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Distributed Generalized Nash Equilibrium Seeking for Constrained N-Cluster Games with Second-Order Dynamics

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Abstract— **This paper studies N-cluster games with second-order dynamics, wherein the players' decisions are restricted by local set constraints and nonlinear coupled inequality constraints. The presence of second-order dynamics coupled with constraints leads to difficulties in the design and analysis of generalized Nash equilibrium (GNE) seeking algorithms, since it may be impossible to directly determine the decisions of players based on their control inputs. To facilitate the autonomous execution of N-cluster game tasks through second-order players, by employing state feedback, projection, primal-dual, dynamic average consensus, and passivity methods, a distributed algorithm is proposed to find the variational GNE of the studied games, under which the players' decisions can satisfy the set constraints all the time. Additionally, the algorithm's convergence is rigorously analyzed, and its efficacy is validated by a simulation example.**

I. INTRODUCTION

Resource allocations and non-cooperative games reflect the collaborative and competitive characteristics among agents, respectively, with applications across various fields like smart grids and sensor networks [1], [2]. In resource allocations, agents work together to minimize a collective cost function [3]. In non-cooperative games, each player strives to minimize their own cost function selfishly [4]. Noteworthily, the simultaneous presence of collaboration and competition among agents is common in fields like smart grids and multi-party politics [2], [5], and this complex interaction can be effectively modeled by N-cluster games.

In N-cluster games, multiple clusters exist, with each cluster containing several players. Players in the same cluster collaborate to minimize this cluster's cost, i.e., the sum of the cost of all players within the same cluster, by competing against other clusters. The inter-cluster competition persists until a Nash equilibrium (NE) is reached, where no cluster can unilaterally reduce its cost by altering its strategy. Recently, various algorithms have been developed to find the NE or generalized NE (GNE) of N-cluster games. For example, a subgradient-based algorithm was introduced in [6] to handle non-smooth N-cluster games. To mitigate communication and computation costs, the authors in [7] introduced an interference graph-based algorithm. However, these algorithms require that each player has full access to other players' decisions, which is often impractical in engineering scenarios due to privacy concerns. Thus, several distributed algorithms based on partial-decision information have been devised. For instance, a projection-based algorithm for N-cluster games with local constraints was designed in [8]. An average consensus-based algorithm was developed in [9] for N-cluster games with consistency constraints. A finite-time consensus-based algorithm was presented in [10] for N-cluster games with local and linear equality coupled constraints.

In engineering practices, various physical systems like vehicles [11] and generators [2], can be effectively characterized by second-order dynamics. Additionally, constraints are frequently encountered in physical systems in consideration of security, capacity, and inherent physical limitations. Hence, to enable autonomous execution of distributed tasks by physical systems, it is imperative to account for both constraints and dynamics in the design of distributed algorithms, as done in [2], [12]. Motivated by above discussions, this paper aims to investigate N-cluster games with second-order dynamics, local and nonlinear coupled inequality constraints.

The contributions of this paper are:

- Compared with the N-cluster games investigated by [6]–[10], players' dynamics are additionally considered in this paper. The cost functions and constraints are more general than those in [2], [4], [12]. Furthermore, unlike the N-cluster games studied in [6], [10], which assumed that each player's cost function is unaffected by the decisions of other players in the same cluster, while the cost functions of players considered in this paper rely on the decisions of all players. This formulation is more comprehensive and accurately captures the collaborative characteristics within clusters.
- A distributed variational GNE (vGNE) seeking algorithm is proposed in this paper for constrained N-cluster games with second-order dynamics. Compared with the full-decision information-based algorithms [6], [7], [12], only local information is required in the proposed algorithm. Additionally, to guarantee that the decisions of players satisfy set constraints all the time, the projection method used in this paper avoids the nonsmooth analysis and simplifies the convergence analysis in the meantime, while the barrier function approach [12] may not.
- Based on the Lyapunov stability theory and LaSalle's invariance principle, the designed algorithm is rigor-

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TABLE I: Notations

Symbols	Explanations
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegtive real numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\nabla f(\cdot)$	the gradient of a function $f(\cdot)$
$\times$ or $\prod$	Cartesian product
$\otimes$	Kronecker product
$[n]$	$\{1, 2, \dots, n\}$ for positive integer n
$\mathrm{diag}\{(k_i)_{i\in[n]}\}$	a diagonal matrix with $k_1, \dots, k_n$ being its principal diagonal elements
$\mathrm{col}((x_i)_{i\in[n]})$	$[x_1^T, \dots, x_n^T]^T$
$\|x\|_2$ or $\|x\|$	the Euclidean norm of a vector x
$\|X\|$	the spectral norm of matrix X
X^T	the transpose of matrix X
1_n	an n-dimensional vector with all its elements being 1
0_n	an n-dimensional vector with all its elements being 0
I_n	the $n \times n$ identity matrix
$\lambda_{\min}(X)$	the smallest eigenvalue of matrix X
$\lambda_2(X)$	the second smallest eigenvalue of matrix X

ously analyzed and can drive all players' decisions to the exact vGNE of N-cluster games, instead of merely approaching the neighborhood of the vGNE [12].

