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Consensus of networked hyperbolic systems via event-triggered boundary feedback control

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Abstract— **This paper investigates the consensus problem of networked hyperbolic partial differential equation (PDE) systems for reaching an agreement over the whole spatial domain via event-triggered boundary feedback control. Consensus controllers are proposed for each PDE system based on the boundary information of its neighboring systems, where both centralized and distributed event-triggered strategies are designed in order to reduce the controller updating frequency. By employing the Lyapunov technique, sufficient conditions with respect to system matrices, event-triggered conditions and the undirected communication topology are obtained to ensure consensus of the networked systems, and it is proved that the Zeno behavior can also be avoided. Finally, the consensus control of a three-lane freeway traffic flow system modeled by Aw-Rascle-Zhang Equations is given as an application example, and the numerical simulation is carried out to validate the theoretical results.**

I. INTRODUCTION

Partial differential equations (PDEs) are the governing equations for systems with temporal and spatial variables, such as heat exchange, chemical processes, traffic flow and gas flow [1]. In recent decades, benefiting from the computer and communication technologies, networked control systems (NCSs) have brought widespread concerns due to their lower cost, and remote control capabilities, and broad applications including industrial automation, internet of vehicles, robots, smart grids [2]–[4] and so on. However, the existing research of NCSs mainly focuses on lumped parameter systems described by ordinary differential equations (ODEs), and there is a lack of research on networked PDEs. It is challenging but meaningful to investigate the control problem of networked PDEs.

For networked multi-agent systems, consensus means that the states of all agents reach an agreement [5]. Consensus problem for networked heat processes was investigated under undisturbed boundary conditions and disturbed boundary conditions [6]. Bipartite consensus for networked wave PDEs was investigated in [7], where the interactions among PDEs can be antagonistic. Aguilar et. al. [8] considered the tracking consensus problem for networked wave PDEs. Concerning on the consensus of systems with unknown bounded actuator delays , a transport PDE was used to model the actuator delay in [9].

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It can be observed that the results on consensus of networked PDEs mentioned above merely focus on continuous control, which may not be applicable due to the fact that the onboard energy is limited. To reduce the onboard resource consumption, event-triggered control is an efficient way, and researchers have achieved fruitful results on consensus of ODE systems via event-triggered control [10]–[12]. To the best of authors' knowledge, little attention has been paid to the consensus of networked PDEs based on event-triggered control. Although Zhao et. al. [13] employed event-triggered boundary control to address the bipartite consensus problem of networked parabolic PDEs, the research on event-triggered control of networked hyperbolic PDEs is still open.

Therefore this paper is devoted to investigating the consensus problem for networked linear hyperbolic PDEs via event-triggered boundary feedback control. Both centralized and distributed event-triggered strategies are designed in the boundary controller for the networked PDEs to achieve an agreement over the whole spatial domain. By using graph theory and Lyapunov technique, sufficient conditions for ensuring the asymptotic consensus of the networked hyperbolic PDEs are obtained, and it is also proved that the Zeno behavior can be avoided. Additionally, an application to the consensus control of a three-lane freeway traffic flow system is carried out with numerical simulations to validate the theoretical results.

The rest of this paper is organized as follows. Section II gives preliminaries and problem formulation. Consensus analyses based on event-triggered boundary control under undirected communication topologies are presented in Section III. Section IV shows an application to the consensus control of a multi-lane ARZ traffic flow system. Section V concludes this paper.

Notation: R, R^n and $R^{m\times n}$ respectively represent the sets of real numbers, n-dimensional real column vectors and $m\times n$ real matrices. Given a matrix A, A^{-1} and A^T denote the inverse and transpose matrix of A. $\|A\|_2 = \sqrt{\lambda_{\max}(A^TA)}$. $A \le (<)\, 0$ denotes that A is a negative semi-definite (definite) matrix. $s(A)$ denotes the spectral radius of matrix A. Given a diagonal matrix $D = \mathrm{diag}\{d_1, ..., d_n\}$, denote $|D| = \mathrm{diag}\{|d_1|, ..., |d_n|\}$. $\mathbf{0} = [0, \cdots, 0]^T$ and $\mathbf{1} = [1, \ldots, 1]^T$. $\otimes$ represents the Kronecker product. Given a function $f(x) \in L^2([0, L]; R^n)$, we define its L^2-norm as $\|f\|_{L^2} = \sqrt{\int_0^L f^T(x)f(x)dx}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Graph theory

For a network of N hyperbolic dynamical systems, the interaction among systems is described as an undirected graph $G = \{V, E, \mathcal{A}\}$, with $V = \{v_1, \ldots, v_N\}$ being the node set , $E \subset V \times V$ being the edge set, and $\mathcal{A} = (a_{ij})_{N\times N}$ being the adjacency matrix. $\varepsilon_{ij} = (v_i, v_j) \in E$ and $a_{ji} = 1$ if node j could receive information from node i, else $a_{ji} = 0$. Particularly, $a_{ii} = 0$. G is undirected means that $\mathcal{A} = \mathcal{A}^T$. The Laplacian matrix of G is denoted by $\mathcal{L} = \mathrm{diag}\{\sum_{j=1}^N a_{1j}, \ldots, \sum_{j=1}^N a_{Nj}\} - \mathcal{A}$. Denote λ_i as the eigenvalue of Laplacian matrix $\mathcal{L}$ with $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

