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Strictly Positive Realness-Based Feedback Gain Design Under Imperfect Input-Output Feedback Linearization in Prioritized Control Problem

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Abstract— **The prioritized control problem is a process to find a control strategy for a dynamical system with prioritized multiple outputs, so that it can operate outside its nonsingular domain. Singularity typically leads to imperfect inversion in the prioritized control problem, which in turn results in imperfect input-output feedback linearization. In this paper, we propose a method based on the Kalman-Yakubovich-Popov lemma that compensates nonlinear feedback terms caused by the imperfect inversion of the prioritized control problem. In order to realize this idea, we prove existence of a feedback gain matrix that gives a strictly positive real transfer function whose output matrix is identical to the feedback gain matrix. Our proof is constructive so that a set of such matrices can be found. Also, we provide a numerical approach that gives a larger set of feedback gain matrices and validate the result with numerical examples.**

I. Introduction

Priority is a strategy to distribute a limited resource to multiple tasks. The study of priority for control systems started in the robotics society in 1980s in order to find a control strategy for redundant robotic systems. Here, a system is called *redundant* when the number of control inputs is larger than the number of outputs. Thus, we can consider a secondary task for a redundant system. We do not specify what a *task* is but only assume that a task can be represented by a set of output variables. The early effort was made to formulate a control input that makes a priority structure of a redundant system, in which a secondary task is performed without affecting a primary task [1], [2], [3], [4]. Later, this idea was extended for arbitrary finite number of tasks [5], [6], [7]. The study of priority has been used and expanded in many areas such as constrained control [8], [9], task switching [10], [11], optimal control [12], [13], machine learning [14], [15], [16], etc.

Notwithstanding extensive research on this subject, only a few of them focus on analytic properties such as trajectory existence, output tracking, and stability of a prioritized control system. Here, *prioritized control system* designates a control system that has a hierarchy structure generated by priority. Antonelli [17] analyzed output tracking of a prioritized kinematic system. Sentis *et al.* [18] showed asymptotic stability of robot postures when using prioritized whole-body control structures. Ott *et al.* [19] analyzed asymptotic stability and output passivity of prioritized multi-task compliance control in terms of conditional stability, and Dietrich *et al.* [20], [21] improved this result for output regulation and tracking. Basso and Pettersen [22] derived sufficient conditions for the feedback linearization when a task-priority operational space pre-feedback control law is applied.

Yet, in spite of 50 years of research, there is not a widely accepted formal definition of the problem concerning priority for control systems. It might be a reason why analytic properties of prioritized control systems have not been studied more actively. Roughly speaking, the *prioritized control problem* is a process to find a control strategy for a dynamical system with multiple outputs along with the priority relation between outputs. As an effort to formalize this rough definition, An and Lee [23] proposed a generalization of the prioritized inverse kinematics problem in the form of multiobjective optimization with the lexicographical ordering. Recently, An *et al.* [24] expanded this generalization to the input-output feedback linearization.

Usually, the prioritized control problem contains two subproblems: orthogonalization and inversion [25]. The role of orthogonalization is to ensure the priority relation between outputs and that of inversion is to find a control input that realizes the required behavior of outputs. A difficulty in solving the prioritized control problem is to handle singularity that occurs whenever there are conflicts between outputs. Specifically, if a prioritized control system operates in a vicinity of a singular point, orthogonalization can be discontinuous and inversion can be imperfect. Therefore, prioritized control problem can be considered as an effort to extend the domain of a dynamical system to singular points, and a challenge is to find a solution to handle those degenerate properties caused by singularity. Nevertheless, previous works have not included singularity in their analysis and the challenge still remains unaddressed.

In this paper, we propose an original method to handle a part of the imperfect inversion in the prioritized control problem. The imperfect inversion, caused by singularity, typically results in imperfect input-output feedback linearization, so that the closed loop system of the output has trailing nonlinear terms. The key idea is to separate the right hand side of the closed loop system of the output into three parts: linear feedback, nonlinear feedback, and nonlinear interconnection. Then, we apply the Kalman-Yakubovich-Poppov lemma to the closed system without the nonlinear interconnection in order to find a positive definite function whose derivative is negative definite. This idea requires that a transfer function, whose output matrix is identical to the feedback gain matrix,

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should be strictly positive real. Therefore, we prove that there exists a feedback gain matrix that makes the aforementioned transfer function strictly positive real. Also, we find a set of such matrices, provide a way to find a larger set of such matrices numerically, and validate the result with numerical examples.

II. Issues in Prioritized Control Problem

A. Background on Prioritized Control Problem

We quickly recall the prioritized control problem [24]. Consider a multivariate nonlinear input-affine system with multiple vector outputs

$$\dot{x} = f(x) + G(x)u \tag{1a}$$
$$y_i = h_i(x) \quad (1 \le i \le k) \tag{1b}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h_i : \mathbb{R}^n \to \mathbb{R}^{p_i}$ for $1 \le i \le k$ are sufficiently smooth on $\mathbb{R}^n$; n, m, and p_i are dimensions of the state x, input u, and the i-th output y_i, respectively; and k is the total number of vector outputs. We assume $p = p_1+\cdots+p_k \le m \le n$ and that for each y_i there exists $r_i \in \mathbb{N}$ such that $L_G L_f^j h_i(x) = 0$ for all $0 \le j \le r_i-2$ and $x \in \mathbb{R}^n$ and $J_i(x) = L_G L_f^{r_i-1} h_i(x)$ has full rank *almost everywhere* on $\mathbb{R}^n$ where $L_f^j h_i(x) = (\partial(L_f^{j-1}h_i)/\partial x)f(x)$, $L_G L_f^j h_i(x) = (\partial(L_f^j h_i)/\partial x)G(x)$, and $L_f^0 h_i(x) = h_i(x)$. In this paper, we restrict our discussion to the case $p \le m$ but the main result also holds for the case $p > m$ [26]. We also clarify that the smoothness assumption on f, G, h_i, and J holds for a large class of systems such as Lagrangian mechanics. Let $\kappa_i(x) = L_f^{r_i} h_i(x)$. Then, we have

$$y_i^{(r_i)} = \frac{d^{r_i} y_i}{dt^{r_i}} = \kappa_i(x) + J_i(x)u \quad (1 \le i \le k). \tag{2}$$

The reason for not combining outputs as a single vector $y \coloneqq \mathrm{col}(y_1, \ldots, y_k)$ is to consider priority relations between outputs $y_1, \ldots, y_k$. (We will use notations h, κ, and J as the same meaning of y. Also, $\bullet_{i:j}$ will denote $\mathrm{col}(\bullet_i, \ldots, \bullet_j)$ unless otherwise stated.) The concept of priority is closely related to the output controllability of (1) that depends on the rank condition of the input gain matrix $J(x)$. If $J(x)$ has full rank at some $x \in \mathbb{R}^n$, then the system (1) has a vector relative degree at x and the output y is controllable in a neighborhood of x [27, Chapter 5]. If $J(x)$ is rank deficient, then we cannot always control all outputs at x simultaneously. In this case, we introduce priority relations between outputs and then try to control higher priority outputs prior to the lower priority outputs. (We assume that the outputs are listed in the order of priority such that y_1 has the highest priority and y_k has the lowest priority.)