The structure of this paper is as follows: Section II provides the preliminaries. Section III introduces the problem formulation. In Section IV, a distributed vGNE-seeking algorithm and its convergence analysis are given. Section V is a simulation example. Finally, Section VI concludes the paper.

II. Preliminaries

This section presents some preliminaries and clarifies the notations used in this paper, as detailed in Table I.

A. Graph Theory

Denote by $\mathcal{G} := \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ as an undirected graph, with $\mathcal{V} := [N]$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = [a_{ij}]_{N\times N}$ being the vertex set, the edge set, and the adjacency matrix, respectively. Moreover, a_{ij} is the weight of $\{i, j\}$. Specifically, $a_{ij} > 0$ if $\{i, j\} \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. For a connected undirected graph $\mathcal{G}$, its Laplacian matrix L is symmetric (i.e., $L^T = L$) and satisfies $L1_N = 0_N$, which indicates that $\lambda_{\min}(L) = 0$ and $\lambda_2(L) > 0$. More details can be found in [13].

B. Projection Operator

Given that C is a nonempty closed convex set, the following results hold.

Denote by $\mathcal{N}_C(x) = \{y : \langle y, v - x\rangle \le 0, \forall v \in C\}$ the normal cone to C at x, and by $\mathcal{T}_C(x) = \{z \in \mathbb{R}^n \mid z^T y \le 0, \forall y \in \mathcal{N}_C(x)\}$ the tangent cone to C at x. The projection of x to C is $\mathcal{P}_C(x) = \arg\min_{y\in C} \|x - y\|$. Moreover, for any $x \in C$ and $y \in \mathbb{R}^n$, the projection of y over set C at the point x is defined as $\Pi_C(x, y) := \mathcal{P}_{\mathcal{T}_C(x)}(y) = \lim_{\delta\to 0^+} \frac{\mathcal{P}_C(x+\delta y)-x}{\delta}$ [14]. Two useful lemmas are given below.

Lemma 1: [15, Lemma 2.11]: For a closed convex set $C \subseteq \mathbb{R}^n$, one has $\langle x - \mathcal{P}_C(x), \mathcal{P}_C(x) - y\rangle \ge 0, \forall x \in \mathbb{R}^n, \forall y \in C$.

Lemma 2: [4, Lemma 1]: Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $x, y \in C$, then $(x - y)^T \Pi_C(x, z) \le (x - y)^T z, \forall z \in \mathbb{R}^n$.

III. Problem Formulation

Consider an N-cluster game composed of N clusters. Cluster $j \in [N]$ contains n_j players with second-order dynamics. The second-order dynamics of player $i \in [n_j]$ in cluster $j \in [N]$ can be described as:

$$\begin{cases} \dot{x}_i^j = v_i^j \\ \dot{v}_i^j = u_i^j \end{cases} \tag{1}$$

where u_i^j and $x_i^j \in \mathbb{R}$ are the control input and the decision of player i in cluster j, respectively. Let $x := \mathrm{col}(x^j, x^{-j})$ denote the decisions of all players, where $x^j := \mathrm{col}((x_i^j)_{i\in[n_j]})$ is cluster j's decision, and $x^{-j} := \mathrm{col}((x^r)_{r\in[N]\setminus\{j\}})$ represents the decision of all clusters except cluster j. The feasible decision set of x is $\Lambda := \{x \in \mathbb{R}^n \mid h(x) \le 0, x \in C\}$, where $C := \prod_{j=1}^N C^j$, $C^j := \prod_{i=1}^{n_j} C_i^j$ with $C_i^j \subseteq \mathbb{R}$ being the local decision constraint of player i in cluster j, $h(x) := \sum_{j\in[N]} \sum_{i\in[n_j]} h_i^j(x_i^j)$ with $h_i^j : \mathbb{R} \to \mathbb{R}$ being a nonlinear function, and $n := \sum_{j\in[N]} n_j$. Player i in cluster j is equipped with a cost function $f_i^j(x^j, x^{-j})$, and cluster j's cost function is $f^j(x^j, x^{-j}) := \sum_{i\in[n_j]} f_i^j(x^j, x^{-j})$. This N-cluster game can be denoted as $\Gamma(\mathcal{N}, f, \Lambda)$ with $\mathcal{N} := [N]$ and $f := \{f^1, \dots, f^N\}$. The aim of each cluster $j \in [N]$ in $\Gamma(\mathcal{N}, f, \Lambda)$ is to selfishly minimize its cost function f^j, i.e., to seek the GNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$, which is defined as:

Definition 1: [16]: The decision $x^* := \mathrm{col}((x^{j*})_{j\in[N]})$ is a GNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ if

$$x^{j*} \in \arg\min_{x^j} f^j(x^j, x^{-j*}),\ \mathrm{s.t.}(x^j, x^{-j*}) \in \Lambda,\ \forall j \in [N].$$

To proceed, some mild assumptions for N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ are given below.

