot{\chi}(s) &= -\theta(s)(H \otimes I_n)\tilde{f}(s) - (H \otimes I_n)(\tilde{\vartheta}(s) \otimes I_n)\chi(s) \\ &\quad - \theta(s)(C_0H \otimes I_n)(\tilde{\vartheta}(s) \otimes I_n)\chi(s), \end{aligned} \quad (30a)$$

$$\dot{\omega}(s) = (K \otimes I_n)[r(s) - (\sigma \otimes I_n)(\omega(s) - \hat{\omega}(s))], \quad (30b)$$

$$\dot{\hat{\omega}}(s) = (K \otimes I_n)[(\delta \otimes I_n)(\omega(s) - \hat{\omega}(s))], \quad (30c)$$

where $K = \mathrm{diag}\{k_1, \cdots, k_N\}$, $\sigma = \mathrm{diag}\{\sigma_1, \cdots, \sigma_N\}$, $\delta = \mathrm{diag}\{\delta_1, \cdots, \delta_N\}$ and $\tilde{\vartheta}(s) = \mathrm{diag}\{\tilde{\vartheta}_1(s), \cdots, \tilde{\vartheta}_N(s)\}$.

Next, we present the adaptively distributed prescribed-time synchronization result of CDNs over a digraph.

Theorem 2: Based on Assumptions 1 and 2 and for CDNs in (1)-(2), a digraph $\bar{\mathcal{G}}$ and any initial conditions $x_i(0)$, the developed distributed pinning controller in (28) ensures that prescribed-time synchronization is achieved in a pre-defined time T, i.e., $\lim_{t\to T} x_i(t) = z_0(t)$, and the synchronization is maintained (i.e., $x_i(t) = z_0(t)$ for all $t \geq T$). Moreover, $u_i(t)$ is uniformly bounded for all $t \in [0, \infty)$.

Proof: Since $r_i(s) = \chi_i^T(s)\chi_i(s) \geq 0$, it follows from (30b) and (30c) that for scalars $\omega_i(s)$, $\hat{\omega}_i(s)$, $r_i(s)$, $i \in \mathcal{V}$ and $\omega_i(0) \geq \hat{\omega}_i(0) > 1$, we can obtain

$$\dot{\tilde{\omega}}_i(s) = -k_i(\delta_i + \sigma_i)\tilde{\omega}_i(s) + k_i r_i(s),\ \tilde{\omega}_i(s) = \omega_i(s) - \hat{\omega}_i(s),$$

which gives that by Comparison Lemma (Lemma 3.4 of [22]), $\tilde{\omega}_i(s) = e^{-k_i(\delta_i+\sigma_i)s}\tilde{\omega}_i(0) + \int_0^s e^{-k_i^2(\delta_i+\sigma_i)(s-\tau)} r_i(\tau)d\tau \geq 0$. Thus, we can obtain that $\omega_i(s) \geq \hat{\omega}_i(s) > 1$ and $\hat{\omega}_i(s)$ is monotonically increasing for $r_i(s) \geq 0$ and $\omega_i(0) \geq \hat{\omega}_i(0) > 1$. Furthermore, $\omega_i(s)$ and $\hat{\omega}_i(s)$ can converge towards certain finite values if $\chi_i(s)$ approaches zero.

Based on the above observation, we choose

$$V(s) = \frac{1}{3}\sum_{i=1}^{N} \gamma_i (1 + r_i(s))^3 \omega_i(s) + \sum_{i=1}^{N} \frac{1}{k_i}(\omega_i(s) - \omega_i^*)^2 + \sum_{i=1}^{N} \frac{\sigma_i}{k_i\delta_i}(\hat{\omega}_i(s) - \omega_i^*)^2, \tag{31}$$

where $\gamma_i > 0$ is defined in Lemma 1 and $\omega_i^* > 0$ is a scalar.

Then, by following a similar procedure in Theorem 1, we can further obtain that

$$\begin{aligned}\dot{V}(s) \leq& -\frac{\lambda_2}{4}\sum_{i=1}^{N} \chi_i^T(s)[\omega_i^2(s)(\vartheta_i^2(s) - 1) + \vartheta_i^2(s)(\omega_i^2(s) \\ &- 1)]\chi_i(s) - (2\omega^* - \lambda_3)\chi^T(s)\chi(s) \\ \leq& -(2\omega^* - \lambda_3)\chi^T(s)\chi(s) < 0 \end{aligned} \tag{32}$$

provided that $\omega^* = \min_{i\in\mathcal{V}}\{\omega_i^*\}$ with ω_i^* being a positive scalar selected in the Lyapunov function (31) rather than the controller, and $\omega^* > \lambda_3/2$ for $\lambda_3 = \lambda_1 + \frac{k_*^2}{12\lambda_2^3} + \frac{4}{\lambda_2} > 0$.