In order to realize this idea, we orthogonalize the input gain matrix as in [23, Lemma 1]

$$\underbrace{\begin{bmatrix} J_1 \\ \vdots \\ J_k \end{bmatrix}}_{J(x)\in\mathbb{R}^{p\times m}} = \underbrace{\begin{bmatrix} L_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ L_{k1} & \cdots & L_{kk} & 0 \end{bmatrix}}_{L_e(x)=[L_{ij}(x)]\in\mathbb{R}^{p\times m}} \underbrace{\begin{bmatrix} Q_1 \\ \vdots \\ Q_{l+1} \end{bmatrix}}_{Q_e(x)\in\mathbb{R}^{m\times m}} \tag{3}$$

where L_e is lower triangular, Q_e is orthogonal, and $L_{ij}(x) \in \mathbb{R}^{p_i\times p_j}$. Then, we can rewrite (2) as

$$y_i^{(r_i)} = \kappa_i(x) + \sum_{j=1}^{i} L_{ij}(x)Q_j(x)u_j \quad (1 \le i \le k) \tag{4}$$

where $u_i = P_i(x)u$, $P_i = Q_i^T L_{ii}^+ L_{ii} Q_i$, and $\bullet^+$ is the pseudoinverse of a matrix $\bullet$. (We clarify that the orthogonalization (3) has a special property that allows us to write (4). See [23, Lemma 1] for details.) The orthogonal projectors $P_i(x)$ decompose the input space $\mathbb{R}^m$ into mutually orthogonal subspaces $\mathcal{R}(P_i(x))$ and $u_i \in \mathcal{R}(P_i(x))$ is the i-th input that can be utilized for the control of the i-th output y_i under the priority relations. Thus, the prioritized control problem usually finds u_i for each y_i first and then define the *prioritized control input* as $u = u_1 + \cdots + u_k$.

A formal definition of the prioritzed control input along with some examples can be found in [23], [24], [28]. Among them, the *canonical* prioritized control input is given as

$$u_i = (Q_i^T L_{ii}^{+(\lambda_i)})(x) \left(v_i - \kappa_i(x) - \sum_{j=1}^{i-1} (L_{ij}Q_j)(x)u_j \right) \tag{5}$$

where $v_i \in \mathbb{R}^{p_i}$ is the i-th external control input that can be determined freely for y_i, $\lambda_i : \mathbb{R}^n \to [0, \infty)$ is the damping function, and $\bullet^{+(c)} = \bullet^T(\bullet\,\bullet^T + c^2 I)^+$ is the damped pseudoinverse of a matrix $\bullet$ with the damping constant $c \ge 0$. By applying (5) into (4), we can formulate

$$y_i^{(r_i)} = P_i^\circ(x)v_i + N_i^\circ(x)\kappa_i^\circ(x, v_{1:i-1}) \tag{6a}$$
$$= v_i - N_i^\circ(x)(v_i - \kappa_i^\circ(x, v_{1:i-1})) \tag{6b}$$

where $P_i^\circ = L_{ii}L_{ii}^{+(\lambda_i)}$, $N_i^\circ = I_{p_i} - P_i^\circ$, and $\kappa_i^\circ = \kappa_i + \sum_{j=1}^{i-1} L_{ij}Q_j u_j$. (Here, we write $\kappa_i^\circ(x, v_{1:i-1})$ to show that κ_i° depends on $x, v_1, \ldots, v_{i-1}$.) From (6), we can clearly see how the orthogonalization (3) realizes the idea of priority. Since

$$\mathrm{rank}(J_{1:i}(x)) = \sum_{j=1}^{i} \mathrm{rank}(P_i(x)) = \sum_{j=1}^{i} \mathrm{rank}(P_i^\circ(x)) \tag{7}$$

holds for all $1 \le i \le k$ and $x \in \mathbb{R}^n$ by [23, Lemma 1], the triangular structure of (3) ensures that whenever there are conflicts between outputs, higher priority outputs occupy the shared part of the control input.

B. Imperfect Input-Output Feedback Linearization

Let $p_{i:j} = p_i + \cdots + p_j$, $\rho_{i:j}(x) = \mathrm{rank}(P_i(x)) + \cdots + \mathrm{rank}(P_j(x))$, and $\Omega_{i:j} = \rho_{i:j}^{-1}(p_{i:j})$. Since $J(x)$ is assumed to be sufficiently smooth on $\mathbb{R}^n$, each $\Omega_{1:i} \subset \mathbb{R}^n$ is open and $\Omega_{1:k} \subset \Omega_{1:k-1} \subset \cdots \subset \Omega_{1:1}$. The motivation of studying the prioritized control problem is to extend the domain of the system (1) from $\Omega_{1:k}$ to $\Omega_{1:i}$ for some $i < k$. Since $J_{1:i}(x)$ has full rank on $\Omega_{1:i}$ by (7), we see that $L_{jj}(x) \in \mathbb{R}^{p_j\times p_j}$ has full rank for all $1 \le j \le i$ and $x \in \Omega_{1:i}$. It follows that if we let $\lambda_j(x) = 0$ in (6) for all $1 \le j \le i$ and $x \in \Omega_{1:i}$, we can establish linear input-output relations $y_j^{(r_j)} = v_j$ for $1 \le j \le i$ on $\Omega_{1:i}$, no matter what happens in lower priority outputs

$y_{i+1}, \dots, y_k$. While it appears that we have successfully addressed the prioritized control problem, letting $\lambda_{1:i}(x) = 0$ leads to divergence of the canonical prioritized control input (5) in the boundary of $\Omega_{1:i}$, rendering it impractical in many applications. Thus, we are forced to define positive damping functions $\lambda_1(x), \dots, \lambda_i(x)$ on $\Omega_{1:i}$ in many cases. However, positive damping functions make the linear input-output relations imperfect because the nonlinear terms $N_j^\circ(v_j - \kappa_j^\circ)$ remain. Although, we can compromise the nonliear terms with the control input by adjusting the damping functions, those trailing terms make the analysis of the imperfect input-output feedback linearization (6) nontrivial.