Assumption 1: There exists $x \in C$ such that $h(x) < 0$ holds.

Assumption 2: For each $j \in [N]$, $i \in [n_j]$, C_i^j is a nonempty, compact, and convex set. $h_i^j(x_i^j)$ is convex and twice continuously differentiable. $f_i^j(x^j, x^{-j})$ is twice continuously differentiable and convex in x^j for any fixed x^{-j}. Besides, $\nabla_{x_i^j} f_k^j(x) := \frac{\partial f_k^j(x)}{\partial x_i^j}$ is Lipschitz continuous in x, i.e., there exists $l > 0$ such that $\|\nabla_{x_i^j} f_k^j(x) - \nabla_{x_i^j} f_k^j(x')\| \le l\|x - x'\|, \forall x, x' \in \mathbb{R}^n$.

Assumption 3: $\mathcal{F}(x)$ is strongly monotone in x, i.e., $\forall x, y \in C$, there exists $\mu > 0$ such that $(x - y)^T(\mathcal{F}(x) - \mathcal{F}(y)) \ge \mu\|x - y\|^2$, where $\mathcal{F}(x) := \mathrm{col}((\nabla_{x_i^j} f^j(x^j, x^{-j}))_{j\in[N], i\in[n_j]})$.

Assume that all players communicate with each other through undirected graph $\mathcal{G}^0$, and players in cluster j communicate with each other through undirected graph $\mathcal{G}^j$. Moreover, for each $j \in [N]$, $\mathcal{G}^j$ is a subgraph of $\mathcal{G}^0$.

Assumption 4: $\mathcal{G}^0, \mathcal{G}^1, \ldots, \mathcal{G}^N$ are undirected and connected graphs.

Suppose that Assumptions 1 and 2 hold, according to [16, Theorem 4.6], x^* is a GNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ iff, $\forall j \in [N]$, there exists Lagrange multipliers $\vartheta^{j*} \in \mathbb{R}$ such that the classical KKT conditions for N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$

$$\begin{aligned} &0_{n_j} \in \nabla_{x^j} f^j(x^*) + \nabla_{x^j} h(x^*)^T \vartheta^{j*} + \mathcal{N}_{C^j}(x^{j*}) \\ &h(x^*) \leq 0,\ \vartheta^{j*} h(x^*) = 0,\ \vartheta^{j*} \geq 0 \end{aligned} \tag{2}$$

are satisfied.

From (2), one can observe that the Lagrange multipliers may be different for each cluster for the same coupled constraint, which indicates that system (2) is ill-posed. Thus, this paper focus on vGNE, which is a special GNE that satisfies (2) with the same Lagrange multiplier, i.e., $\vartheta^{j*} = \vartheta^{k*} = \bar{\vartheta}^*$ for all $j, k \in [N]$. It is worth mentioning that the vGNEs of N-cluster games enjoy economic justifiability and computational convenience (see [17]). This paper aims to propose a distributed algorithm for second-order players to seek the vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$.

Remark 1: Assumptions 1-4 are standard in distributed games, which are widely made in [2], [4], [7]–[9]. Assumption 1 implies that the Slater's is satisfied. Assumption 2 guarantees the existence of the vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ (see [18, Proposition 2.2]). Under Assumption 3, the strong monotonicity of $\mathcal{F}(x)$ indicates that there exists a unique vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ (see [19, Theorem 2.3.3]).

Remark 2: N-cluster game considers both cooperation and competition among players, while only competition or cooperation among players was discussed in [1], [3], [4]. Moreover, N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$ is more general than related problems investigated in [2], [4], [6]–[10], [12], since it considers not only strategy constraints but also the second-order dynamics of players. Furthermore, in $\Gamma(\mathcal{N}, f, \Lambda)$, the cost function of every player relies on the decisions of all players, while the cost function of every player in [6], [10] only depends on its own decision and the decisions of other clusters, which implies that N-cluster games studied in this paper can characterize the cooperation within clusters better.

IV. Main Results

A section proposes a distributed vGNE-seeking algorithm and provides a rigorous analysis of its convergence.

A. Algorithm Design

The distributed vGNE seeking algorithm for player i in cluster j is designed in Algorithm 1.

