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A Two-Layer Opinion Dynamics Model Coupling Static and Bounded-Confidence Interactions *

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***Abstract*— In this paper, we present a new bounded-confidence model of opinion dynamics where agents with a substantially different opinion may still be able to influence each other along a static graph. Our intention is to account for hard and fast ties present due to physical or social proximity. This additional feature allows weak but persistent interaction between disagreeing agents. Albeit simple, the model remains difficult to analyse due to its separation of time scales. We introduce an appropriate notion of stability, give some properties of the system and formulate a conjecture supported by numerical simulations.**

I. INTRODUCTION

a) Research context and motivation: For quite a while now, information diffusion and opinion dynamics over networks have been under the scope of a mathematical formalisation. Among the large body of literature, the class of *bounded-confidence* (BC) models has been intensively studied and refined in many ways since the seminal works of Hegsellman and Krause [1] and Deffuant and Weisbuch [2]. In a BC model, agents over the network interact only if their respective opinion is close enough. On the contrary, if two agents stand too far (in opinion), they dismiss one each other. This mechanism accurately encompasses a structural principle of social psychology, namely *selective exposure*, i.e., the trend one has to dismiss dissonant information. Nonetheless, many other behaviours shall be incorporated in order to make a step closer to a more realistic behavioural description. For instance, some authors include some group pressure [3] in the modelling, others consider multi-dimensional dynamics to take into account the multiplicity of topics in a discussion [4]. For a recent survey of opinion dynamics and BC models, the interested reader may refer to [5] or [6]. In this model, we propose an extension of the Hegsellman-Krause (HK) model in a different direction: in addition to the opinion-driven interaction, the agents also influence one each other over a sparse network. In the context of opinion dynamics, the added network may represent the connectivity derived from hardly avoidable links such as familial, professional, or neighbourhood, just to name a few. This is why it is reasonable to assume the graph constant. We also take it sparse based on the premise that physical interactions are very limited because of time and attention. To this title, see the current literature related to Dunbar number [7], which has already taken as a core hypothesis of some opinion dynamics' model [8]. By contrast, the BC interaction, being free of any physical or social constraint is dense and volatile. These features remind *online* information seeking or news-feed scrolling, behaviours heavily biased by the preexisting opinion of the agent in consideration. It is clear that on the Internet some leader-follower dynamics emerge and the influence is not symmetric, but for sake of simplicity, we restrain the analysis to leaderless communities.

b) Related literature: In the initial HK model, the interaction only depends on the respective interacting agents. Recently, various works incorporate a graph in the model. In [9] and [10], the graph represents the physical limitation, is kept constant and is viewed as a model parameter. In these papers, the authors undertake deep mathematical analysis to estimate the termination-time of the algorithm regarding the graph structure. In a more empirical way, numerical investigations have also been conducted to understand BC dynamics over various networks [11]. Using similar methods, another variant is proposed in [12] allowing the graph to be adaptive: the graph changes throughout time in co-evolution with the opinion distribution in accordance with well-understood principles of social psychology, more precisely *selective exposure* [13] and *homophily* [14]. Finally, another attempt to acknowledge the plurality of confidence degrees in social interaction is to define for each agent an individualised bounded confidence value. In this frame known as *heterogeneous models*, the parameter is not a scalar anymore but a whole vector, leading to specific mathematical difficulties (see for instance [15] or [16]).

Although close to all of these models and pertaining to the same research trend, the one presented in the current paper is clearly distinct: the latter is indeed a superimposition of a BC dynamics and a (weak but persistent) interaction across a sparse and constant network. In this sense, the physical layer is not a limitation but an additional influence. Furthermore, interactions take place synchronously, which is not the case in the works [11] and [12] quoted above, designed in the Deffuant-Weisbuch fashion.

c) Paper contributions and overview: The paper contains three main contributions. Firstly, we construct a two-