In this paper, we present our analysis results of strictly positive real transfer functions that are closely related to the nonlinear terms in (6). To see this, let h_{ij} be the j-th component of h_i, $\xi_{ij} = \mathrm{col}(L_f^0 h_{ij}(x), L_f^1 h_{ij}(x), \dots, L_f^{r_i-1} h_{ij}(x))$, and $\xi_i = \mathrm{col}(\xi_{i1}, \dots, \xi_{ip_i})$. Then, we can rewrite (6) as

$$\dot{\xi}_i = A_i \xi_i + B_i P_i^\circ(x) v_i + B_i N_i^\circ(x) \kappa_i^\circ(x, v_{1:i-1}) \tag{8a}$$

$$y_i = C_i \xi_i \tag{8b}$$

where (A_i, B_i, C_i) is the controllable canonical form representation of p_i chains of r_i integrators. (Indeed, (8) is the partial normal form of (1) applied by the canonical prioritized control input (5). We do not consider the internal dynamics of (1) in this paper.) Now, let $v_i = -K_i \xi_i$ with some feedback gain matrix $K_i \in \mathbb{R}^{p_i \times p_i r_i}$ for $1 \le i \le k$ and consider the simultaneous stabilization of the multiple outputs $y_1, \dots, y_k$. An important property of the canonical prioritized control input is that for every $1 \le i \le k$, every $x \in \mathbb{R}^n$, and every damping functions $\lambda_i(x) \in [0, \infty)$, there exists $\sigma_i \ge 0$ such that

$$M_i(x) \coloneqq P_i^\circ(x) - \sigma_i I_{p_i} = M_i(x)^T \ge 0. \tag{9}$$

Then, we may tackle the simultaneous output stabilization problem of (8) with the following closed loop system

$$\begin{aligned}\dot{\xi}_i = {} & \underbrace{(A_i - \sigma_i B_i K_i)\xi_i}_{\text{linear feedback}} + \underbrace{B_i(\sigma_i I_{p_i} - P_i^\circ(x))K_i \xi_i}_{\text{nonlinear feedback}} \\ & + \underbrace{B_i N_i^\circ(x) \kappa_i^\circ(x, v_{1:i-1})}_{\text{nonlinear interconnection}} \end{aligned} \tag{10a}$$

$$y_i = C_i \xi_i. \tag{10b}$$

Obviously, it is not always possible to stabilize all outputs simultaneously on $\Omega_{1:i}$ because $J(x)$ can be singular on $\Omega_{1:i}$. Thus, we stabilize only higher priority outputs $y_1, \dots, y_i$ on $\Omega_{1:i}$. For that purpose, we use the additional property of (9) that for every $1 \le i \le k$, every compact $\mathcal{C} \subset \Omega_{1:i}$, and every $\lambda_1(x), \dots, \lambda_i(x) \in [0, \infty)$, there exist $\sigma_1, \dots, \sigma_i > 0$ satisfying (9) on $\mathcal{C}$. The right-hand side of (10a) consists of three parts: linear feedback, nonlinear feedback, and nonlinear interconnection between outputs. Our strategy to the simultaneous output stabilization of $y_1, \dots, y_i$ on a compact set $\mathcal{C} \subset \Omega_{1:i}$ consists of two problems:

1) to find the feedback gain matrices $K_1, \dots, K_i$ that stabilize (10) without the nonlinear interconnection;
2) to find a stability condition when the nonlinear interconnection is added as a perturbation to the first problem.

In this paper, we focus on the first problem. Indeed, if we solve the first problem, then a simple way to handle the nonlinear interconnection is to assume that $N_1^\circ(x), \dots, N_i^\circ(x)$ are sufficiently small on $\mathcal{C}$. (This condition can be achieved by letting $\lambda_1(x), \dots, \lambda_i(x)$ sufficiently small.) However, if we want to find a tighter stability condition for the second problem, we need to use the structure of $\kappa_i^\circ(x, v_{1:i-1})$ that also depends on the internal dynamics of (1). The stability condition of the whole system including the nonlinear interconnection and the internal dynamics based on the result of this paper can be found in our recent work [26].

III. Main Result

We consider a linear system with linear and nonlinear feedbacks

$$\dot{\xi} = (A - \sigma BK)\xi + Bu \tag{11a}$$

$$y = K\xi \tag{11b}$$

$$u = -M(x)y \tag{11c}$$

where $r = r_1 + \dots + r_p$, $\xi = \mathrm{col}(\xi_1, \dots, \xi_p) \in \mathbb{R}^r$, $\xi_i \in \mathbb{R}^{r_i}$, $A = \mathrm{diag}(A_1, \dots, A_p) \in \mathbb{R}^{r \times r}$, $B = \mathrm{diag}(B_1, \dots, B_p) \in \mathbb{R}^{r \times p}$, $A_i = \begin{bmatrix} 0 & I_{r_i - 1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{r_i \times r_i}$, $B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{r_i}$, $\sigma > 0$, $K \in \mathbb{R}^{p \times r}$, $y, u \in \mathbb{R}^p$, and $M(x) = M(x)^T \ge 0$ on $\mathcal{C} \subset \mathbb{R}^n$. (We assume that a linear feedback $u' = -K\xi$ is already applied to the system $\dot{\xi} = A\xi + \sigma Bu' + Bu$ in (11a).) Note that (11) represents (10a) without the interconnection. (We dropped the index i in (10) in order to simplify the notation but considered different relative degrees for each component of y_i in (10) in order to make the main result more applicable.) Thus, if we find K that stabilizes ξ of (11), then $K_i = K$ will also stabilize the output y_i of (10) without the nonlinear interconnection term.

The transfer function of (11a)–(11b) from u to y given by

$$H(s) = K(sI_r - A + \sigma BK)^{-1} B \tag{12}$$

has a property that the output matrix is identical to the feedback gain matrix. Thus, poles and zeros of each entry of $H(s)$ are coupled by the matrix K. This unique characteristic of $H(s)$ requires additional efforts in the analysis compared to the usual case in which the transfer function is given as $C(sI_r - A + \sigma BK)^{-1} B$ with an output matrix C [29], [30], [31]. We present our analysis result of a strictly positive real $H(s)$ in Theorem 1. Let $A_c = A - \sigma BK$.

Theorem 1: For all $\sigma > 0$ there is $K \in \mathbb{R}^{p \times r}$ satisfying

(A) (A_c, B) is controllable,
(B) (A_c, K) is observable, and
(C) $H(s)$ is strictly positive real.

Once we find K satisfying Theorem 1, we can analyze stability of ξ in (11) by using the Kalman-Yakubovich-Popov (KYP) lemma [32, Lemma 6.3] along with the condition $M(x) = M(x)^T \ge 0$ on $\mathcal{C} \subset \mathbb{R}^n$ as follows. By Theorem 1 and the KYP lemma, there exist a positive constant c and matrices K, $P = P^T > 0$, and L such that

$$PA_c + A_c^T P = -L^T L - 2cP \tag{13a}$$

$$PB = K^T. \tag{13b}$$

Let $V(\xi) = \frac{1}{2}\xi^T P \xi$. Then, we have

$$\begin{aligned}\dot{V}(\xi) &= \frac{1}{2}\xi^T(PA_c + A_c^T P)\xi - \xi^T PBM(x)K\xi \\ &= \frac{1}{2}\xi^T(-L^T L - 2cP)\xi - \xi^T K^T M(x)K\xi \\ &\le -cV(\xi) \end{aligned} \tag{14}$$

for all $\xi \in \mathbb{R}^r$ and $x \in \mathcal{C}$.