B. Problem formulation

Consider a group of hyperbolic systems of the following form

$$\partial_t \xi_i(t,x) + \Lambda \partial_x \xi_i(t,x) = M\xi_i(t,x), i = 1, \ldots, N, \quad (1)$$

where $\xi_i : [0, +\infty) \times [0, L] \to R^n$, Λ and M are $n \times n$ real matrices. Λ is a diagonal matrix with nonzero diagonal elements, i.e. $\Lambda = \mathrm{diag}\{\gamma_1, \ldots \gamma_n\}$, where $\gamma_i > 0, \forall i \in \{1, \ldots, m\}$, and $\gamma_i < 0, \forall i \in \{m+1, \ldots, n\}$. Let $\Lambda^+ = \mathrm{diag}\{\gamma_1, \ldots, \gamma_m\}$, $\Lambda^- = \mathrm{diag}\{|\gamma_{m+1}|, \ldots, |\gamma_n|\}$, $\Lambda = \mathrm{diag}\{\Lambda^+, -\Lambda^-\}$, and $|\Lambda| = \mathrm{diag}\{\Lambda^+, \Lambda^-\}$. The boundary input and output of system (1) are denoted by

$$\xi_{i,in}(t) = \begin{bmatrix} \xi_i^+(t,0) \\ \xi_i^-(t,L) \end{bmatrix}, \xi_{i,out}(t) = \begin{bmatrix} \xi_i^+(t,L) \\ \xi_i^-(t,0) \end{bmatrix},$$

where $\xi_i^+(t,x) = [\xi_i^1(t,x), \ldots, \xi_i^m(t,x)]^T \in R^m$, and $\xi_i^-(t,x) = [\xi_i^{m+1}(t,x), \ldots, \xi_i^n(t,x)]^T \in R^{n-m}$.

For each hyperbolic system, the boundary condition is designed as

$$\xi_{i,in}(t) = A\xi_{i,out}(t) + Bu_i(t), \quad (2)$$

where $A \in R^{n\times n}$, $B \in R^{n\times r}$ is of full row rank. $u_i(t)$ denotes the control input, which will be designed later.

Definition 1 (Consensus of networked hyperbolic systems): Networked hyperbolic system (1)-(2) is said to achieve consensus if for any $i, j = 1, \ldots, N$ and $x \in [0, L]$, the following condition holds:

$$\lim_{t\to+\infty} \|\xi_i(t,x) - \xi_j(t,x)\|_{L^2} = 0. \quad (3)$$

The main objective of this paper is to design the event-triggered boundary control inputs for the networked hyperbolic system (1)-(2) to achieve consensus, so as to save the onboard energy. Fig. 1 shows the event-triggered control framework for networked hyperbolic systems. Before giving detailed controllers, we first define the following local information for each hyperbolic system:

$$q_i(t) = \sum_{j=1}^N a_{ij}(\xi_{j,out}(t) - \xi_{i,out}(t)). \quad (4)$$

An assumption on the communication topology G of the networked hyperbolic system (1)-(2) is stated as follows.

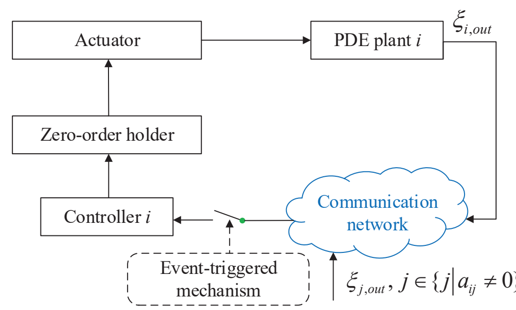

Fig. 1. Event-triggered control framework.

Assumption 1: The communication topology G is undirected and connected.

Two triggering mechanisms will be considered in this paper. The first one is the centralized event-triggered strategy described by

$$u_i(t) = Kq_i(t_k), t \in [t_k, t_{k+1}), \quad (5)$$

where $K \in R^{r\times n}$ denotes the control gain. The triggering instants $\{t_k\}, k = 0, 1, \ldots$ for all the N hyperbolic systems are determined by the following event-triggered condition

$$t_{k+1} = \inf_{t>t_k} \{t \in R_{\ge 0} | f(e(t), q(t)) > 0\}, \quad (6)$$

where f is a triggering function to be given later, $q(t) = [q_1^T(t), \ldots q_N^T(t)]^T$, and $e(t) = q(t_k) - q(t), t \in [t_k, t_{k+1})$.