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layers network model: the first layer is the classical bounded-confidence interaction, and the second one corresponds to a fixed and symmetric sparse graph. To the best of our knowledge, this model has never been presented before. From this construction, a slow time scale emerges, and mirrors in some ways the long term mutual influence arising from local interaction. Secondly, we introduce a new notion of stability appropriate for this model and use it to characterise some properties. We hope this stability concept can also be of interest in a different context. Thirdly, we state a conjecture, thus pinpointing the critical challenge of the discrete-time slow-fast model dynamics. The paper is organised as follows. In Section II, we describe the model. In Section III, we define the suited notion of stability for the model analysis and prove a property. In Section IV, we formulate a conjecture on the system behaviour. Section V is dedicated to numerical simulations. Finally, Section VI concludes the paper by summarising the main findings and listing some research perspectives.

d) Notations: In the sequel, we systematically note $[K]$ for $\{1,...,K\}$ as a shorthand. In the linear space $\mathbb{R}^K$, we define the canonical basis $(e_k)_{k\in[K]}$ and note $\mathbb{1} := \sum_{j=1}^{K} e_j \in \mathbb{R}^K$. We also make use of the infinite norm $||x||_\infty := \max_{k\in[K]} |x_k|$. $\mathcal{P}$ is the set of probability measures supported on $[0,1]$, and $\delta_x \in \mathcal{P}$ is a Dirac measure at point $x \in [0,1]$. For $u \in \mathbb{R}^K$, define the diagonal matrix $\text{diag}(u) \equiv \sum_j u_j e_j e_j^T$. We note $\mathbb{1}_U$ the indicator of any set U: $\mathbb{1}_U(x) = \delta_x(U) = 1$ if $x \in U$ and 0 if $x \notin U$. Finally, in the graph $\mathcal{G}_A$ represented by the matrix A, the degree of node j is noted $\deg_A(j)$ or more lightly d_j.

II. MODEL DESCRIPTION

Let be a population of agents of size K, each agent is indexed by an integer $k \in [K] \equiv \{1,2,...,K\}$. At each time $t \in \mathbb{N}$, the scalar value $X_k(t) \in [0,1]$ corresponds to the opinion of agent k at time t. The vector $X(t) \in [0,1]^K$ is the *opinion profile*. In addition, we set a fixed unweighted graph $\mathcal{G}_A$ represented by a matrix $A \in \{0,1\}^{K^2}$, supposed to be sparse, symmetric and connected: $a_{kj} = a_{jk}$ (symmetry) and $\max_{j\in[K]} \deg_A(j) = o(1)$ as $K \longrightarrow +\infty$ (sparsity), $\deg_A(j)$ being the degree of agent j within the graph A. The static graph A represents the *imposed* interaction layer that cannot be dismissed regardless of the opinion gap between two agents. Thus agents connected through the adjacency matrix interact and influences each other opinion even if their opinions are *very* different. At the same time, a second layer of influence occurs as *chosen* interactions based on opinion levels only. Indeed, agents with close opinions interact and influence each other through this second layer. But as each individual level of opinion is dynamic over time, this latter interaction graph is also dynamic, which is not the case for the physical interactions. This second layer of interaction follows an HK model with parameter $r \in [0,1]$, i.e. two agents with opinion levels close to each other by a value lower than r interact. We denote by $\phi^r_{kj}(t) \overset{\text{def}}{=} \mathbb{1}_{\{|X_k(t)-X_j(t)|<r\}}$ the indicator function that takes one if agents k and j interact at time t based on their opinion levels. Thus, the overall system can be represented by a two-layer graph model as illustrated on Figure (1) for 8 agents with $r = 0.1$ at a particular time t with specific opinion level for each agent. In this example, agents 4 and 5 mutually influence both at the rigid layer (i.e. $a_{45} = 1$) and also at the dynamical layer because their opinion levels are close to each other, i.e. $|X_4(t)-X_5(t)| = 0.01 < 0.1$. On the contrary, agent 5 interacts with agent 6 only on the physical layer. Similarly, agents 6 and 3 are not interacting physically, but due to their opinion levels they interact at the social layer. Considering the interactions throughout the state-dependent symmetric matrix $\phi^r(t)$, then a parameter $\lambda \in [0,1]$ stands as *the influence parameter for BC interaction* in detriment of the physical interaction with intensity $(1-\lambda)$. Based on the two coupled interactions, the discrete-time evolution of the opinion level of an agent k is thus given by:

$$X_k(t+1) = \frac{\sum_j \left(\lambda\phi^r_{kj}(t) + (1-\lambda)a_{kj}\right) X_j(t)}{\lambda\sum_j \phi^r_{kj}(t) + (1-\lambda)d_k}. \quad (1)$$