We must clarify that (14) does not guarantee $\xi(t) \to 0$ as $t \to \infty$ for a solution $\xi(t)$ of (11) because the theorem holds only for $x \in \mathcal{C}$. Indeed, the system (11) does not include the evolution of x, so we need additional information. For example, if $x = \phi(\xi) \in \mathcal{C}$ holds for all $\xi \in \mathbb{R}^r$, then (14) guarantees $\xi(t) \to 0$ as $t \to \infty$. For the original problem (10), we need an internal dynamics $\dot{\eta} = f^\circ(\eta,\xi) + G^\circ(\eta,\xi)u$ along with a diffeomorphism $(\eta,\xi) = \Phi(x)$ and should analyze stability of (η,ξ) together [26]. As we stated earlier, we do not expand our discussion to this subject in this paper.

The proof of Theorem 1 to be presented in Section IV is constructive, listing up Conditions 1–9 that explicitly give a set of $K \in \mathbb{R}^{p\times r}$ satisfying Theorem 1. In the following, we summarize and restate the conditions in a way that additional information on the desired location of poles and zeros of $H(s)$ is pointed out.

(D1) Let $z_{ia} = \alpha_{ia} + j\beta_{ia} \in \mathbb{C}$ for $1 \le i \le p$ and $1 \le a \le r_i - 1$ be such that $\alpha_{ia} < 0$; $z_{ib} = \alpha_{ia} - j\beta_{ia}$ for some $1 \le b \le r_i - 1$; and

$$\varepsilon_{i0} = \min_{1\le a\le r_i-1} |\alpha_{ia}| > \max_{1\le a,b\le r_i-1} |z_{ia} - z_{ib}| = \varepsilon_{i1}. \tag{15}$$

(D2) Let $c_{ia} \in \mathbb{R}$ be such that

$$\prod_{a=1}^{r_i-1}(s - z_{ia}) = s^{r_i-1} + \sum_{a=1}^{r_i-1} c_{ia}s^{a-1} \tag{16}$$

and define $c_i = \max\{1, |c_{i1}|, \ldots, |c_{i,r_i-1}|\}$ and $d_i = (1 + \sqrt{1 + 1/c_i})/\sigma$.

(D3) Define constants $m_{ia}, M_{ia} \in (0,\infty)$ for $1 \le a \le 3$ as:

$$m_{i1} = \left(\frac{r-1}{\sum_{a=1}^{r_i-1}(1 + \beta_{ia}^2/\alpha_{ia}^2)^{(r-1)/2}}\right)^{1/(r-1)}$$

$$m_{i2} = \max_{1\le a\le r_i-1}\left(|\alpha_{ia}|\cot\frac{\pi}{4(r_i-1)} + |\beta_i|\right)$$

$$m_{i3} = \left(1 + \frac{3\sqrt{3}}{16}\right)\left(1 + \frac{\pi}{8}\right)\sum_{a=1}^{r_i-1}(|\alpha_{ia}| + |\beta_{ia}|)$$

$$M_{i1} = \max\{m_{i2}, m_{i3}\}/(\sigma m_{i1}) \tag{17a}$$

$$M_{i2} = (1 + \sigma c_i d_i)/(\sigma - 1/d_i) \tag{17b}$$

$$M_{i3} = (1 + \sigma c_i d_i)^2/(\sigma(\varepsilon_{i0} - \varepsilon_{i1})). \tag{17c}$$

(D4) Define $K \in \mathbb{R}^{p\times r}$ as

$$k_{ir_i} > \max\{M_{i1}, M_{i2}, M_{i3}\} \tag{18a}$$

$$k_{ia} = k_{ir_i}c_{ia} \quad (1 \le a \le r_i - 1) \tag{18b}$$

$$K_i = \mathrm{row}(k_{i1}, k_{i2}, \ldots, k_{ir_i}) \tag{18c}$$

$$K = \mathrm{diag}(K_1, K_2, \ldots, K_p). \tag{18d}$$

Theorem 2: For every $\sigma > 0$ and every $K \in \mathbb{R}^{p\times r}$ defined by (D1)–(D4), Items (A)–(C) of Theorem 1 hold. Also, if $p_{ia} \in \mathbb{C}$ for $1 \le i \le p$ and $1 \le a \le r_i$ satisfies

$$\prod_{a=1}^{r_i}(s - p_{ia}) = s^{r_i} + \sigma\sum_{a=1}^{r_i} k_{ia}s^{a-1},$$

then, without loss of generality, $|p_{ia} - z_{ia}| < \varepsilon_{i0}$ for $1 \le a \le r_i - 1$ and $p_{ir_i} + \sigma k_{ir_i} + \sum_{a=1}^{r_i-1} z_{ia} \to 0$ as $k_{ir_i} \to \infty$.

Although Theorem 2 gives a set of K satisfying Theorem 1, the condition on K is quite conservative. Thus, it can be beneficial for practical applications to have a numerical method that finds a larger set of K satisfying Theorem 1. The following corollary can be used for that purpose.

Corollary 3: Let $z_{ia} = \alpha_{ia} + j\beta_{ia} \in \mathbb{C}$ for $1 \le i \le p$ and $1 \le a \le r_i - 1$ be such that $\alpha_{ia} < 0$ and $z_{ib} = \alpha_{ia} - j\beta_{ia}$ for some $1 \le b \le r_i - 1$. Let $c_{ia} \in \mathbb{R}$ be as in (16). Then, there exists $k_{ir_i} > 0$ for $1 \le i \le p$ such that Theorem 1 holds for $K \in \mathbb{R}^{p\times r}$ defined by (18b)–(18d). Such k_{ir_i} can be found by checking (C1)–(C3).

We will show in Section IV that (C2) holds for all $k_{ir_i} > M_{i1}$ and (C3) holds for all $k_{ir_i} > M_{i0} = \sigma^{-1}\sum_{a=1}^{r_i-1}|\alpha_{ia}|$. The lower bound M_{i0} of k_{ir_i} for (C3) is tight (see Section IV-D) and M_{i1} for (C2) is somewhat conservative but acceptable. Since $M_{i1} > M_{i0}$, we may check (C1) and (C2) numerically by using M_{i0} as a minimum value of k_{ir_i} or only (C1) by using M_{i1}. We will discuss it in Section V.

IV. Proof of Main Result

A. Overview

For the proofs of Theorems 1–2 and Corollary 3, we will find Conditions 1–9 on the feedback gain matrix $K \in \mathbb{R}^{p\times r}$. Conditions 1–7 will be used to prove Theorem 1 and Corollary 3 and Conditions 1–9 will be used for Theorem 2.