Algorithm 1 : Fully Distributed vGNE Seeking Algorithm

Initialization: For any $k, i \in [n_j]$, $j \in [N]$, set $x_i^j(0), \tilde{x}_i^j(0) \in C_i^j$; $\sum_{k\in[n_j]} \eta_{k,i}^j(0) = 0$; $\alpha > \frac{c}{2}$, $\beta > \frac{2+(c+2)\mu}{2\mu\lambda_2(\mathbb{L})}$, $\gamma > \frac{4(1+\sqrt{n})^2 l_1^2 + 4\mu l_1 + \mu}{4\mu\lambda_2(L^0)}$.

Gradients Estimation: The dynamic average consensus algorithm is employed for player i in coalition j to estimate $\frac{1}{n_j}\sum_{k=1}^{n_j} \nabla_{x_i^j} J_k^j(\hat{x}^{j\_k})$ by $\psi_{i,i}^j$:

$$\dot{\eta}_{k,i}^j = -\beta \sum\nolimits_{l\in\mathcal{N}_k^j} (\psi_{k,i}^j - \psi_{l,i}^j) \tag{3a}$$

$$\psi_{k,i}^j = \eta_{k,i}^j + n_j \nabla_{x_i^j} J_k^j(\hat{x}^{j\_k}),\ \eta_{k,i}^j(0) = 0 \tag{3b}$$

where $j\_k = \sum_{l=0}^{j-1} n_l + k$ with $j \in [N]$, $k \in [n_j]$ and $n_0 = 0$, which is related to player k in coalition j; $\hat{x}^{j\_k} = \mathrm{col}((\hat{x}_{r\_s}^{j\_k})_{r\in[N], s\in[n_r]})$ with $\hat{x}_{j\_k}^{j\_k} = \tilde{x}_k^j$ and $\hat{x}_{r\_s}^{j\_k}$ being the estimate of player k in cluster j on $\tilde{x}_s^r$ of player s in cluster r for any $j\_k \neq r\_s$; $\tilde{x} = \mathrm{col}((\tilde{x}_i^j)_{j\in\mathcal{I}^0, i\in\mathcal{I}^j})$ with $\tilde{x}_i^j := x_i^j + v_i^j$; $\mathcal{N}_k^j$ is the neighboring set of player k in coalition j in graph $\mathcal{G}^j$.

Update of Lagrangian Multipliers: According to KKT conditions (2), the Lagrangian multipliers ϑ_i^j are updated as follows:

$$\dot{\vartheta}_i^j = -\vartheta_i^j + \tilde{\vartheta}_i^j \tag{4a}$$

$$\dot{\lambda}_i^j = -\sum\nolimits_{r\_s\in\mathcal{N}_{j\_i}^0} (\tilde{\vartheta}_i^j - \tilde{\vartheta}_s^r) \tag{4b}$$

where $\tilde{\vartheta}_i^j = \mathcal{P}_{\mathbb{R}_+}(\vartheta_i^j + h_i^j(\tilde{x}_i^j) + \sum_{r\_s\in\mathcal{N}_{j\_i}^0} (\lambda_i^j - \lambda_s^r) + \dot{\lambda}_i^j)$.

Design of Control Laws u_i^j: The control of player i in coalition j is based on the following control law

$$\begin{aligned} u_i^j = \Pi_{C_i^j}\Big(&\tilde{x}_i^j, -\alpha v_i^j - \psi_{i,i}^j - \nabla h_i^j(\tilde{x}_i^j)^T \tilde{\vartheta}_i^j \\ &- \gamma \sum\nolimits_{r\_s\in\mathcal{N}_{j\_i}^0} (\hat{x}_{j\_i}^{j\_i} - \hat{x}_{j\_i}^{s\_r})\Big) - v_i^j \end{aligned} \tag{5}$$

$$\dot{\hat{x}}_{-j\_i}^{j\_i} = -\gamma \sum\nolimits_{r\_s\in\mathcal{N}_{j\_i}^0} (\hat{x}_{-j\_i}^{j\_i} - \hat{x}_{-j\_i}^{r\_s}) \tag{6}$$

where $\hat{x}^{j\_i} = (\hat{x}_{j\_i}^{j\_i}, \hat{x}_{-j\_i}^{j\_i}) = \mathrm{col}((\hat{x}_{r\_s}^{j\_i})_{r\in[N], s\in[n_r]})$, $\mathcal{N}_{j\_i}^0$ is the neighboring set of player i in cluster j in graph $\mathcal{G}^0$, $\mathbb{L} := \mathrm{diag}\{(I_{n_j} \otimes L^j)_{j\in[N]}\}$ with L^j being the Laplacian matrix of graph $\mathcal{G}^j$, and L^0 is the Laplacian matrix of graph $\mathcal{G}^0$.

Remark 3: When simultaneously considering the dynamics and set constraints of players, the algorithms in [4] can only guarantee that set constraints are asymptotically satisfied, and the barrier function-based algorithm in [12] requires that the set constraints possess piecewise-smooth boundaries. Due to the employment of barrier function method, the vGNE sought by the algorithm proposed in [12] may not be the vGNE of the original game, as vGNE may lie in the constraint boundaries. Furthermore, using barrier functions necessitates nonsmooth analysis, which complicates the convergence analysis of the algorithm. In contrast, in the scenario with more general cost functions and constraints, Algorithm 1 not only keeps the players' decisions always in the set constraints, but also can drive the players' decisions to the exact vGNE of the studied game.