Since $V(s)$ in (31) is non-negative and $V^{'}(s) \leq 0$ based on (32) for $s \in [0, \infty)$, the obtained error system in (30) is globally stable. Further, all states $\chi_i(s)$, $\omega_i(s)$ and $\hat{\omega}_i(s)$ are bounded for $s \in [0, \infty)$. Thus, by using the invariance-like theorem [22], it can be concluded that $\lim_{s\to\infty} \chi(s) = 0_{Nn}$, i.e., $\lim_{s\to\infty} \chi_i(s) = 0_n$, $i \in \mathcal{V}$ asymptotically. In addition, it is noted that $t \to T$ when $s \to \infty$ according to the time transformation function in (5) and hence, by using Lemma 2, it is concluded that $\lim_{t\to T} e_i(t) = 0_n$. Since $e(t) = -(H \otimes I_n)\tilde{x}(t)$ and H is positive definite by Assumption 1, we have $\lim_{t\to T}(x_i(t) - z_0(t)) = 0_n$.

Next, similar to the proof of Theorem 1, we can show that the synchronization can be maintained and the control input $u_i(t)$ is uniformly bounded for all $t \in [0, \infty)$. ■

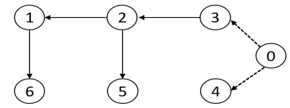

Fig. 1. The directed communication topology of six nodes and one target.

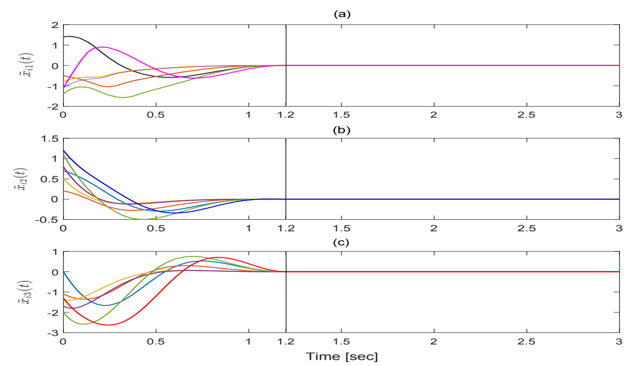

Fig. 2. The trajectories of prescribed-time synchronization errors $\tilde{x}_i(t) = x_i(t) - z_0(t), i = 1, 2, \cdots, 6$ of Chua's circuit systems under the developed distributed prescribed-time pinning controller in (6) over the directed graph.

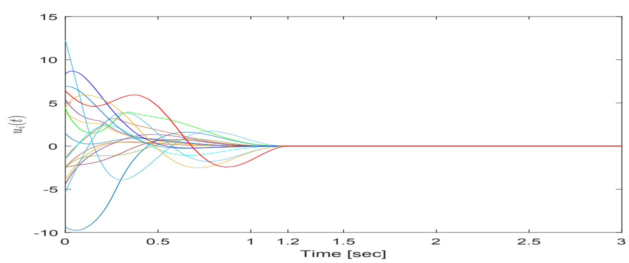

Fig. 3. The plot of control inputs $u_i(t)$ under the proposed controller (6).

IV. Numerical Example

We consider a complex dynamical network (1) governed by the following Chua's circuit system (e.g., [1], [10]),

$$\begin{aligned} f(x_i(t)) &= \begin{bmatrix} a & b & 0 \\ 1 & -1 & 1 \\ 0 & \beta & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g(x_{i1}(t)) \\ &= \begin{pmatrix} -ax_{i1}(t) + bx_{i2}(t) + \alpha g(x_{i1}(t)) \\ x_{i1}(t) - x_{i2}(t) + x_{i3}(t) \\ -\beta x_{i2}(t) \end{pmatrix}, \end{aligned} \tag{33}$$

where $x_i(t) = \mathrm{col}(x_{i1}(t), x_{i2}(t), x_{i3}(t))$, $\alpha = 27/7$, $\beta = 14.28$, $a = 19/7$, $\beta = 9$, and $g(x_{i1}) = |x_{i1} + 1| - |x_{i1} - 1|$.

Consider a directed graph of six nodes ($i = 1, \cdots, 6$) and one target ($i = 0$) as depicted in Fig. 1, which satisfies Assumption 1. Based on some calculations, we verify that $f(x_i(t))$ satisfies Assumption 2. In light of (33), the dynamics of $z_0(t)$ are given by $\dot{z}_0(t) = f(z_0(t))$, where $z_0(t) = \mathrm{col}(z_{01}(t), z_{02}(t), z_{03}(t))$ with $z_0(0) = \mathrm{col}(0.1, 0.3, 0.5)$.