Condition 1: Let $K_i = \begin{bmatrix} k_{i1} & \cdots & k_{ir_i}\end{bmatrix} \in \mathbb{R}^{1\times r_i}$ and $K = \mathrm{diag}(K_1, \ldots, K_p) \in \mathbb{R}^{p\times r}$.

Condition 2: Let $k_{i1} \ne 0$ for $1 \le i \le p$.

Since A, B, and K are block diagonal, $A_c = A - \sigma BK = \mathrm{diag}(A_{c1}, \ldots, A_{cp})$ and

$$A_{ci} = A_i - \sigma B_i K_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\sigma k_{i1} & -\sigma k_{i2} & \cdots & -\sigma k_{ir_i}\end{bmatrix}.$$

Also, $H(s) = \mathrm{diag}(H_1(s), \ldots, H_p(s))$ and

$$H_i(s) = \frac{\sum_{a=1}^{r_i} k_{ia}s^{a-1}}{s^{r_i} + \sigma\sum_{a=1}^{r_i} k_{ia}s^{a-1}}. \tag{19}$$

Thus, the proof is complete if we show that for every $1 \le i \le p$ and $\sigma > 0$ there exists $K_i \in \mathbb{R}^{1\times r_i}$ such that (A_{ci}, B_i) is controllable, (A_{ci}, K_i) is observable, and $H_i(s)$ is strictly positive real. Since $H_i(s)$ is a strictly proper transfer function and $H_i(0) = 1/\sigma > 0$ by Condition 2, $H_i(s)$ is strictly positive real if and only if [32, Lemma 6.1]

(C1) poles of $H_i(s)$ have negative real parts;
(C2) $\mathrm{Re}(H_i(j\omega)) > 0$ for all $\omega \in \mathbb{R}$;

(C3) either $H_i(\infty) > 0$ or $H_i(\infty) = 0$ and

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}(H_i(j\omega)) > 0.$$

Fix $1 \le i \le k$ and $\sigma > 0$. In the rest of the proof, we drop the index i in order to simplify the notation.

B. Proof of Items (A) and (B)

We can easily check that (A_c, B) is controllable from the controllability matrix

$$\begin{aligned}&\begin{bmatrix} B & A_cB & \cdots & A_c^{r-1}B\end{bmatrix}\\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & -\sigma k_r\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & -\sigma k_r & \cdots & (-\sigma k_r)^{r-2} & (-\sigma k_r)^{r-1}\end{bmatrix}.\end{aligned}$$

Since the observability matrix of (A, K) is given by

$$\begin{bmatrix} K\\ KA\\ \vdots\\ KA^{r-1}\end{bmatrix} = \begin{bmatrix} k_1 & k_2 & \cdots & k_r\\ 0 & k_1 & \cdots & k_{r-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & k_1\end{bmatrix},$$

Condition 2 implies that (A, K) is observable. By [33, Lemma 9.18] , there exists $M \in \mathbb{R}^r$ so that real or complex conjugate eigenvalues of $A - MK$ can be assigned arbitrarily. Let $N \in \mathbb{R}^r$ and $M = \sigma B + N$. Since $A_c - NK = A - MK$ and (A, K) is observable, [33, Lemma 9.18] implies that (A_c, K) is also observable.

C. Proof of (C2)

Condition 3: Let $k_r > 0$.

By Condition 3, we can consider the factorization of the polynomial

$$s^{r-1} + \sum_{i=1}^{r-1} \frac{k_i}{k_r} s^{i-1} = \prod_{i=1}^{r-1} (s - z_i) \tag{20}$$

and let $z_i = \alpha_i + j\beta_i \in \mathbb{C}$. By Condition 2, $z_i \ne 0$ for all $1 \le i \le r-1$. Then, we can rewrite (19) as

$$H(s) = \frac{1}{\sigma}\left[1 + \frac{s}{\sigma k_r}\prod_{i=1}^{r-1}\left(1 - \frac{\alpha_i + j\beta_i}{s}\right)^{-1}\right]^{-1}$$

for all $s \notin \{0, z_1, \ldots, z_{r-1}\}$.

Condition 4: Let $\alpha_i < 0$ for all $1 \le i \le r-1$.

By Condition 4, $j\omega \notin \{z_1, \ldots, z_{r-1}\}$. Let $\omega \in \mathbb{R} \setminus \{0\}$, $\gamma_i = 1 - \beta_i/\omega$, $\delta_i = \alpha_i/\omega$, and

$$P(j\omega) = 1 + \frac{j\omega}{\sigma k_r}\prod_{i=1}^{r-1}\frac{1}{\gamma_i + j\delta_i}. \tag{21}$$

Then, $H(j\omega) = 1/(\sigma P(j\omega))$ and

$$\mathrm{Re}(H(j\omega)) = \frac{\mathrm{Re}(P(j\omega))}{\sigma|P(j\omega)|^2} > 0 \iff \mathrm{Re}(P(j\omega)) > 0.$$

Let $\tau_i = |\gamma_i + j\delta_i|$, $\tau = \prod_{i=1}^{r-1}\tau_i$, $\theta_i = \angle(\gamma_i + j\delta_i) = \mathrm{atan2}(\delta_i, \gamma_i)$, and $\theta = \sum_{i=1}^{r-1}\theta_i$. Then, we have

$$\mathrm{Re}(P(j\omega)) = 1 + \frac{\omega\sin\theta(\omega)}{\sigma k_r \tau(\omega)}. \tag{22}$$

Since $\tau_i(\omega) \to 1$ and $\theta_i(\omega) \to 0$ as $\omega \to \pm\infty$, we expand $\sin\theta$ around $\theta = 0$ as

$$\sin\theta = \sum_{i=1}^{r-1} \mathrm{atan2}(\delta_i, \gamma_i) + r(\theta)$$

where $r(\theta) = -(\theta^2/2)\sin\theta_0$ for some $\theta_0 \in [-\theta, \theta]$ by the Taylor's theorem [34, Theorem 5.15]. Since $\gamma_i(\omega) \to 1$ and $\delta_i(\omega) \to 0$ as $\omega \to \pm\infty$, we expand $\mathrm{atan2}(\delta_i, \gamma_i)$ around $(\delta_i, \gamma_i) = (0, 1)$. Observe that if $|\omega| \ge |\alpha_i| + |\beta_i|$, then $\gamma_i > 0$ and $\delta_i \in [-\gamma_i, \gamma_i]$ such that $\delta_i/\gamma_i \in [-1, 1]$ and

$$\mathrm{atan2}(\delta_i, \gamma_i) = \arctan(\delta_i/\gamma_i) = \delta_i/\gamma_i + r_0(\delta_i/\gamma_i)$$

where $r_0(\delta_i/\gamma_i) = -(z/(1+z^2)^2)(\delta_i/\gamma_i)^2$ for some $z \in [-1, 1]$ by the Taylor's theorem. Therefore, we have