The second one is distributed event-triggered strategy given by

$$u_i(t) = Kq_i(t_k^i), t \in [t_k^i, t_{k+1}^i), \quad (7)$$

where $K \in R^{r\times n}$ denotes the control gain. Let $\bar{e}_i = q_i(t_k^i) - q_i(t)$, and $\bar{e}(t) = [\bar{e}_1^T(t), ..., \bar{e}_N^T(t)]^T$. The triggering instants $\{t_k^i\}, k = 0, 1, \ldots$ for hyperbolic system i are determined by the following event-triggered condition

$$t_{k+1}^i = \inf_{t>t_k^i} \{t \in R_{\ge 0} | f_i(\bar{e}_i(t), q_i(t)) > 0\}, \quad (8)$$

where f_i is a triggering function that will be discussed in the next section.

Two problems associated with the centralized and distributed event-triggered strategies are explicitly defined as follows.

Problem 1 (Consensus under centralized event-triggered strategy): The consensus control problem for networked hyperbolic system (1)-(2) under centralized event-triggered strategy is to design u_i in (5) for hyperbolic system i and triggering function f in (6) for all hyperbolic systems such that consensus is achieved and Zeno behavior is excluded.

Problem 2 (Consensus under distributed event-triggered strategy): The consensus control problem for networked hyperbolic system (1)-(2) under distributed event-triggered strategy is to design u_i in (7) and triggering function f_i in (8) for hyperbolic system i such that consensus is achieved and Zeno behavior is excluded.

III. Main results

A. Centralized event-triggered strategy

We first consider Problem 1, where all the hyperbolic systems will update their control inputs simultaneously at triggering instants $t_k, k = 0, 1, ...$ determined by (6). The triggering function f in (6) is designed as

$$f(e(t), q(t)) = d\|e(t)\|^2 - \frac{\alpha}{\|L\|^2}\|q(t)\|^2 - c\exp(-ht), \quad (9)$$

where d, α, c and h are all positive numbers.

Theorem 1: Under Assumption 1, Problem 1 is solvable by adopting the controller (5) and triggering instants determined by (6) with triggering function f defined in (9) if the following two conditions are satisfied:

1) There exist a diagonal matrix Q, and positive constants ω and θ satisfying

$$\max_{i=2,\cdots,N} |1-\lambda_i\theta|^2 A^TQA - \exp(-L)Q + \omega A^TQA \leq 0, \quad (10)$$

$$-Q(x) + M^TQ(x)|\Lambda|^{-1} + |\Lambda|^{-1}Q(x)M < 0, \forall x \in [0, L], \quad (11)$$

where $Q(x) = \mathrm{diag}\{\exp(-x)I_m, \exp(x-L)I_{n-m}\}Q$, and $|1-\lambda_i\theta| < \exp(-L/2)/s(A)$;

2) Control gain $K = \theta B^T(BB^T)^{-1}A$ and parameters in (9) satisfy $d = \frac{1}{b}(\theta + \theta^2\|L\|) + \theta^2$, $0 < \alpha < \omega - b(\theta + \theta^2\|L\|)$ with $0 < b < \frac{\omega}{\theta+\theta^2\|L\|}$, $c > 0$ and $h > 0$.

Proof: Let $m = [m_1, \ldots, m_N]^T$ be a non-negative left eigenvector of $\mathcal{L}$ associated with eigenvalue 0 and $m^T\mathbf{1} = 1$. Denoting $\xi^* = \sum_{i=1}^{N} m_i\xi_i$, one obtains $\partial_t\xi^* + \Lambda\partial_x\xi^* = M\xi^*, i = 1, \ldots, N$. Owing to $m^TL = 0$, boundary condition w.r.t. ξ^* can be written as $\xi^*_{in} = A\xi^*_{out}$. Letting $\varepsilon_i = \xi_i - \xi^*$, one can get $\partial_t\varepsilon_i + \Lambda\partial_x\varepsilon_i = M\varepsilon_i$, $i = 1, \ldots, N$. Let $\varepsilon = [\varepsilon_1^T, \cdots, \varepsilon_N^T]^T$, $\varepsilon_{in} = [\varepsilon^T_{1,in}, \ldots, \varepsilon^T_{N,in}]^T$, and $\varepsilon_{out} = [\varepsilon^T_{1,out}, \ldots, \varepsilon^T_{N,out}]^T$. (1) and (2) can be rewritten as

$$\partial_t\varepsilon + (I_N \otimes \Lambda)\partial_x\varepsilon = (I_N \otimes M)\varepsilon, \quad (12)$$

$$\varepsilon_{in} = (I_N \otimes A - \mathcal{L} \otimes BK)\varepsilon_{out} + (I_N \otimes BK)e. \quad (13)$$