Recall that d_k is the degree of agent k: $d_k = \sum_j a_{kj}$. The parameter r is standard in BC models: it is regarded as the *bounded-confident* parameter. In what follows, the superscript will be omitted for sake of simplicity: $\phi^r = \phi$. As the BC parameter is kept constant throughout the text, this slight abuse does not contravene in any way the legibility. It has also to be noted that if $\lambda = 1$, we recover the standard HK model. Finally, we can put the opinion level dynamics in a matrix form, and it gives:

$$X(t+1) = \text{sto}\Big(\lambda\phi(t) + (1-\lambda)A\Big)X(t), \quad (2)$$

where sto(.) is the row-wise normalisation operator defined by $\text{sto}(M) := \left[\text{diag}(M\mathbb{1})\right]^{-1}M$ for any matrix M for which the last expression is well-defined. Note that here, the inverse operation is indeed well-defined because $\phi_{kk} = 1$ implying that $\sum_j (\lambda\phi + (1-\lambda)A)_{kj}$ is always strictly positive. We then obtain a highly non-linear dynamical system analysed in the next section.

Remark 1: Like many Opinion Dynamics model, the one defined above is not confined to social context but may also describe robotic systems addressing *rendez-vous* issues: suppose that agent k is a moving robot in a cooperative system, and the values $X_k(t)$ are coordinates in space. Then, r shall be seen as interaction signal range, and static bonds correspond to long-range communication.

III. ANALYSIS

A. Consensus

In the case of the pure BC type models (here when $\lambda = 1$), it is well known that the converging points of the opinion dynamics are clustered configurations. Due to the presence of the physical layer term, the situation is slightly different for the dynamics defined by equation (1). But even in our multi-layer context, an asymptotic global consensus is achieved as it is proved in next proposition.

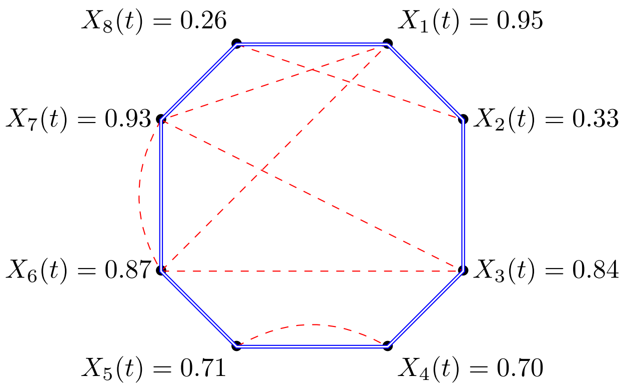

Fig. 1. We represent the two types of interaction (physical in blue and social in red) for an arbitrary opinion profile with $K = 8$ and $r = 0.1$ at a given time t. *In thick blue line, the static physical graph A, here a cycle graph. In dashed red line, the fleeting links based on opinion distance* $\phi^r_{kj}(t),\ k, j \in [\![1; 8]\!]$.

Proposition 2: If $\lambda < 1$ and the graph $\mathcal{G}_A$ is connected, then asymptotic global consensus is achieved:

$$\exists c \in [0, 1], \forall k \in [K], X_k(t) \underset{t\to\infty}{\longrightarrow} c. \tag{3}$$

Proof: The result is a simple application of a well-known theorem on consensus for time-varying networks: see for example theorem 12.4 of [17] (chapter 12) or theorem 3.1 of [18] (chapter 3). For sake of completeness, the result is given now. Consider the following discrete-time linear time-varying dynamics in $\mathbb{R}^K$, $K \geq 1$:

$$X(t+1) = A(t)X(t), \tag{4}$$

where for all $t \geq 1$:

- $A(t)$ is stochastic;
- $a_{ii}(t) > 0$ and there exists some positive scalar $\epsilon > 0$ such that $\forall (i, j),\ a_{ij}(t) > 0 \implies a_{ij}(t) > \epsilon$;
- there exists some T such that the graph $\bigcup_{j=1}^{T} \mathcal{G}_{A(t+j)}$ contains a globally reachable node;

then, we have

$$\exists c, \forall k, X_k(t) \longrightarrow c \text{ as } t \longrightarrow +\infty. \tag{5}$$