$$\omega\sin\theta = \sum_{i=1}^{r-1}\left(\frac{\alpha_i}{\gamma_i} + wr_0\left(\frac{\delta_i}{\gamma_i}\right)\right) + \omega r(\theta) \tag{23}$$

for all $|\omega| \ge \omega_0 = \max_{1\le i\le r-1}(|\alpha_i| + |\beta_i|)$ where

$$|r(\theta)| \le \theta^2/2 \tag{24a}$$

$$|r_0(\delta_i/\gamma_i)| \le (3\sqrt{3}/16)(\delta_i/\gamma_i)^2. \tag{24b}$$

If there exist constants $M_1 > 0$ and $M_2 < \infty$ satisfying $M_1 \le \inf_{\omega\ne 0}\tau(\omega)$ and $M_2 \ge \sup_{\omega\ne 0}|\omega\sin\theta(\omega)|$, we can find a lower bound of (22) as

$$\mathrm{Re}(P(j\omega)) \ge 1 - \frac{|\omega\sin\theta|}{\sigma k_r \tau} \ge 1 - \frac{M_2}{\sigma k_r M_1}. \tag{25}$$

Since $\lim_{\omega\to 0}\tau_i^2(\omega) = \infty$, $\lim_{\omega\to\pm\infty}\tau_i^2(\omega) = 1$, and

$$\frac{d\tau_i^2(\omega)}{d\omega} = \frac{2}{\omega^2}\left(\beta_i - \frac{\alpha_i^2 + \beta_i^2}{\omega}\right),$$

we have $\min_{\omega\ne 0}\tau_i(\omega) = |\alpha_i|/\sqrt{\alpha_i^2 + \beta_i^2}$ and

$$\min_{\omega\ne 0}\tau(\omega) \ge M_1 = \left(\frac{r-1}{\sum_{i=1}^{r-1}(1 + \beta_i^2/\alpha_i^2)^{(r-1)/2}}\right)^{1/(r-1)}$$

by Condition 4 and the inequality between the geometric mean and the harmonic mean. Let

$$\omega_1 = \max_{1\le i\le r-1}\left(|\alpha_i|\cot\frac{\pi}{4(r-1)} + |\beta_i|\right).$$

Obviously, $|\omega\sin\theta| \le \omega_1$ for all $0 < |\omega| < \omega_1$. By using (23), (24), and the fact that

$$|\theta| \le \sum_{i=1}^{r-1}\left|\arctan\frac{\alpha_i}{\omega - \beta_i}\right| \le \sum_{i=1}^{r-1}\arctan\frac{|\alpha_i|}{|\omega| - |\beta_i|} \le \frac{\pi}{4}$$

for all $|\omega| \ge \omega_1$, we can find an upper bound of $|\omega\sin\theta|$ as

$$\begin{aligned}|\omega\sin\theta| &\le \sum_{i=1}^{r-1}\left(\frac{|\alpha_i|}{1 - \frac{|\beta_i|}{|\alpha_i| + |\beta_i|}} + \frac{3\sqrt{3}}{16}(|\alpha_i| + |\beta_i|)\left|\frac{\delta_i}{\gamma_i}\right|\right)\\ &\quad + \frac{1}{2}\left(1 + \frac{3\sqrt{3}}{16}\right)|\theta|\sum_{i=1}^{r-1}(|\alpha_i| + |\beta_i|)\\ &\le \left(1 + \frac{3\sqrt{3}}{16}\right)\left(1 + \frac{\pi}{8}\right)\sum_{i=1}^{r-1}(|\alpha_i| + |\beta_i|) = M_2'\end{aligned}$$

on $|\omega| \geq \omega_1$. Since $|\omega \sin\theta| \leq M_2 = \max\{\omega_1, M_2'\} < \infty$ for all $\omega \neq 0$, (C2) follows from (25) and the following condition.

Condition 5: Let $k_r > M_2/(\sigma M_1)$.

D. Proof of (C3)

Since $H(s)$ is strictly proper, $H(\infty) = 0$. From (21),

$$\lim_{\omega\to\infty}\left|\frac{P(j\omega)}{\omega}\right| = \lim_{\omega\to\infty}\left|\frac{1}{\omega} + \frac{j}{\sigma k_r}\prod_{i=1}^{r-1}\frac{1}{\gamma_i + j\delta_i}\right| = \frac{1}{\sigma k_r}.$$

From (22), (23), and (24), we also have

$$\begin{aligned}&\lim_{\omega\to\infty}\mathrm{Re}(P(j\omega))\\ &= \lim_{\omega\to\infty}\left(1 + \frac{1}{\sigma k_r \tau}\sum_{i=1}^{r-1}\left(\frac{\alpha_i}{1-\beta_i/\omega} + \omega r_0\right) + \frac{\omega r}{\sigma k_r \tau}\right)\\ &= 1 + \frac{1}{\sigma k_r}\sum_{i=1}^{r-1}\alpha_i.\end{aligned}$$

Therefore, (C3) follows from

$$\begin{aligned}\lim_{\omega\to\infty}\omega^2\mathrm{Re}(H(j\omega)) &= \lim_{\omega\to\infty}\frac{1}{\sigma}\left|\frac{P(j\omega)}{\omega}\right|^{-2}\mathrm{Re}(P(j\omega))\\ &= \sigma k_r^2\left(1 + \frac{1}{\sigma k_r}\sum_{i=1}^{r-1}\alpha_i\right)\end{aligned}$$

and the next condition.

Condition 6: Let $k_r > \sigma^{-1}\sum_{i=1}^{r-1}(-\alpha_i)$.

E. Proof of (C1)

Consider the factorization of the denominator of (19) along with (20)

$$D(s) = s^r + \sigma k_r \prod_{i=1}^{r-1}(s - z_i) = \prod_{i=1}^{r}(s - p_i). \tag{26}$$

Since $p_i \neq 0$ for all $1 \leq i \leq r$ by Condition 2, we have

$$\frac{D(p_i)}{p_i^r} = 1 + \sigma k_r \frac{\prod_{a=1}^{r-1}(p_i - z_a)}{p_i^r} = 0 \tag{27}$$

for all $1 \leq i \leq r$ and $k_r > 0$. By the properties of the root loci [35, Chapter 7], (27) shows that without loss of generality $p_i \to z_i$ for $1 \leq i \leq r-1$ and $\mathrm{Re}(p_r) \to -\infty$ as $k_r \to \infty$. (Indeed, $p_r < 0$ for sufficiently large k_r; thus, we can safely write $p_r \to -\infty$.) By Condition 4, there exists $M_3 < \infty$ such that $\mathrm{Re}(p_i) < 0$ for all $1 \leq i \leq r$ and $k_r > M_3$. Therefore, (C1) follows from the next condition and it completes the proof of Theorem 1 and Corollary 3.

Condition 7: Let $k_r > M_3$.