Construct a candidate Lyapunov function as

$$V(t) = \int_0^L \varepsilon^T(I_N \otimes P(x))\varepsilon dx, \quad (14)$$

where $P(x) = Q(x)|\Lambda|^{-1} \in \mathcal{R}^{n\times n}$ is a positive definite diagonal matrix. The time derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) =& \int_0^L \varepsilon^T[I_N \otimes (-|\Lambda|P(x) + M^TP(x) + P(x)M)]\varepsilon dx \\ & - \varepsilon^T(I_N \otimes \Lambda P(x))\varepsilon\Big|_0^L \\ =& \dot{V}_1 + \dot{V}_2,\end{aligned}$$

where

$$\dot{V}_1 = \int_0^L \varepsilon^T[I_N \otimes (-|\Lambda|P(x) + M^TP(x) + P(x)M)]\varepsilon dx,$$

$$\begin{aligned}\dot{V}_2 =& -\varepsilon^T(I_N \otimes \Lambda P(x))\varepsilon\Big|_0^L \\ =& \varepsilon^T_{out}[(I_N - \theta L)^2 \otimes A^TQA - I_N \otimes \exp(-L)Q]\varepsilon_{out} \\ & + 2\varepsilon^T_{out}(I_N \otimes \theta A^TQA)e - 2\varepsilon^T_{out}(\mathcal{L} \otimes \theta^2A^TQA)e \\ & + e^T(I_N \otimes \theta^2A^TQA)e.\end{aligned}$$

By selecting an orthogonal matrix $\Delta = (1_N/\sqrt{N}, \delta_2, \ldots, \delta_N)$ such that $\delta_i^T\mathcal{L} = \lambda_i\delta_i^T$, one has $\Delta^T\mathcal{L}\Delta = J = \mathrm{diag}\{0, \lambda_2, \ldots \lambda_N\}$. Let $\tilde{\varepsilon} = (\Delta \otimes I_n)^T\varepsilon$ and partition $\tilde{\varepsilon} = [\tilde{\varepsilon}_1^T, \ldots, \tilde{\varepsilon}_N^T]^T$, where $\tilde{\varepsilon}_1 = 0$.
For the first item

$$\begin{aligned}&\varepsilon^T_{out}[(I_N - \theta L)^2 \otimes A^TQA - I_N \otimes \exp(-L)Q]\varepsilon_{out} \\ &= \sum_{i=2}^{N} \tilde{\varepsilon}^T_{i,out}[(1-\lambda_i\theta)^2A^TQA - \exp(-L)Q]\tilde{\varepsilon}_{i,out} \\ &\leq -\omega\left\|A^TQA\right\|\|\varepsilon_{out}\|^2,\end{aligned}$$

in which the last inequality follows from (10). According to the event based rule (6) and the triggering function (9), we have

$$\begin{aligned}&2\varepsilon^T_{out}(I_N \otimes \theta A^TQA)e \\ &\leq b\theta\left\|A^TQA\right\|\|\varepsilon_{out}\|^2 + \frac{\theta}{b}\left\|A^TQA\right\|\|e\|^2, \\ &-2\varepsilon^T_{out}(\mathcal{L} \otimes \theta^2A^TQA)e \\ &\leq b\theta^2\|\mathcal{L}\|\left\|A^TQA\right\|\|\varepsilon_{out}\|^2 + \frac{\theta^2}{b}\|\mathcal{L}\|\left\|A^TQA\right\|\|e\|^2, \\ &e^T(I_N \otimes \theta^2A^TQA)e \leq \theta^2\left\|A^TQA\right\|\|e\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\dot{V}_2 \leq& [-\omega + b(\theta + \theta^2\|L\|)]\left\|A^TQA\right\|\|\varepsilon_{out}\|^2 \\ & + d\left\|A^TQA\right\|\|e\|^2 \\ \leq& c\left\|A^TQA\right\|\exp(-ht).\end{aligned}$$

According to (11), there exists a positive constant ℓ such that $-|\Lambda|P(x) + M^TP(x) + P(x)M + \ell P(x) \leq 0$. Hence

$$\dot{V}(t) \leq -\ell V(t) + c\left\|A^TQA\right\|\exp(-ht). \quad (15)$$

Letting $\dot{\varphi}(t) = -\ell\varphi(t) + c\left\|A^TQA\right\|\exp(-ht)$, $\varphi(0) = V(0)$, one has $0 \leq V(t) \leq \varphi(t)$. From the structure of $\varphi(t)$, we obtain

$$\varphi(t) = \exp(-\ell t)\varphi(0) + \frac{c\left\|A^TQA\right\|}{\ell - h}[\exp(-ht) - \exp(-\ell t)].$$

It is obvious that $\varphi(t)$ converges to zero asymptotically. Hence $\lim_{t\to\infty} V(t) = 0$, i.e., $\lim_{t\to\infty} \varepsilon(t, x) = 0$.