It is clear that dynamics given by Equation (1) verifies the previous assumptions: all the matrices $M(t)$ are stochastic; furthermore, $a_{kk}(t) \geq \frac{1}{K+d}$, and $a_{kj}(t) > 0 \implies a_{kj}(t) > \frac{1}{K+d}$; and finally the third hypothesis of the cited theorem above is also verified because we have assumed the graph $\mathcal{G}_A$ to be connected. Applying the result stated above terminates the proof of the existence of a global consensus for our multi-layer dynamic. ■

The time to reach this consensus may be unreasonably long when K grows to infinity. This is why in essence the simple convergence result is not satisfying to appreciate accurately the dynamical process in its whole lifetime. In addition, another important think to notice is that one has for almost all k, $\sum_j \phi_{kj} = \Theta(K) \gg \max_k \deg_A(k)$ as $K \longrightarrow +\infty$. In addition, by a simple computation, we get

$$X_k(t+1) = \frac{\sum_j \phi_{kj}}{\sum_j \phi_{kj} + d_k} \frac{\sum_j \phi_{kj} X_j}{\sum_j \phi_{kj}} + \frac{\sum_j a_{kj} X_j}{\sum_j \phi_{kj} + d_k}.$$

The dynamics can then be reformulated as

$$X(t+1) = \vec{\eta}(t)\text{sto}\big(\phi(t)\big)X(t) + \vec{\epsilon}AX(t), \tag{6}$$

where $\epsilon_k(t) \underset{K\to\infty}{=} o(1)$ and $\eta_k(t) \underset{K\to\infty}{\longrightarrow} 1$. This is why a time-scale separation occurs: at the first period of time, the HK dynamics prevails until X reaches a stable state of the HK dynamics. Then after, a new stage in the evolution of the system occurs. In order to quantify it more precisely, define the *clustering time*: it is the first time instant after which the opinion profile forms clusters:

$$T^r_{\text{clust}} := \inf \Big\{ t > 0 : \exists p \geq 1, C : [K] \mapsto [p], \\ C(k) = C(j) \iff |X_k(t) - X_j(t)| < r \Big\}. \tag{7}$$

Here above, C the *partition map*. By the last definition, from T_{clust} on Equation (7), for any BC parameter $r \in [0, 1]$, from T^r_{clust} on, the matrix $\phi^r(t)$ is block-diagonal (up to an index permutation) with the following blocks $\mathbb{K}_{l_j},\ j \in [p]$: $\phi = \text{Diag}\left(K_{l_1}, ..., K_{l_p}\right)$, where each K_{l_j} corresponds to a complete graph of size l_j, $\left(K_{l_j}\right)_{lm} = 1$.

B. Set Stability

In order to capture more precisely the overall system's behaviour, one has to introduce a specific notion of stability. Hereafter, we propose a natural generalisation of stability properties of dynamics to set points. Let first define the neighbourhood of any set U. For any $\epsilon > 0$ and subset $U \subset \mathbb{R}^K$, we define the $\epsilon-$neighbourhood of U as $U^\epsilon := \left\{x \in \mathbb{R}^K : \inf_{y\in U} ||x - y|| < \epsilon\right\}$.

Definition 3 (Set-stability): For any dynamical system $(X(t))_{t\in\mathbb{N}}$ with initial point $X(0)$, a set U is set-stable if it verifies:

$$\exists \eta > 0,\ \forall X(0) \in U^\eta : X(t) \underset{t\to\infty}{\longrightarrow} U. \tag{8}$$

But an accurate notion of stability is a weaker one that can be called *weak set-stability*.

Definition 4 (Weak set-stability): For a dynamical system $(X(t))_{t\in\mathbb{N}}$ with initial point $X(0)$, a set U is weakly set-stable if it verifies:

$$\forall x \in U, \exists \eta_x > 0, ||x - X(0)|| < \eta_x \implies X(t) \underset{t\to\infty}{\longrightarrow} U. \tag{9}$$

In words, the last formula (equation 9) means that the set U has various degrees of robustness to disturbance, depending on the initial value taken. On the contrary, the preceding one (equation 8) states a fixed level of tolerance to disturbance.