Remark 4: In comparison with the algorithms presented in [6], [7], [12], Algorithm 1 is under the partial-decision information and does not need decisions and gradient-related information of all players. Instead, learning strategies are applied to estimate these factors.

B. Convergency Analysis

The convergence of Algorithm 1 is analyzed in this subsection. By (1), Algorithm 1 can be rewritten as the following compact form:

$$\dot{x} = v \tag{7a}$$

$$\dot{\hat{x}} = \mathcal{R}^T \Pi_C(\tilde{x}, -\alpha v - \Psi - \mathbb{H}(\tilde{x})^T \tilde{\vartheta} \tag{7b}$$

$$- \gamma \mathcal{R}(L^0 \otimes I_n)\hat{x}) - \gamma \mathcal{S}^T \mathcal{S}(L^0 \otimes I_n)\hat{x} \tag{7c}$$

$$\dot{\eta} = -\beta \mathbb{L}\psi \tag{7d}$$

$$\psi = \eta + \mathbb{F}(\hat{x}), \ \eta(0) = 0_{\sum_{j=1}^N n_j^2} \tag{7e}$$

$$\dot{\vartheta} = -\vartheta + \mathcal{P}_{\mathbb{R}^n_+}(\vartheta + \boldsymbol{h}(\tilde{x}) + L^0\lambda + \dot{\lambda}) \tag{7f}$$

$$\dot{\lambda} = -L^0 \tilde{\vartheta} \tag{7g}$$

where $x := \mathrm{col}((x^j)_{j\in[N]})$, $x^j := \mathrm{col}((x_i^j)_{i\in[n_j]})$, $v := \mathrm{col}((v^j)_{j\in[N]})$, $v^j := \mathrm{col}((v_i^j)_{i\in[n_j]})$, $\hat{x} := \mathrm{col}((\hat{x}^j)_{j\in[N]})$, $\hat{x}^j := \mathrm{col}((\hat{x}^{j\text{-}i})_{i\in[n_j]})$, $\hat{x}^{j\text{-}i} := \mathrm{col}((\hat{x}^{j\text{-}i}_{r\text{-}s})_{r\in[N],s\in[n_r]})$, $\mathcal{R} := \mathrm{diag}\{(\mathcal{R}_i^j)_{j\in[N],i\in[n_j]}\}$, $\mathcal{R}_i^j := [0^T_{j\text{-}i-1} \ 1 \ 0^T_{n-j\text{-}i}]$, $\mathcal{R}\hat{x} := \tilde{x} = x + v$, $\tilde{x} := \mathrm{col}((\tilde{x}^j)_{j\in[N]})$, $\tilde{x}^j := \mathrm{col}((\tilde{x}_i^j)_{i\in[n_j]})$, $\Psi := \mathrm{col}((\psi^j_{i,i})_{j\in[N],i\in[n_j]})$, $\psi := \mathrm{col}((\psi^j)_{j\in[N]})$, $\psi^j := \mathrm{col}((\psi^j_{k,i})_{i\in[n_j],k\in[n_j]})$, $\mathbb{H}(\tilde{x}) := \nabla h(\tilde{x}) = \mathrm{diag}\{(\nabla h_i^j(\tilde{x}_i^j))_{j\in[N],i\in[n_j]}\}$, $\vartheta := \mathrm{col}((\vartheta^j)_{j\in[N]})$, $\vartheta^j := \mathrm{col}((\vartheta_i^j)_{i\in[n_j]})$, $\tilde{\vartheta} := \mathrm{col}((\tilde{\vartheta}^j)_{j\in[N]})$, $\tilde{\vartheta}^j := \mathrm{col}((\tilde{\vartheta}_i^j)_{i\in[n_j]})$, $\eta := \mathrm{col}((\eta^j)_{j\in[N]})$, $\eta^j := \mathrm{col}((\eta^j_{k,i})_{i\in[n_j],k\in[n_j]})$, $\lambda := \mathrm{col}((\lambda^j)_{j\in[N]})$, $\lambda^j := \mathrm{col}((\lambda_i^j)_{i\in[n_j]})$, $\boldsymbol{h}(\tilde{x}) := \mathrm{col}((\boldsymbol{h}^j(\tilde{x}^j))_{j\in[N]})$, $\boldsymbol{h}^j(\tilde{x}^j) := \mathrm{col}((h_i^j(\tilde{x}_i^j))_{i\in[n_j]})$, $\mathbb{F}_i^j(\hat{x}^j) := \mathrm{col}((\nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k}))_{k\in[n_j]})$, $\mathbb{F}(\hat{x}) := \mathrm{col}((\mathbb{F}_i^j(\hat{x}^j))_{j\in[N],i\in[n_j]})$, $\mathcal{S} := \mathrm{diag}\{(\mathcal{S}_i^j)_{j\in[N],i\in[n_j]}\}$,