Next, the avoidance of Zeno behavior will be proved. For $t \in [t_k, t_{k+1})$, we have

$$\|\dot{e}(t)\| \leq \|M\|\|e(t)\| + \varpi, \quad (16)$$

where $\varpi = \sup_{t\in[t_k,t_{k+1})} \|M\|\|q(t_k)\| + \|\Lambda\|\|\partial_x q(t)\|$. Consider a non-negative function $\phi : [0, \infty) \to R$ satisfying

$$\dot{\phi} = \|M\|\phi + \varpi, \quad \phi(0) = \|e(t_k)\| = 0. \quad (17)$$

Therefore, we can obtain that $\|e(t)\| \leq \phi(t - t_k)$, where $\phi(t) = \frac{\varpi}{\|M\|}(\exp(\|M\|t) - 1)$ is the solution of (17). From (9), one has that $f \leq 0$ is satisfied if $\|e(t)\|^2 \leq \frac{c\exp(-ht)}{d}$. Additionally, the interval τ_k between two adjacent triggering instants is greater than the solution $\bar{\tau}$ of the following equation

$$\frac{c\exp(-h(t_k+\bar{\tau}))}{d} = \frac{\varpi^2}{\|M\|^2}(\exp(\|M\|\,\bar{\tau})-1)^2,$$

which is equivalent to

$$\bar{\tau} = \frac{1}{\|M\|}\ln(1+\frac{\|M\|}{\varpi}\sqrt{\frac{c\exp(-h(t_k+\bar{\tau}))}{d}}). \tag{18}$$

Assume that Zeno behavior happens, i.e. $\lim_{k\to\infty} t_k = t^*$ with t^* being a positive constant. According to the definition of limits of sequences, for any $\epsilon > 0$, there exists an integer $\mathbb{N} > 0$ such that for $k \geq \mathbb{N}$, $t^* - \epsilon < t_k \leq t^*$. Let $\epsilon = \frac{1}{2\|M\|}\ln(1+\frac{\|M\|}{\varpi}\sqrt{\frac{c\exp(-ht^*)}{d}})$. It follows from (18) that $\tau_k \geq 2\epsilon$. Therefore, $t_{k+1} \geq t_k + \tau_k > t^* + \epsilon$. This contradicts the fact that $t_{k+1} \leq t^*$ for $k \geq \mathbb{N}$. Therefore, Zeno behavior will not occur. The proof of Theorem 1 is thus completed. ■

Focusing on the centralized event-triggered controller (5) and associated triggering function (6), the global state measurement error $e(t)$ needs to be known by each hyperbolic system. We will present a distributed event-triggered strategy based on local measurement error $\bar{e}_i(t)$ instead of global information $e(t)$.

B. Distributed event-triggered strategy

Under the distributed event-triggered strategy, each hyperbolic system updates the controller at its own triggering instants using the state information of neighbours. The triggering function f_i of (8) is designed as

$$f_i(\bar{e}_i(t), q_i(t)) = \tilde{d}\|\bar{e}_i(t)\|^2 - \frac{\tilde{\alpha}}{4}\sum_{j=1}^{N} a_{ij}\|\xi_{j,out}(t) - \xi_{i,out}(t)\|^2 - \tilde{c}\exp(-\tilde{h}t), \tag{19}$$

where $\tilde{d}$, $\tilde{\alpha}$, $\tilde{c}$ and $\tilde{h}$ are all positive numbers.

Theorem 2: Under Assumption 1, Problem 2 is solvable by adopting the controller (7) and triggering instants determined by (8) with triggering function f_i defined in (19) if the following two conditions are satisfied:

1) There exist a diagonal matrix Q, and positive constants ω and θ satisfying

$$\max_{i=2,\cdots,N} |1-\lambda_i\theta|^2 A^TQA - \exp(-L)Q + \omega A^TQA \leq 0, \tag{20}$$

$$-Q(x) + M^TQ(x)|\Lambda|^{-1} + |\Lambda|^{-1}Q(x)M < 0, \forall x \in [0, L], \tag{21}$$

where $Q(x) = \text{diag}\{\exp(-x)I_m, \exp(x-L)I_{n-m}\}Q$, $\theta > 0$ and $|1-\lambda_i\theta| < \exp(-L/2)/s(A)$;

2) Control gain $K = \theta B^T(BB^T)^{-1}A$ and parameters in (19) satisfy $\tilde{d} = N[\frac{1}{\tilde{b}}(1+2N\theta)\theta + \theta^2]$, $0 < \tilde{\alpha} < \omega - \tilde{b}(1+2N\theta)\theta$ with $0 < \tilde{b} < \frac{\omega}{(1+2N\theta)\theta}$, $\tilde{c} > 0$ and $\tilde{h} > 0$.