Note that in general, standard set-stability considers a neighbourhood of the whole set.

With this in mind, we can now focus on the set of equilibria $\mathfrak{E}$ for the pure HK dynamics ($\lambda = 1$), that is defined as the set of following points:

$$\mathfrak{E} = \Big\{ X \in [0, 1]^K : \forall (k, j) \in [K]^2, \\ X_k = X_j \text{ or } |X_k - X_j| > r \Big\}.$$

Rather than considering the opinion profile $(X_k)_{k\in[K]}$, one can consider the empirical measure $\mu_t^X := \frac{1}{K}\sum_{j=1}^K \delta_{X_j(t)} \in \mathcal{P}$. Using this definition, the opinion dynamic of agent k given by Equation (1) can be rewritten as:

$$X_k(t+1) = \frac{\lambda \int_s s\mathbb{1}_{\left(|X_k(t)-s|<r\right)}\mu_t^X(ds) + \frac{1-\lambda}{K}\sum_j a_{kj}X_j(t)}{\lambda \int_s \mathbb{1}_{\left(|X_k(t)-s|<r\right)}\mu_t^X(ds) + \frac{1-\lambda}{K}d_k}. \tag{10}$$

Considering the previous measure-theoretic description of the opinion dynamics, the set of *stable* equilibria can be described as follows:

$$\mathcal{S} := \Big\{\mu \in \mathcal{P} : \mu = \sum_{k=1}^N \alpha_k\delta_{v_k} \text{ with } \alpha_k \geq 0,\ N \geq 1 \text{ and } k \neq j \implies |v_k - v_j| > r\Big\}. \tag{11}$$

Each value v_k represents a cluster point and the value N corresponds to the number of clusters. In the linear space coordinates, one may reformulate it as

$$\mu^X = \sum_k \alpha_k\delta_{v_k} \in \mathcal{S} \iff \forall p \in [K]\ \exists k, X_p = v_k.$$

Actually, the set of clustered configurations is a disjoint union of subsets S_j such that:

$$\mathcal{S} = \bigcup_{j=1}^{M_r} \mathcal{S}_j, \tag{12}$$

where S_j is the set of probability measures formed with j clusters: $\mu = \sum_{k=1}^j \alpha_k\delta_{v_k} \in \mathcal{S} \iff \mu \in \mathcal{S}_j$. For any BC parameter $r \in [0,1]$, M_r is the maximum number of clusters possible to build by the opinion dynamics. Note that each set S_j is a manifold of dimension j. In particular, S_1 is the set of global consensus configurations, S_2 is the set where there are two clusters, etc. In a similar fashion, we now characterise the *unstable* equilibria with j clusters:

$$U_j = \Bigg\{\mu \in \mathcal{P} : \mu = \sum_{k=1}^j \alpha_k\delta_{v_k} :\ |v_p - v_q| \geq r, \text{ such that } \exists(p^*, q^*) \text{ with } p^* \neq q^* \text{ and } |v_{p^*} - v_{q^*}| = r\Bigg\}. \tag{13}$$

It is clear that these configurations are unstable, but are only sensitive to specific disturbances: when a significant proportion of agents of a same cluster move. A different definition of stability can be found in [19]. Relying on the definitions given above, the next proposition characterises the set stability of the equilibria for the HK model, i.e. when $\lambda = 1$.

Proposition 5: For the pure HK dynamics ($\lambda = 1$), for $j \geq 2$, the sets S_j are weakly set-stable but not strongly set-stable.