A. Distributed Prescribed-Time Synchronization

The proposed distributed prescribed-time controller in (6) is performed over the digraph in Fig. 1 with $c = 2$, $m = 2.5$ and $T = 1.2$sec. The simulation result is depicted in Fig. 2 under initial conditions $X1 \triangleq x_i(0) = \mathrm{col}(1.5, 1, 0.5;\ -0.4, 0.5, -0.6;\ -0.7, 0.8, -0.9;\ -1, 1.1, -1.2;\ -1.3, 1.4, -1.5;\ -1, 1.5, -0.8)$, where the plot of synchronization errors $\tilde{x}_i(t)$

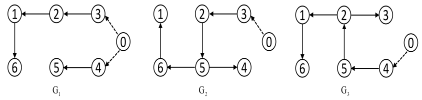

Fig. 4. Different directed topologies.

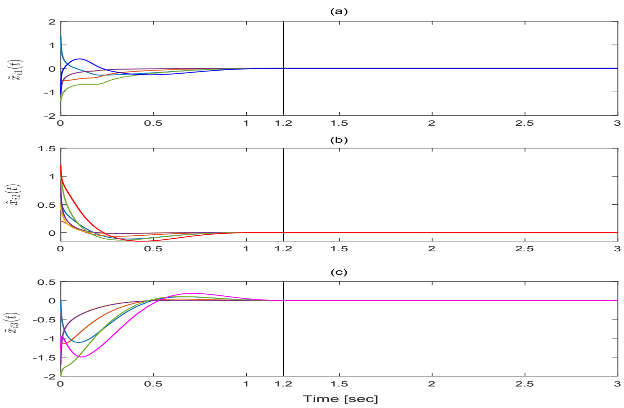

Fig. 5. The trajectories of prescribed-time synchronization errors $\tilde{x}_i(t) = x_i(t) - z_0(t)$ for Chua's circuit systems under the proposed adaptively distributed prescribed-time controller in (28) over the directed graph.

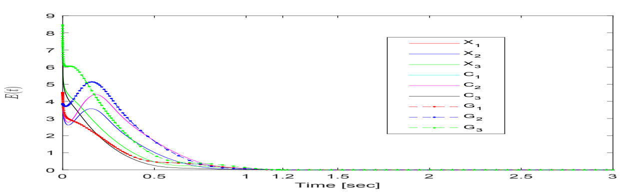

Fig. 6. The plot of $E(t)$ with different cases under the controller in (28).

of Chua's circuit systems and the target under the controller (6) is presented. It can be seen from Fig. 2 that the network can be synchronized within the prescribed time $T = 1.2$ sec, and the synchronization can be maintained for $t \geq 1.2$ sec. Moreover, Fig. 3 depicts the evolution of the control input $u_i(t), i = 1, 2, \cdots, 6$ and these control signals are uniformly bounded for all the time as we have analysed.

B. Distributed Prescribed-Time & Adaptive Synchronization

The developed distributed prescribed-time and adaptive pinning controller in (28) is conducted with the same simulation setting in the subsection IV-A. The initial state values of dynamic gains are set as $m_i(0) = \hat{m}_i(0) = 2$. The three different cases are tested: firstly, the three sets of initial state values are tested: (1) $X1$, (2) $X2 = -X1$ and (3) $X3 = 2X2$; secondly, there different control parameters are tested: (a) $C_1 \triangleq \{c, m\} = \{1.5, 2.5\}$, (b) $C_2 \triangleq \{c, m\} = \{2, 3\}$ and (c) $C_3 \triangleq \{c, m\} = \{4, 4.5\}$; thirdly, different digraphs are tested: (a) G_1, (b) G_2, and (b) G_3 in Fig. 4. The result is depicted in Fig. 5. Moreover, the plot of the error states $E(t) = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{n} |\tilde{x}_{ij}(t)|^2}$, $i = 1, 2, \cdots, 6, j = 1, 2, 3,$ under different cases is shown in Fig. 6. As observed, the network is synchronized in a prescribed time $T = 1.2$ sec, and this synchronization can be maintained for $t \geq 1.2$ sec. The evolution of $E(t)$ under different cases is shown in Fig. 6, where the time T does not depend on any initial states, control parameters and digraphs, which verifies the analysis.

V. Conclusion

In this paper, we have presented prescribed-time synchronization algorithms for CDNs, where each node's strategy is updated over a directed graph. Under the developed distributed pinning controllers, synchronization is obtained in a prescribed time. The convergence time T is pre-specified and user-defined according to synchronization requirements, and does not depend on any initial conditions, control parameters and communication structures. Further, the distributed design does not need known global information as analyzed.