Proof: Construct a candidate Lyapunov function as

$$V(t) = \int_0^L \varepsilon^T(I_N \otimes P(x))\varepsilon dx, \tag{22}$$

where $P(x) = Q(x)|\Lambda|^{-1} \in \mathcal{R}^{n\times n}$. Following a similar process, we can conclude that the time derivative of $V(t)$ is

$$\dot{V}(t) = \dot{V}_1 + \dot{V}_2,$$

where

$$\dot{V}_1 = \int_0^L \varepsilon^T[I_N \otimes (-|\Lambda|P(x) + M^TP(x) + P(x)M)]\varepsilon dx,$$

$$\begin{aligned}\dot{V}_2 =& \varepsilon_{out}^T[(I_N-\theta L)^2 \otimes A^TQA - I_N \otimes \exp(-L)Q]\varepsilon_{out} \\ &+ 2\varepsilon_{out}^T(I_N \otimes \theta A^TQA)\bar{e} - 2\varepsilon_{out}^T(\mathcal{L} \otimes \theta^2 A^TQA)\bar{e} \\ &+ \bar{e}^T(I_N \otimes \theta^2 A^TQA)\bar{e}.\end{aligned}$$

For the first item

$$\begin{aligned}&\varepsilon_{out}^T[(I_N-\theta L)^2 \otimes A^TQA - I_N \otimes \exp(-L)Q]\varepsilon_{out} \\ &\leq -\omega\left\|A^TQA\right\|\sum_{i=1}^N \|\varepsilon_{i,out}\|^2.\end{aligned}$$

According to the event based rule (8) and the triggering function (19), we have

$$\begin{aligned}&2\varepsilon_{out}^T[I_N \otimes \theta A^TQA]\bar{e} \\ &\leq \theta\tilde{b}\left\|A^TQA\right\|\sum_{i=1}^N\|\varepsilon_{i,out}\|^2 + \frac{\theta}{\tilde{b}}\left\|A^TQA\right\|\sum_{i=1}^N\|\bar{e}_i\|^2, \\ &-2\varepsilon_{out}^T[L \otimes \theta^2A^TQA]\bar{e} \\ &\leq 2\tilde{b}\theta^2N\left\|A^TQA\right\|\sum_{i=1}^N\|\varepsilon_{i,out}\|^2 + \frac{2\theta^2N}{\tilde{b}}\left\|A^TQA\right\|\sum_{i=1}^N\|\bar{e}_i\|^2, \\ &\bar{e}^T(I_N \otimes \theta^2A^TQA)\bar{e} \leq \sum_{i=1}^N \theta^2\left\|A^TQA\right\|\|\bar{e}_i\|^2,\end{aligned}$$

where Young's inequality and symmetry of the graph are applied. Therefore

$$\begin{aligned}\dot{V}_2(t) \leq& [-\omega + \tilde{b}(1+2N\theta)\theta]\left\|A^TQA\right\|\sum_{i=1}^N\|\varepsilon_{i,out}\|^2 \\ &+ \frac{\tilde{d}}{N}\left\|A^TQA\right\|\sum_{i=1}^N\|\bar{e}_i\|^2.\end{aligned}$$

By applying (19) one has

$$\frac{\tilde{d}}{N}\sum_{i=1}^N\|\bar{e}_i(t)\|^2 \leq \tilde{\alpha}\sum_{i=1}^N\|\varepsilon_i(t)\|^2 + \tilde{c}\exp(-\tilde{h}t).$$

Therefore

$$\dot{V}_2(t) \leq \tilde{c}\left\|A^TQA\right\|\exp(-\tilde{h}t).$$

According (21), there exists a positive constant ℓ such that $-|\Lambda|P(x) + M^TP(x) + P(x)M + \ell P(x) \leq 0$. Hence

$$\dot{V}(t) \leq -lV(t) + \tilde{c}\left\|A^TQA\right\|\exp(-\tilde{h}t). \tag{23}$$

Following with a similar program with Theorem 1, one has $\lim_{t\to\infty}\varepsilon(t,x) = 0$.

Zeno behavior can be excluded by the method of contradiction, which is similar to the proof of Theorem 1 and hence omitted here. ■

IV. Application to Freeway Traffic Control

In this section, we provide an application to a multi-lane traffic consensus control problem. Consider a freeway traffic flow system consisting of N lanes under boundary event-triggered control. The objective is to make the traffic flow states of all lanes consistent at any position along

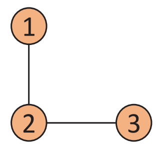

Fig. 2. Communication topology of the three-lane freeway traffic flow system.

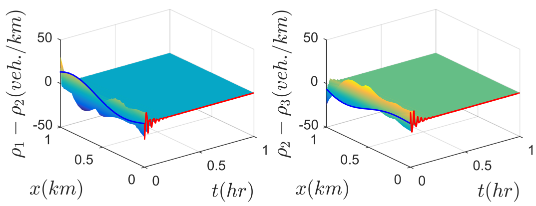

Fig. 3. The evolutions of density deviations under the centralized event-triggered strategy.

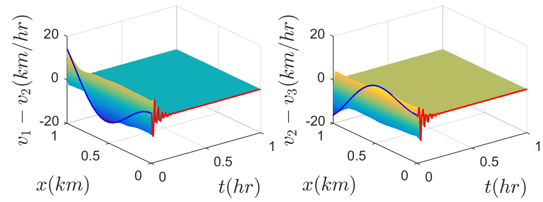

Fig. 4. The evolutions of velocity deviations under the centralized event-triggered strategy.