Proof: We first show weak set-stability. For, fix a point $x \in S_l, l \leq M_r$. Without loss of generality, let us index by order the cluster values $v_1 < v_2 < ... < v_{l-1} < v_l$, and set $\beta := \min_{k\in[K]} |v_{k+1} - v_k|$ which is by construction larger than r. This means that we have a margin of $m = \frac{\beta - r}{2}$. Formally, this implies that for all y in the ball $\Big\{u : ||u - x||_\infty < \min(\frac{m}{2}, \frac{r}{3})\Big\}$, the clusters remain:

$$x_k = x_j \implies |y_k - y_j| < \frac{2r}{3} \text{ and}$$
$$x_k \neq x_j \implies |y_k - y_j| > r.$$

The first equation means that two agents k and j initially in the same cluster cannot be moved away one from each other by a distance superior of r. The second inequality means that two agents from two distinct clusters cannot be brought closer to a distance inferior of r. To show this, consider two agents k and j with $x_k = v_p$ and $x_j = v_{p+1}$. Bringing closer y_k and y_j together leads to $y_k = v_p + m$ and $y_j = v_{p+1} - m$ but

$$y_j - y_k = (v_{p+1} - v_j) - 2m = \beta - 2m > r. \tag{14}$$

Hence, because of clusterization, the distribution converges to $\mathcal{S}$ in one iteration, which concludes the first part of the proof.

Next, we show that there is no global basin of attraction uniformly for all equilibria. It is sufficient to find a counter-example. Let us consider the following situation: $\mu = \alpha\delta_x + (1-\alpha)\delta_{x+r-\epsilon} \in S_2^\epsilon$. It is clear that this configuration converges to S_1 for all $\epsilon > 0$. This allows to conclude that there is no global basin of attraction for S_j, $j \geq 2$. ■

Remark 6: Note that the counter-example stands only if we have at least two clusters, this is why $\mathcal{S}_1$ is not weakly set-stable.

IV. Cluster cohesiveness

The next statement presented as a conjecture is clear in view of the several numerical experiments, but some pathological cases complicate the rigorous mathematical proof.

Conjecture 7 (cluster cohesiveness): Suppose that the initial distribution is i.i.d according to a given measure with density f:

$$(X_k(0))_{k\in[K]} \sim_{i.i.d} f.$$

Then, for any $t > T_{\text{clust}}$, the clusters maintain with high probability in K, that is the probability that the next property does not occur vanishes when K grows:

$$|X_k(t) - X_j(t)| < r \implies |X_k(s) - X_j(s)| < r\ \forall s \geq t. \tag{15}$$

The conjecture means that two individuals in a cluster remain in the same cluster for ever. Then, there is no *cluster fission* in the particular sense that we define now: consider a cluster $C \subset [K]$ at time $t \in \mathbb{N}$. We say a $s-$fission (s for significant) has occurred if at time $t' \geq t$ we have

$$\exists U_1, U_2 \subset C, (k,j) \in U_1 \times U_2 \implies |X_k - X_j| > r \text{ and}$$
$$\min\left(\frac{|U_1|}{|U_2|}, \frac{|U_2|}{|U_1|}\right) \geq s.$$

For such a fission to occur, two clusters C_l and C_r shall stand each of which on either side of the focal one C according to

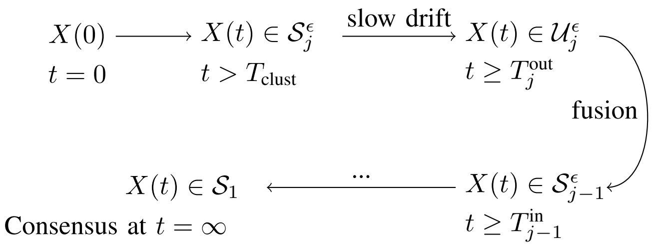

Fig. 2. This scheme recaps the main steps of the evolution according the hitting times defined earlier. T_{clust} is probably reached very quickly. The fusion period is also very short.

a very specific distance. More precisely, agents in C shall be influenced by one and only one of the two lateral clusters. Otherwise, suppose there exists a significant part of agents pertaining to C under the two opposite influences of C_1 and C_2, or on the contrary they are not influenced at all by members of C_l, C_r. In both cases, this mass of agents remain stable after this step, and may bring back to the initial position other agents having significantly moved.
In the preceding counterexample, the particularity is due to the very precised relative position of the three blocks, and this is why the last case is taken as very unlikely.