References

[1] L. Chua, M. Komuro, T. Matsumoto, "The double scroll family," *IEEE Trans. Circuits Syst.*, 33(11): 1072–1118, 1986.

[2] C. Wu. Synchronization in complex networks of nonlinear dynamical systems. Singapore: World Science, 2007.

[3] W. He, F. Qian, J. Lam, G. Chen, Q. Han, J. Kurths, "Quasi-synchronization of heterogeneous dynamic networks via distributed impulsive control: error estimation, optimization and design," *Automatica*, 62(5): 249–262, 2015.

[4] J. Wang, H. Wu, T. Huang, S. Ren, J. Wu, "Passivity and output synchronization of complex dynamical networks with fixed and adaptive coupling strength," *IEEE Trans. on Neural Netw. Learn. Syst.*, 29(2), 364–376, 2020.

[5] L. Gambuzza, M. Frasca, V. Latora, "Distributed control of synchronization of a group of networked nodes," *IEEE Trans. Autom. Control*, 64(1): 365–372, 2019.

[6] M. Coraggio, P. Delellis, M. Bernardo, "Convergence and synchronization in networks of piecewise-smooth systems via distributed discontinuous coupl -ing," *Automatica*, vol. 129, p. 109596, 2021.

[7] O. Suarez, C. Vega, E. Sanchez, G. Chen, D. Rodriguez, "Neural sliding-mode pinning control for output synchronization for uncertain general complex networks," *Automatica*, vol. 112, p. 108694, 2020.

[8] X. Liu, J. Cao, W. Yu, Q. Song, "Nonsmooth finite-time synchronization of switched coupled neural networks", *IEEE Trans. Cybern.*, 46(10): 2360–2371, 2016.

[9] X. Yang, J. Lu, "Finite-time synchronization of coupled networks with Markovian topology and impulsive effects", *IEEE Trans. Autom. Control*, 61(8): 2256–2261, 2016.

[10] X. Yang, J. Lam, D. W. C. Ho, Z. Feng, "Fixed-time synchronization of complex networks with impulsive effects via nonchattering control," *IEEE Trans. Autom. Control*, 62(11): 5511–5521, 2017.

[11] X. Liu, T. Chen, "Finite-time and fixed-time cluster synchronization with or without pinning," *IEEE Trans. Cybern.*, 48: 240–252, 2018.

[12] X. Liu, D. W. C. Ho, Q. Song, W. Xu, "Finite-/fixed-time pinning synchronization of complex networks with stochastic disturbances," *IEEE Trans. Cybern.*, 49(6): 2398–2403, 2019.

[13] S. Zhu, J. Zhou, J. Lü, J. Lu, "Finite-time synchronization of impulsive dynamical networks with strong nonlinearity," *IEEE Trans. Autom. Control*, 66(8): 3550–3561, 2021.

[14] X. Liu, T. Chen, "Fixed-time synchronization of complex dynamical networks: a novel and economical mechanism," *IEEE Trans. Cybern.*, 52(6): 4430–4440, 2022.

[15] Y. Wang, Y. Song, D. J Hill, M. Krstic, "Prescribed-time consensus and containment control of networked multi-agent systems," *IEEE Trans. on Cybern.*, 49(4): 1138–1447, 2018.

[16] C. Wei, J. Luo, Z. Yin, J. Yuan, "Leader-following consensus of second-order multi-agent systems with arbitrarily appointed-time prescribed performance," *IET Control Theory & Applications*, 12(6): 22 76–2286, 2018.

[17] Y. Zhao, Y. Zhou, Y. Liu, G. Wen, P. Huang, "Fixed-time bipartite synchronization with a pre-appointed settling time over cooperative-antagonistic networks," *Automatica*, vol. 123, p. 109301, 2021.

[18] C. Hu, H. He, H. Jiang, "Fixed/preassigned-time synchronization of complex networks via improving fixed-time stability," *IEEE Trans. on Cybern.*, 51(6): 2882–2892, 2021.

[19] B. Wei, F. Xiao, Y. Shi, "Fully distributed synchronization of dynamic networked systems with adaptive nonlinear couplings," *IEEE Trans. Cybern.*, 50(7): 2926–2934, 2020.

[20] B. Wang, "Distributed error estimation for quasi-synchronization of heterogeneous dynamical networks", *IEEE Trans. Autom. Control*, 67 (5): 2489–2496, 2022.

[21] P. Benner, R. Findeisen, D. Flockerzi, U. Reichl, K. Sundmacher. Large-scale networks in engineering and life sciences. Springer, 2014.

[22] H. K. Khalil, Nonlinear Systems, 3rd ed, Prentice-Hall, 2002.