$$\mathcal{S}_i^j = \begin{bmatrix} I_{(j\text{-}i)-1} & 0^T_{(j\text{-}i)-1} & 0_{((j\text{-}i)-1)\times(n-j\text{-}i)} \\ 0_{(n-j\text{-}i)\times((j\text{-}i)-1)} & 0^T_{(n-j\text{-}i)} & I_{(n-j\text{-}i)} \end{bmatrix}.$$

First, the equilibrium point (EP) of Algorithm 1 is analyzed, leading to the following result.

Lemma 3: Under Assumptions 1, 2 and 4, consider the N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$. If $(x^*, \hat{x}^*, \eta^*, \psi^*, \vartheta^*, \lambda^*)$ is an EP of (7), then x^* is a vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$.

Proof: 1) An EP $(x^*, \hat{x}^*, \eta^*, \psi^*, \vartheta^*, \lambda^*)$ of 1

$$0_n = v^* \tag{8a}$$

$$0_{n^2} = \mathcal{R}^T \Pi_C(\tilde{x}^*, -\alpha v^* - \Psi^* - \mathbb{H}(\tilde{x}^*)^T \tilde{\vartheta}^*$$

$$- \gamma \mathcal{R}(L^0 \otimes I_n)\hat{x}^*) - \gamma \mathcal{S}^T \mathcal{S}(L^0 \otimes I_n)\hat{x}^* \tag{8b}$$

$$0_{\tilde{n}} = -\beta \mathbb{L}\psi^* \tag{8c}$$

$$\psi^* = \eta^* + \mathbb{F}(\hat{x}^*), \ \eta(0) = 0_{\tilde{n}} \tag{8d}$$

$$0_n = -\vartheta^* + \mathcal{P}_{\mathbb{R}^n_+}(\vartheta^* + \boldsymbol{h}(\tilde{x}^*) + L^0\lambda^*) \tag{8e}$$

$$0_n = -L^0 \tilde{\vartheta}^* \tag{8f}$$

where $\tilde{n} := \sum_{j=1}^N n_j^2$.

On the basis of the definition of $\mathcal{R}$ and $\mathcal{S}$, one has

$$\begin{aligned} \mathcal{R}^T\mathcal{R} + \mathcal{S}^T\mathcal{S} = I_{n^2}, \quad \mathcal{R}\mathcal{R}^T = I_n \\ \mathcal{R}\mathcal{S}^T = 0_{n\times n(n-1)}, \quad \mathcal{S}\mathcal{S}^T = I_{n(n-1)}. \end{aligned} \tag{9}$$

By left multiplying $\mathcal{S}$ to (8b), one has $(L^0 \otimes I_n)\hat{x}^* = 0_n^2$, which together with (8a) indicates that $\hat{x}^* = 1_n \otimes x^*$. Based on and $\eta(0) = 0_{\tilde{n}}$, one has $\sum_{k\in[n_j]} \eta^j_{k,i}(0) = 0$, $\forall i \in [n_j]$. Moreover, according to the property of undirected graphs, for all $j \in [N]$, $k, i \in [n_j]$, one has $\sum_{k\in[n_j]} \dot{\eta}^j_{k,i}(t) = 0$ for all $t \geq 0$, by left multiplying $\mathrm{diag}\{(I_{n_j} \otimes 1^T_{n_j})_{j\in[N]}\}$ to $\dot{\eta}(t) = -\beta\mathbb{L}\psi(t)$. Thus, $\forall j \in [N]$

$$\sum\nolimits_{k\in[n_j]} \eta^j_{k,i}(t) = \sum\nolimits_{k\in[n_j]} \eta^j_{k,i}(0) = 0. \tag{10}$$

It follows from (8c) that $\psi^{j*}_{k,i} = \psi^{j*}_{l,i}$, $\forall k, l, i \in [n_j]$. Besides, according to (10), by left multiplying $\mathrm{diag}\{(I_{n_j} \otimes 1^T_{n_j})_{j\in[N]}\}$ to (8d), one obtains $\sum_{k\in[n_j]} \psi^{j*}_{k,i} = n_j \sum_{k\in[n_j]} \nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k*})$, which implies that $\psi^{j*}_{k,i} = \psi^{j*}_{l,i} = \sum_{k\in[n_j]} \nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k*})$. Subsequently, it follows from $\hat{x}^* = 1_n \otimes x^*$ that $\Psi^* = \mathcal{F}(x^*)$. Then, based on (8b), one has

$$0_n = \mathcal{F}(x^*) + \mathbb{H}(x^*)^T \tilde{\vartheta}^* + \mathcal{N}_C(x^*). \tag{11}$$