Figure 2 is a schematic diagram representing the main steps of the dynamics. We define the following important time in the dynamics. For all $j \in \{1, \dots, M_r\}$, let be:

$$T_j^{\text{in}} := \inf \{s > 0 : X(s) \in (S_j)^\epsilon\}, \text{ and} \quad (16)$$

$$T_j^{\text{out}} := \inf \{s > T_j^{\text{in}} : X(s) \notin (S_j)^\epsilon\}. \quad (17)$$

It is clear that $T_j^{\text{in}} < T_j^{\text{out}} < T_{j-1}^{\text{in}}$. Here we get forced to state it using set approximation because sets U_j and S_j are configurations at equilibrium for the HK dynamics *only*, but because of opinion exchange over static graph the opinion profile $X(t)$ gets pushed out of these sets until consensus, revealing the inherent difficulties of slow-fast discrete time systems.

V. Numerical Experiments

In this section, we display some illustrative numerical simulations. They have been conducted with $K = 500$ agents (excepted for the fission example where we have taken $K = 450$) and the graph taken is a simple line graph: $a_{kj} = \mathbb{1}_{\{|k-j| \leq 1\}}$. Initial distribution is taken uniform (figures 3 and 4) or according to a measure with a continuous density $f(x) = 12(x - \frac{1}{2})^2 \mathbb{1}_{[0,1]}(x)$ (figure 5). This latter initial distribution may model cases when the agents are facing a burning topic: the population is thus significantly polarised and there is very few people with a neutral opinion. We use as a benchmark the system uniformly distributed at initial time and with line graph as static graph A (see figure 3). This is because we consider the line graph as a simple physical graph, and that we make no specific assumption on the underlying topic generating the uniform distribution.

A. Time-scale separation

In every case, the time-scale separation is clearly visible at long term: quickly after $t = 0$ (see figure 4), clusters form and curves of same clusters are indistinguishable ($t > T_{\text{clust}}$). At the macroscopic size (cf. figures 3 and 5), one can only observe a few trajectories, and the clusters merge all together progressively to finally reach a global consensus .In the case with an non uniform initial distribution, the time to reach a clustered configuration is larger, and there are some isolated agents at the first steps, but the global picture is very similar to the uniform case.

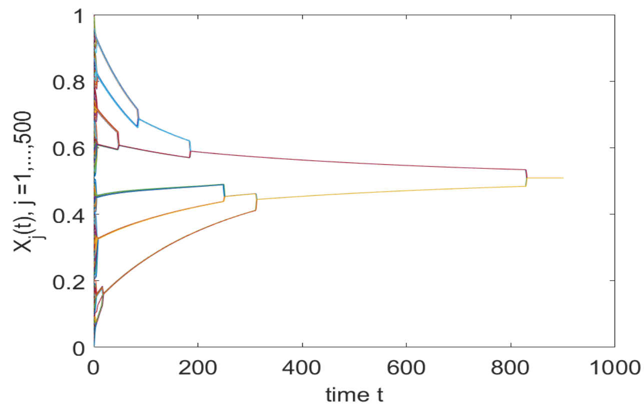

Fig. 3. The individual trajectories for $K = 500$, $\lambda = 0.8$ and $r = 0.05$ for $t = 0, 1, ..., 900$. The initial distribution is taken uniform: $X_k(0) \sim_{i.i.d}$ Unif[0,1].

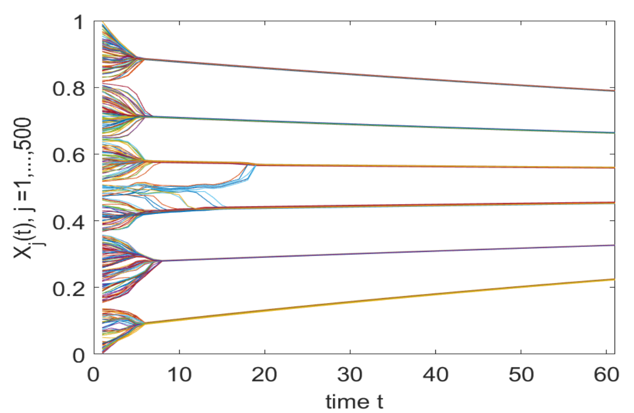

Fig. 4. The same system that the one above, but taken from a shorter time-horizon: $t = 0, 1, ..., 60$. *In the first dozens of steps, the dynamics is quasi stable and the cluster slowly drift toward fusion (see figure 4). Before $t = 50$, one observe only one fusion and six clusters remain.*