F. Speed of Divergence of p_r

In Section IV-E, we showed $p_r \to -\infty$ as $k_r \to \infty$. We can also find the speed of divergence of p_r. By (26),

$$\frac{\sigma k_r}{p_r} = -\prod_{i=1}^{r-1}\left(1 - \frac{z_i}{p_r}\right)^{-1} \to -1 \quad (k_r \to \infty)$$

and

$$\begin{aligned}\frac{D(p_r)}{p_r^{r-1}} &= p_r + \sigma k_r\left(1 - \frac{z_1}{p_r}\right)\left(1 - \frac{z_2}{p_r}\right)\cdots\left(1 - \frac{z_{r-1}}{p_r}\right)\\ &= p_r + \sigma k_r - \frac{\sigma k_r}{p_r}(z_1 + \cdots + z_{r-1} + r'(p_r)) = 0\end{aligned}$$

where $r'(p_r) \to 0$ as $p_r \to -\infty$. Therefore,

$$p_r + \sigma k_r + \sum_{i=1}^{r-1} z_i = \left(1 + \frac{\sigma k_r}{p_r}\right)\sum_{i=1}^{r-1} z_i + \frac{\sigma k_r}{p_r} r'(p_r) \to 0 \tag{28}$$

as $k_r \to \infty$.

G. Finding M_3

Let $d_1, \ldots, d_r > 0$, $D = \mathrm{diag}(d_1, \ldots, d_r)$, and

$$\begin{aligned}M &= [m_{ab}] = (DA_cD^{-1})^T\\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -\sigma(d_r/d_1)k_1\\ d_1/d_2 & 0 & \cdots & 0 & -\sigma(d_r/d_2)k_2\\ 0 & d_2/d_3 & \cdots & 0 & -\sigma(d_r/d_3)k_3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & d_{r-1}/d_r & -\sigma k_r\end{bmatrix}.\end{aligned}$$

Obviously, the eigenvalues of M coincides with A_c, i.e., the poles $p_1, \ldots, p_r$. Let $d_i = k_r$ and $c_i = k_i/k_r$ for $1 \leq i \leq r-1$. The Gershgorin discs of M [36, Theorem 6.1.1] is given as

$$\mathcal{D}_i = \begin{cases}\{s \in \mathbb{C} : |s| \leq \sigma d_r |c_1|\}, & i = 1\\ \{s \in \mathbb{C} : |s| \leq 1 + \sigma d_r |c_i|\}, & 2 \leq i \leq r-1\\ \{s \in \mathbb{C} : |s + \sigma k_r| \leq k_r/d_r\}, & i = r.\end{cases}$$

Let $c_0 = \max\{1, |c_1|, \ldots, |c_{r-1}|\}$. Then,

$$\mathcal{D}_i \subset \mathcal{D}_0 = \{s \in \mathbb{C} : |s| \leq 1 + \sigma d_r c_0\}$$

for $1 \leq i \leq r-1$. By [36, Theorem 6.1.1] , if $d_r > 1/\sigma$ and $k_r > (1 + \sigma d_r c_0)/(\sigma - 1/d_r)$, then $D_r \cap \bigcup_{i=1}^{r-1}\mathcal{D}_i = \emptyset$ such that $p_1, \ldots, p_{r-1} \in \mathcal{D}_0$ and $p_r \in \mathcal{D}_r$. We can easily check that $d_r = (1 + \sqrt{1 + 1/c_0})/\sigma$ minimizes the following problem

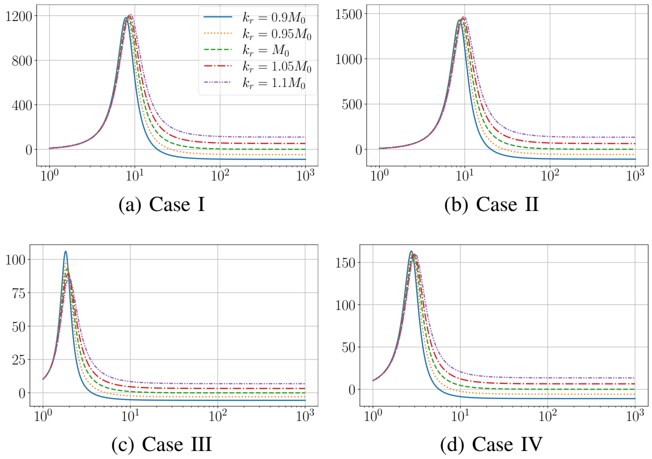
$$\min_{d > 1/\sigma}\frac{1 + \sigma c_0 d}{\sigma - 1/d} = \frac{1 + \sigma c_0 d_r}{\sigma - 1/d_r} = M_4.$$

Thus, if $k_r > M_4$, then $|p_i| \leq 1 + \sigma d_r c_0$ for $1 \leq i \leq r-1$ and $p_r < -\sigma d_r c_0 < 0$.

Condition 8: Let $M_3 \geq M_4$ and

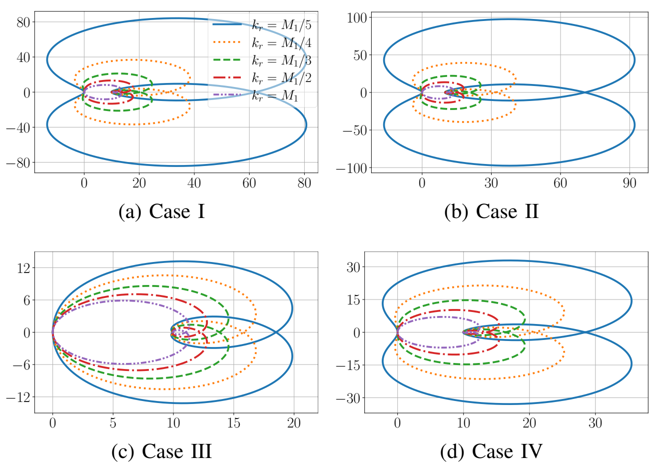
$$\varepsilon_0 = \min_{1 \leq i \leq r-1}|\mathrm{Re}(z_i)| > \max_{1 \leq a,b \leq r-1}|z_a - z_b| = \varepsilon_1.$$

Since $|p_i - z_i| < \varepsilon_0$ implies $\mathrm{Re}(p_i) < 0$, if we find an upper bound of k_r for the condition $|p_i - z_i| \geq \varepsilon_0$, then every k_r greater than that bound will guarantee $\mathrm{Re}(p_i) < 0$. From (26), Conditions 7 and 8, and the inequality between