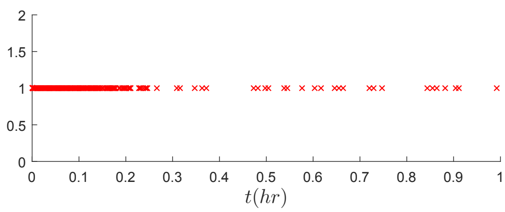

Fig. 5. The triggering instants under the centralized event-triggered strategy.

the freeway, which can avoid unnecessary lane-changing behaviors so as to improve the traffic efficiency.

A. Multi-lane traffic flow model

The dynamics of each lane is described by ARZ model [14], [15] as

$$\begin{cases} \partial_t \rho_i + \partial_x (v_i \rho_i) = 0, \\ \partial_t v_i + (v_i - \rho_i p'(\rho_i)) \partial_x v_i = \dfrac{V(\rho_i) - v_i}{\tau}, \end{cases} \tag{24}$$

where $\rho_i(x,t)$ and $v_i(x,t)$ $(i = 1,\dots,N)$ respectively denote the vehicle density and average speed. τ represents the relaxation time related to driving behavior. $V(\rho_i) = v_f(1 - \frac{\rho_i}{\rho_m})$ is the Greenshields model of which v_f denotes the free speed and ρ_m is the maximal density. The traffic pressure is defined as $p(\rho_i) = v_f - V(\rho_i) = \frac{v_f \rho_i}{\rho_m}$. By denoting $w_i = v_i + \frac{v_f \rho_i}{\rho_m}$ and $z_i = v_i$, (24) can be described under the Riemann coordinate as

$$\begin{cases} \partial_t w_i + z_i \partial_x w_i = \dfrac{v_f - w_i}{\tau}, \\ \partial_t z_i + [(1+\gamma) z_i - \gamma w_i] \partial_x z_i = \dfrac{v_f - w_i}{\tau}. \end{cases} \tag{25}$$

We linearize system (25) around states (w^*, z^*), and the corresponding state (ρ^*, v^*) of (24) satisfies $w^* = v_f$ and $z^* = v^*$. The deviation from (w^*, z^*) is defined as $\tilde{w}_i = w_i - w^*$ and $\tilde{z}_i = z_i - z^*$. Then we get the linearized ARZ model

$$\begin{cases} \partial_t \tilde{w}_i + z^* \partial_x \tilde{w}_i = -\dfrac{\tilde{w}_i}{\tau}, \\ \partial_t \tilde{z}_i + (2z^* - w^*) \partial_x \tilde{z}_i = -\dfrac{\tilde{w}_i}{\tau}. \end{cases} \tag{26}$$

By letting $\xi_i = [\tilde{w}_i, \tilde{z}_i]^T$ one has

$$\partial_t \xi_i(x,t) + \Lambda \partial_x \xi_i(x,t) = M \xi_i(x,t), \tag{27}$$

where $\Lambda = \text{diag}(z^*, 2z^* - w^*)$, $M = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{bmatrix}$. Assume that $2z^* - w^* < 0$, which means that the traffic flow lies in the congestion region [16].

B. Event-triggered boundary feedback control

With the application of Vehicular Ad Hoc Network techniques, the vehicle density $\rho_i(t,L)$ at the downstream boundary and average speed $v_i(t,0)$ at the upstream boundary can be measured and transmitted to neighboring lanes. We assume that velocities and headways of vehicles in each lane can be autonomously regulated. Therefore it is reasonable to set $\rho_i(t,0)$ and $v_i(t,L)$ as control variables. For the centralized case, the boundary condition of lane i is given by

$$\begin{cases} \rho_i(t,L) = k_1(\tilde{\rho}_i(t,L) + \theta \hat{\rho}_i(t,L)) \\ \qquad\qquad + k_2(\tilde{v}_i(t,0) + \theta \hat{v}_i(t,0)) + \rho^*, \\ v_i(t,0) = k_3(\tilde{v}_i(t,0) + \theta \hat{v}_i(t,0)) + v^*, \end{cases} \tag{28}$$

where $\tilde{\rho}_i(t,L) = \rho_i(t,L) - \rho^*$, $\tilde{v}_i(t,0) = v_i(t,0) - v^*$,

$$\begin{cases} \hat{\rho}_i(t,L) = \sum_{j=1}^{N} a_{ij}(\rho_j(t_k,L) - \rho_i(t_k,L)), \\ \hat{v}_i(t,0) = \sum_{j=1}^{N} a_{ij}(v_j(t_k,0) - v_i(t_k,0)), t \in [t_k, t_{k+1}). \end{cases}$$