B. Fission

Although highly improbable when the initial distribution is drawn randomly according to a non-atomic measure, it is possible to construct a configuration when a fission occurs. Take the following initial measure:

$$\mu = \frac{1}{9}(\delta_0 + \delta_1) + \frac{1}{3}(\delta_{0.25} + \delta_{0.75}) + \frac{10}{9}\mathbb{1}_{[0.45,0.55]} \quad (18)$$

with $r = 0.25$, $\lambda = 0.8$. The central block quickly splits because of the two properly aggregated clusters at 0.25

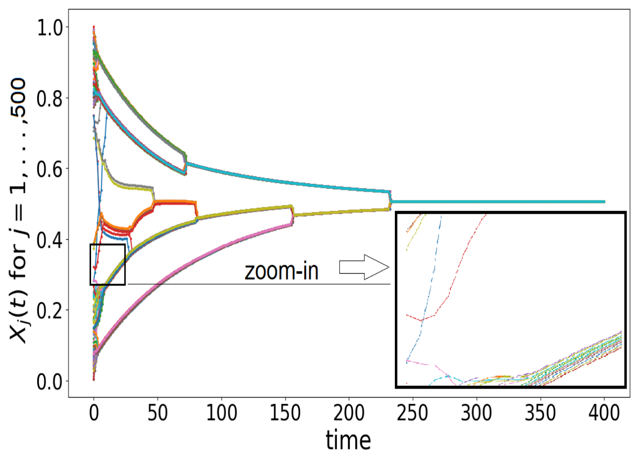

Fig. 5. Here we have taken exactly the same parameters but with a different initial distribution: $X_k(0) \sim_{i.i.d} f$ with $f(x) = 12(x - \frac{1}{2})^2 \mathbb{1}_{[0,1]}(x)$. *As illustrated in the zoom-in box (bottom right), one can see that at the beginning of the dynamics, there are some isolated agents -for instance the red trajectory- that finally reach clusters. It is due to the particular initial distribution with very few agents in the center of the opinion space.*

and 0.75. For the illustrative numerical experiment figure 6 below, we have taken $K = 450$, and a *very small* time-horizon in order to focus on the fission event.

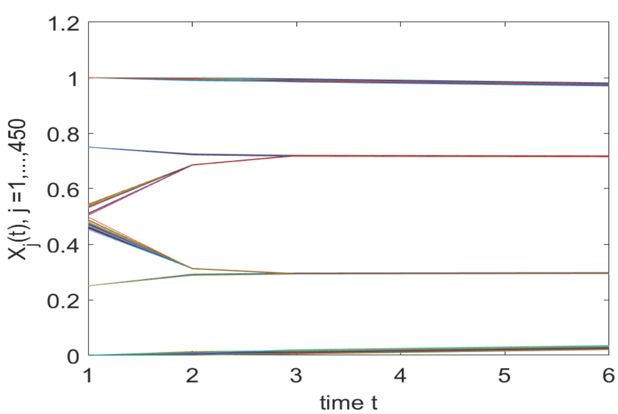

Fig. 6. A fission occurs when a very peculiar initial distribution with atoms is set, here according to equation 18. *We see very clearly the central cluster getting splitted as fission.*

VI. Conclusion

In this paper, we have presented a new BC model of opinion dynamics where agents with a substantially different opinion may still be able to influence each other along a static graph. Although we have been able to identify some interesting properties, the analysis is still incomplete. A mathematical framework of time-scale separation in discrete time could significantly enlighten it. Another approach would be to analyse the asymptotic system in the number of agents, taking $K \longrightarrow +\infty$, leading to an eulerian description of the system, as it has been done in [19], [20]. But as K grows, the sparse graph vanishes, highlighting the inherent difficulty to deal with asymptotic behaviour of sparse graphs, unlike dense graphs theory which has now reached maturity [21]. Just this model driven by only two parameters r and λ, many questions remain unsolved. Specifically, the influence of λ is the whole dynamics may drastically impact the overall dynamics. Another structural feature of the presented model is indeed that updates take place *synchronously*. Investigating the *asynchronous* counterpart would also be of great interest.

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