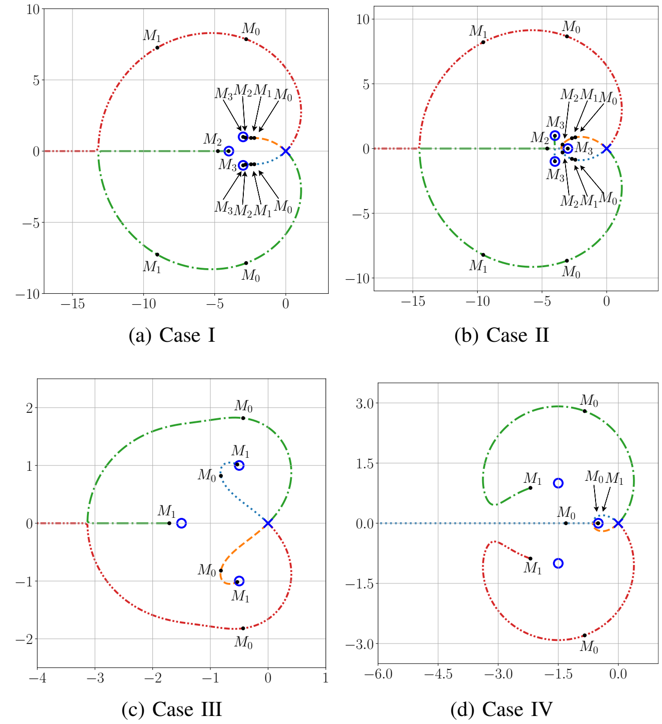
(a) Case I (b) Case II (c) Case III (d) Case IV

Fig. 1. The plots show variations of $\omega^2\mathrm{Re}(H(j\omega))$ for different values of k_r. The horizontal axis is ω and the vertical axis is $\omega^2\mathrm{Re}(H(j\omega))$.



(a) Case I (b) Case II (c) Case III (d) Case IV

Fig. 2. The plots show $H(j\omega)$ on the complex plane for different values of k_r. The horizontal axis is $\mathrm{Re}(H(j\omega))$ and the vertical axis is $\mathrm{Im}(H(j\omega))$.



(a) Case I (b) Case II (c) Case III (d) Case IV

Fig. 3. The plots show the root loci of the characteristic equation (29) with respect to $k_r \in (0, M_3]$ for Cases I and II and $k_r \in (0, M_1]$ for Cases III and IV. The blue X marks are the locations of the poles of $R(s)$ ($k_r = 0$) and the blue O marks are the locations of zeros of $R(s)$ ($k_r = \infty$). The horizontal axis is the real part and the vertical axis is the imagenary part.

TABLE I
VARIOUS LOWER BOUNDS OF k_r FOR THEOREM 2 AND COROLLARY 3

	M_0	M_1	M_2	M_3
Case I	100	229.5	1620	66420
Case II	110	244.8	2060	67550
Case III	25	164.6	N/A	N/A
Case IV	35	115.9	N/A	N/A

the geometric mean and the harmonic mean, we have

$$\begin{aligned} k_r &= \frac{|p_i|}{\sigma}\left(\prod_{a=1}^{r-1}\frac{|p_i - z_a|^{r-1}}{|p_i|^{r-1}}\right)^{-1/(r-1)} \\ &\leq \frac{|p_i|^2}{\sigma}\left(\frac{1}{r-1}\sum_{a=1}^{r-1}\frac{1}{(|p_i - z_i| - |z_i - z_a|)^{r-1}}\right)^{1/(r-1)} \\ &\leq \frac{(1+\sigma d_r c_0)^2}{\sigma(\varepsilon_0 - \varepsilon_1)} = M_5 \end{aligned}$$

for every $|p_i - z_i| \geq \varepsilon_0$. Therefore, the following condition completes the proof of Theorem 2.

Condition 9: Let $M_3 = \max\{M_4, M_5\}$.

V. Numerical Examples

In this section, we verify the analysis results presented in Section III numerically. Specifically, we show how Corollary 3 can be used to find a matrix K that makes the transfer function (12) strictly positive real and compare the results with Theorem 2. As we discussed in Section IV-A, if we assume Conditions 1 and 2, then we only need to check if (19) satisfies (C1)–(C3) for each i. Thus, we restrict our discussion to a scalar transfer function (19). (Also, we omit the index i in order to simplify the notation.) Let $r = 4$ and $\sigma = 0.1$. We need to choose zeros $z_i = \alpha_i + j\beta_i \in \mathbb{C}$ for $1 \leq i \leq 3$, real or complex conjugate pairs, satisfying $\alpha_i < 0$. We consider the following four cases:

- Case I: $z_1 = -4$, $z_2 = -3 + j$, $z_3 = -3 - j$
- Case II: $z_1 = -3$, $z_2 = -4 + j$, $z_3 = -4 - j$
- Case III: $z_1 = -1.5$, $z_2 = -0.5 + j$, $z_3 = -0.5 - j$
- Case IV: $z_1 = -0.5$, $z_2 = -1.5 + j$, $z_3 = -1.5 - j$

where Cases I and II satisfy the condition (15), while Cases III and IV do not. We show various lower bounds of k_r for Theorem 2 and Corollary 3 in Table I. The lower bounds M_2 and M_3 are not available for Cases III and IV because (15) does not hold.

We can easily check the condition (C3) by plotting $\omega^2\mathrm{Re}(H(j\omega))$ as shown in Figure 1. It is clearly seen that the lower bound of k_r for (C3) given by $M_0 = \sigma^{-1}\sum_{i=1}^{r-1}|\alpha_i|$ is tight. We can also check the condition (C2) by plotting $H(j\omega)$ on the complex plane and then by checking if $H(j\omega)$

is on the right half-plane as shown in Figure 2. We see that the lower bound M_1 of k_r for (C2) given by Theorem 2 is a bit conservative, but not too much except for the Case III in which $k_r = M_1/5$ also satisfies (C2). In order to check (C1), we draw the root loci of the characteristic equation

$$1 + kR(s) = 1 + \sigma k_r \frac{\prod_{i=1}^{r-1}(s - z_i)}{s^r} = 0 \qquad (29)$$

for $k = \sigma k_r > 0$ as shown in Figure 3. We can observe that the poles p_i for $1 \leq i \leq 4$ of (19) starts from 0 when $k_r = 0$ and, as $k_r \to \infty$, three of them converge to zeros and one of them diverges to $-\infty$. We also observe from Cases I and II that the lower bound $\max\{M_2, M_3\}$ of k_r for (C1) in Theorem 2 is quite conservative. Indeed, Cases III and IV that do not satisfy (15) require smaller values of k_r to locate the poles near the zeros. Therefore, although Theorem 2 gives a specific set of K satisfying Theorem 1, the numerical approach based on Corollary 3 is recommended for the practical applications.

VI. Conclusion

We proposed a method based on the Kalman-Yakubovich-Popov lemma that compensates the nonlinear feedback term of the imperfect input-output feedback linearization in the prioritized control problem. In order to realize this method, we proved existence of a feedback gain matrix that gives a strictly positive real transfer function in Theorem 1 and found a set of such matrices in Theorem 2. We also provided a way to find a larger set of feedback gain matrices numerically in Corollary 3 and validated the result numerically.

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