Based on (8e), if $\vartheta^* = \tilde{\vartheta}^* = 0_n$, one has $\boldsymbol{h}(x^*) + L^0\lambda^* \leq 0_n$, which yields $h(x^*) \leq 0$ by left multiplying 1_n^T. Obviously, $\tilde{\vartheta}_i^{j*} h(x^*) = 0$. If $\vartheta^* = \tilde{\vartheta}^* > 0_n$, one gets $\boldsymbol{h}(x^*) + L^0\lambda^* = 0_n$, which yields $h(x^*) = 0$ by left multiplying 1_n^T. Therefore, $\tilde{\vartheta}_i^{j*} h(x^*) = 0$. To sum up, it follows from (8e) and (8f) that

$$h(x^*) \leq 0, \ \tilde{\vartheta}_i^{j*} h(x^*) = 0, \ \tilde{\vartheta}_i^{j*} = \tilde{\vartheta}_s^{r*}. \tag{12}$$

Based on (2), it follows from (11) and (12) that x^* is the vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$. ■

Lemma 3 indicates that by Algorithm 1, the second-order player (1) can converge to the vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$, if Algorithm 1 can converge to its EP.

Before analyzing Algorithm 1, a necessary lemma is given below.

Lemma 4: Under Assumption 2, $F(\hat{x})$ is l_1-Lipschitz, where $l_1 := n_{\max} l$, $n_{\max} := \max_{j\in[N]}\{n_j\}$, and

$$F(\hat{x}) := \mathrm{col}\left(\left(\sum\nolimits_{k\in[n_j]} \nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k})\right)_{j\in[N],i\in[n_j],k\in[n_j]}\right).$$

Proof: Define $\hat{y} := \mathrm{col}((\hat{y}^{j\text{-}i})_{j\in[N],i\in[n_j]})$, $\hat{y}^{j\text{-}i} := \mathrm{col}((\hat{y}^{j\text{-}i}_{r\text{-}s})_{r\in[N],s\in[n_r]})$. Then, based on Assumption 3 and the definition of $F(\hat{x})$, one has

$$\begin{aligned} &\|F(\hat{x}) - F(\hat{y})\|^2 \\ &= \sum_{j\in[N]} \sum_{i\in[n_j]} \Big\| \sum_{k\in[n_j]} \nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k}) - \sum_{k\in[n_j]} \nabla_{x_i^j} f_k^j(\hat{y}^{j\text{-}k}) \Big\|^2 \\ &\leq \sum_{j\in[N]} \sum_{i\in[n_j]} \Big(n_j \sum_{k\in[n_j]} \|\nabla_{x_i^j} f_k^j(\hat{x}^{j\text{-}k}) - \nabla_{x_i^j} f_k^j(\hat{y}^{j\text{-}k})\|^2 \Big) \\ &\leq n_{\max}^2 l^2 \sum_{j\in[N]} \sum_{k\in[n_j]} \|\hat{x}^{j\text{-}k} - \hat{y}^{j\text{-}k}\|^2 \\ &\leq n_{\max}^2 l^2 \|\hat{x} - \hat{y}\|^2 \end{aligned}$$

which indicates the l_1-Lipschitz continuity of $F(\hat{x})$. ■

Theorem 1: Suppose that Assumptions 1-4 hold.

(i) The decisions of players satisfy their own local constriants all the time, i.e., $x(t) \in C$ for all $t \in [0, \infty)$.

(ii) Executing Algorithm 1, the decisions of all second-order players converge to the vGNE of N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$.

Proof: Due to the page limitation, the proof is omitted here, which will be provdied in the full paper version. ■

Remark 5: In contrast to the algorithm proposed by [12], which only converges to the neighborhood of vGNE, Algorithm 1 achieves exact convergence to the vGNE of the N-cluster game $\Gamma(\mathcal{N}, f, \Lambda)$.

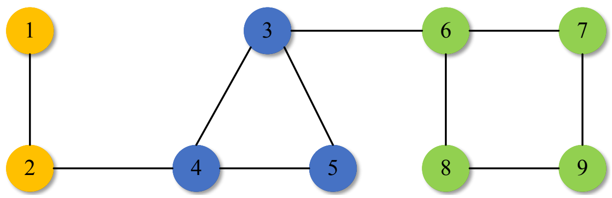

Fig. 1: The communication topology among generators.

TABLE II: Explanations of symbols

Symbols	Explanations
f_i^j	the cost function of generaor i of power plant j
f^j	the cost function of power plant j
P_i^j	the output power of generaor i of power plant j
u_i^j	the control input of generaor i of power plant j
C_i^j	define as $[\underline{P}_i^j,\ \overline{P}_i^j]$
$\overline{P}_i^j$	the upper bound of P_i^j
$\underline{P}_i^j$	the lower bound of P_i^j
d_i^j	the local demand of electricity markets

TABLE III: The parameters of power plants.