It is rewritten under the Riemann coordinate as

$$\xi_{i,in}(t) = A \xi_{i,out}(t) + \theta A \hat{\xi}_{i,out}(t), \tag{29}$$

where $A = \begin{bmatrix} k_1 & -k_1 + \frac{v_f k_2}{\rho_m} + k_3 \\ 0 & k_3 \end{bmatrix}$,

$$\hat{\xi}_{i,out}(t) = \sum_{j=1}^{N} a_{ij}(\xi_{j,out}(t_k) - \xi_{i,out}(t_k)), t \in [t_k, t_{k+1}).$$

For the distributed case, t_k is replaced by t_k^i, and $\hat{\xi}_{i,out}(t)$ in (29) becomes

$$\hat{\xi}_{i,out}(t) = \sum_{j=1}^{N} a_{ij}(\xi_{j,out}(t_k^i) - \xi_{i,out}(t_k^i)), t \in [t_k^i, t_{k+1}^i).$$

C. Numerical simulations

In this subsection, we consider a three-lane freeway traffic flow system whose parameters are given by ρ_m = 160 veh./km, v_f = 120 km/hr, L = 1 km, and τ = 60 s.

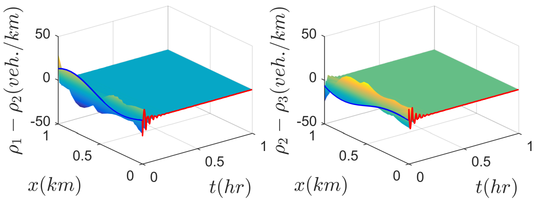

Fig. 6. The evolutions of density deviations under the distributed event-triggered strategy.

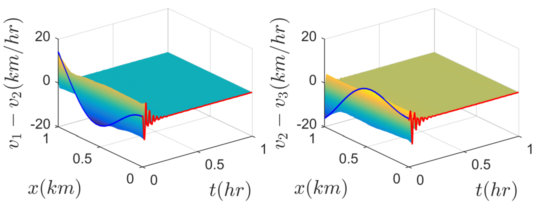

Fig. 7. The evolutions of velocity deviations under the distributed event-triggered strategy.

Setting state $(\rho^*, v^*) = (100, 45)$ satisfying $2z^* - w^* < 0$. Letting $k_1 = 1$, $k_2 = 2.6667$, $k_3 = -1$, one has $A = \text{diag}\{1, -1\}$. Fig. 2 displays the communication topology of the three-lane freeway traffic flow system. Therefore $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $s(A) = 1$. Set $\theta = 0.5$ satisfying $\max\limits_{i=2,3} |1 - \lambda_i \theta| = 0.5 < \exp(-L/2)/s(A) = 0.6065$. Let $\omega = 0.11$. After solving (10) and (11) one obtains $Q = \text{diag}\{0.9346, 0.1410\}$.

For the centralized event-triggered strategy, we select the control parameters $b = 0.07$, $\alpha = 0.02$, $d = 18.1071$, $c = 0.5$ and $h = 1.8$. The simulation results are displayed in Fig. 3-5. The evolutions of density differences and velocity differences under undirected topology G are displayed in Fig. 3 and Fig. 4, from which one can see that consensus is achieved over the whole spatial domain. The triggering instants for each lane under triggering function (9) are presented in Fig. 5, from which we can see that the triggering instants of each lane are finite.

For the distributed event-triggered strategy, the control parameters are given as $\tilde{b} = 0.041$, $\tilde{\alpha} = 0.025$, $\tilde{d} = 147.0915$, $\tilde{c} = 5$ and $\tilde{h} = 2.5$. Fig. 6-8 presented the simulation results. The evolutions of density deviations and velocity deviations under undirected topology G are displayed in Fig. 6 and Fig. 7, respectively. As shown in Fig. 6 and Fig. 7, consensus is achieved under the distributed event-triggered strategy. It is evident that each lane has its own triggering instant sequence that differs from other lanes, and Zeno behavior is excluded from Fig. 8.

V. Conclusions

This paper has addressed the consensus problem for networked linear hyperbolic systems under undirected topologies by using event-triggered mechanisms. The centralized event-triggered strategy and distributed event-triggered strategy are presented in designing the boundary conditions. By employing Lyapunov technique, two sufficient conditions are obtained to reach consensus and exclude Zeno behaviors. Finally, an application to the consensus control of a three-lane traffic flow system modeled by Aw-Rascle-Zhang Equations has been carried out with numerical simulations to verify the correctness of the theoretical results. Our future work will focus on extending the results to more general cases by taking perturbations, directed switching topologies, or dynamic boundary control into account.

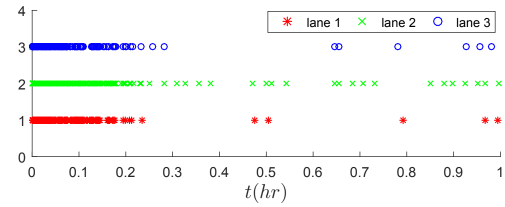

Fig. 8. The triggering instants under the distributed event-triggered strategy.

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