	Generator	a_i^j	b_i^j	c_i^j	$P_i^j(0)$	C_i^j	d_i^j
Plant 1	1 (1)	6	10	2	3	[3 5]	12
	2 (2)	5	15	3	4	[4 6]	11
Plant 2	1 (3)	4	20	4	5	[4 9]	10
	2 (4)	3	25	2	7	[5 10]	7
	3 (5)	2	5	5	8	[6 9]	9
Plant 3	1 (6)	9	15	3	9	[6 12]	5
	2 (7)	7	25	2	9	[7 11]	8
	3 (8)	6	35	4	8	[8 12]	7
	4 (9)	5	25	1	11	[9 12]	11

V. Numerical Examples

Consider an electricity market game of 3 power plants. Power plant $j \in \{1, 2, 3\}$ has n_j generators with $n_1 = 2$, $n_2 = 3$, and $n_3 = 4$. The interactions among those generators are depicted by Fig. 1. In electricity markets, power plant j faces the following problem:

$$\begin{aligned} &\min_{P^j \in \mathbb{R}^{n_j}} f^j(P),\ f^j(P^j, P^{-j}) = \sum_{i=1}^{n_j} f_i^j(P^j, P^{-j}) \\ &\text{s.t.} \quad \sum_{j=1}^{3}\sum_{i=1}^{n_j} P_i^j \geq \sum_{j=1}^{3}\sum_{i=1}^{n_j} d_i^j \\ &\qquad P_i^j \in C_i^j \end{aligned} \tag{14}$$

where the symbols are explained in Table II and $P := \mathrm{col}((P_i^j)_{j\in[3], i\in[n_j]})$. The cost function of generaor i of power plant j is

$$f_i^j(P^j, P^{-j}) = \sigma_i^j(P_i^j) - p(P^j, P^{-j})P_i^j$$

where $\sigma_i^j(P_i^j) = a_i^j + b_i^j P_i^j + c_i^j (P_i^j)^2$ is the generation cost of generaor i of power plant j with a_i^j, b_i^j, c_i^j being the characteristics of generaor i of power plant j; $p(P^j, P^{-j}) = p_0 - \epsilon \sum_{j=1}^{3}\sum_{i=1}^{n_j} P_i^j$ is the electricity price with p_0, ϵ being positive constants.

Neglecting the mechanical and electromagnetic losses, by virtue of feedback linearization, the dynamics of generator i in power plant j can be modeled as second-order system: $\ddot{P}_i^j = u_i^j$ (see [20]).

All parameters of the generators are presented in Table III. Besides, $\alpha = 100$, $\beta = 40$, $\gamma = 50$, $p_0 = 200$ and $\epsilon = 1$. One can easily verify that Assumptions 1-4 are satisfied. Then, the electricity market game (14) can be solved by Algorithm 1. The simulation results are displayed in Fig. 2. As shown in Fig. 2, the output powers of all power plants is convergent, which together with Lemma 1 indicates that the output powers of all power plants converge to the vGNE of electricity market game (14). Moreover, it is clear that the output powers of all power plants satisfy the given local and coupled constraints, which implies that the simulation results validate the validity of Algorithm 1.

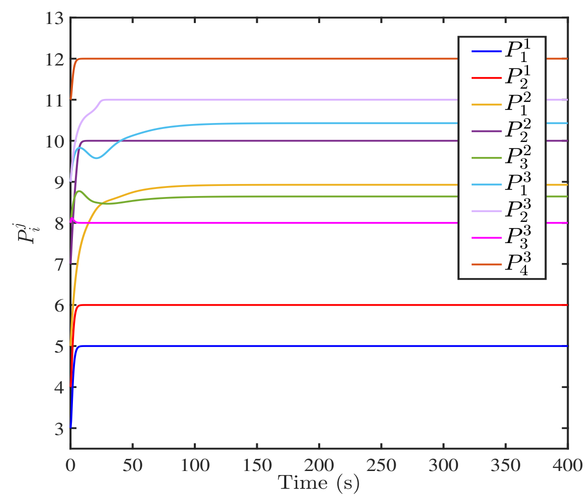

Fig. 2: The evolutions of P_i^j.

VI. Conclusions

This paper delved into the study of N-cluster games involving second-order dynamics, local and nonlinear coupled inequality constraints. To seek the vGNE of N-cluster games distributedly, this paper proposed a distributed algorithm leveraging state feedback, projection, primal-dual, dynamic average consensus, and passivity methodologies. Additionally, the proposed algorithm's convergence was rigorously analyzed. The algorithm guarantees that players' decisions consistently meet set constraints and converge to the exact vGNE of N-cluster games. Finally, the validity of the algorithm was demonstrated through simulations.

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