2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Moment Matching for Nonlinear Systems of Second-Order Equations*

Joel D. Simard[1], Alessio Moreschini[2] and Alessandro Astolfi[3]

Abstract— **In this paper we consider the problem of constructing nonlinear systems of second-order equations that achieve moment matching. In particular, necessary and sufficient conditions are given for which a system of second-order equations achieves moment matching, and a family of systems of second-order equations achieving moment matching is directly constructed by extracting it, via particular choices of the free mappings, from a parameterization of all systems achieving moment matching. The results are specialized for the scenario in which the signal generator is a linear system. Finally, the results of the paper are demonstrated by constructing reduced order models of a two link robotic manipulator in the second-order equation form.**

Index Terms— **Model reduction; moment matching; nonlinear systems; second-order equations**

I. Introduction

Given the model of a complex dynamical system, the primary objective of model order reduction is to construct a model of lower dimension which preserves, or approximates, desired properties of the original model, *e.g.* the steady state response to selected input signals. Several methods to accomplish model order reduction have been developed for linear and nonlinear systems in the last few decades. Such methods include balanced truncation, see [1]–[3], Hankel-operator methods, see [4] and [5], moment matching, see [6]–[10], and the Loewner framework, see [11]–[15].

Systems of second-order equations are ubiquitous in engineering, often arising naturally in the analysis of electrical and mechanical systems, see *e.g.* [16]–[19]. The problem of model order reduction for systems of second-order equations with structure preservation has been challenging. Conventional approaches to model order reduction typically yield a model in the first-order form, hence the structure of the original system is lost. Some model order reduction approaches for systems of second-order equations have been developed by enhancing classical methods for linear systems, see *e.g.* [20]–[23] for balanced truncation methods, and [24]–[26] for second-order Krylov subspace methods. A clustering-based approach to simplify networks of interconnected second-order systems is given in [27].

The authors of [28] have presented a new parameterized family of systems which, for a given dimension, captures all systems of the considered differential-algebraic form matching the tangential data in the Loewner framework for linear and nonlinear systems. In the linear setting, this parameterization has been used to construct systems of second-order equations matching sets of right and left tangential data in [29]. Inspired by these developments in the Loewner framework, the notions of time domain moment and of moment matching have been enhanced to a general class of implicit differential-algebraic systems in [10], and, for a given dimension, a parameterization of all systems that achieve moment matching has been constructed.

The primary objective of this article is the development of a family of nonlinear systems of second-order equations that achieve moment matching, which would allow a designer to preserve the second-order structure in a reduced order model when considering the model order reduction of nonlinear second-order equations. This objective is accomplished by selecting free mappings for the parameterization of systems achieving moment matching in [10] in such a way that the resulting model can be manipulated into a second-order equation form.

The structure of this paper is as follows. In Section II the class of considered nonlinear systems of second-order equations is introduced, and a review of moment matching for systems of nonlinear differential-algebraic equations (DAEs) is given. In Section III we derive conditions for which a nonlinear system of second-order equations achieves moment matching, and a parameterized family of nonlinear systems of second-order equations satisfying the conditions is determined. In Section IV the results are specialized to the scenario in which the signal generator is a linear system. In Section V a demonstrative example is presented wherein the time-domain moment is determined for a two link planar elbow manipulator, and systems of second-order equations achieving moment matching are constructed. Finally, in Section VI some concluding remarks are given.

II. Preliminaries

We use standard notation. The field of real numbers is denoted by $\mathbb{R}$. The set of vectors having n rows with real-valued entries is denoted by $\mathbb{R}^n$, and the set of matrices having n rows and m columns with real-valued entries is denoted by $\mathbb{R}^{n\times m}$.

*This work has been partially supported by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No. 739551 (KIOS CoE); by the Italian Ministry for Research in the framework of the 2017 and of the 2020 Program for Research Projects of National Interest (PRIN), Grant No. 2017YKXYXJ and 2020RTWES4; and by the EPSRC grant "Model Reduction from Data", Grant No. EP/W005557.

[1]Joel D. Simard is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K j.simard18@imperial.ac.uk

[2]Alessio Moreschini is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K a.moreschini@imperial.ac.uk

[3]Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K and with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133 Roma, Italy a.astolfi@imperial.ac.uk

A. Systems of Second-Order Equations

In this article we consider systems of nonlinear second-order equations of the form[1]

$$\overbrace{M(x,\dot{x})\dot{x}}^{\cdot} + D(x,\dot{x})\dot{x} + K(x,\dot{x})x = B(x,\dot{x})u, \quad (1)$$

$$y = C(x,\dot{x},u), \quad (2)$$

with generalized position coordinates $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and smooth mappings $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $C : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Many physical systems and phenomena can be modelled in the form (1)-(2), such as pendulums [30], robotic manipulators [17], electrical and mechanical resonators with nonlinear damping [16], [18], and other nonlinear electromechanical systems, see *e.g.* [19]. In addition, models of the form (1)-(2) often arise as a result of discretizing partial differential equations (PDEs) in vibration analysis [31]. Indeed, as noted in [17], the matrix form of the Euler-Lagrange equations

$$M(q)\ddot{q} + D(q,\dot{q})\dot{q} + K(q) = \tau, \quad (3)$$

arises from any mechanical system the kinetic energy of which is of the form $KE(q) = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$, with the $n \times n$ inertia matrix function $M(\cdot)$ symmetric and positive definite for all $q \in \mathbb{R}^n$, and such that the potential energy $PE(q)$ is independent of $\dot{q}$. Systems of the form (3) can be fit into the family of systems given by (1)-(2) via a suitable change of coordinates, $x := \varphi(q)$, such that $x = 0$ is an equilibrium point (as $K(0)$ is not necessarily zero, for example due to the effects of gravity).

Consider a linear system in second-order form, namely

$$M\ddot{x} + D\dot{x} + Kx = Bu, \quad (4)$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, and matrices $M \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. In the model (4), a sufficient condition for stability of the equilibrium $(x,\dot{x}) = (0,0)$, satisfied by stable electromechanical systems, is that $M \succ 0$, $D \succeq 0$, and $K = K^\top \succeq 0$. These matrices are often referred to as the inertia, damping, and stiffness matrices, respectively [32].

B. Families of DAEs That Achieve Moment Matching

In [10] the authors generalize the notion of time-domain moment for systems of nonlinear differential-algebraic equations (DAEs) of the form

$$\overbrace{E\big(x(t)\big)}^{\cdot} = f\big(x(t),u(t)\big), \quad x(0) = x_0, \quad (5)$$

$$y(t) = h\big(x(t),u(t)\big), \quad (6)$$

with state $x(t) \in \mathbb{R}^\chi$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and smooth mappings $f : \mathbb{R}^\chi \times \mathbb{R}^m \to \mathbb{R}^\chi$, $h : \mathbb{R}^\chi \times \mathbb{R}^m \to \mathbb{R}^p$, $E : \mathbb{R}^\chi \to \mathbb{R}^\chi$ such that $f(0,0) = 0$, $h(0,0) = 0$, and $E(0) = 0$. In general, the Jacobian of the mapping $E(\cdot)$ is singular and has constant rank in a neighbourhood of the origin, so systems of the form (5)-(6) possess both differential equations and algebraic constraints. We assume that the initial condition x_0 is *consistent* with the input $u(\cdot)$, meaning that, locally, there exists a continuously differentiable solution for the initial value problem (5).

An important property for singular systems of the form (5)-(6) is that of *regularity*. A system is said to be regular (with respect to the solution $x(t) = 0$ and $u(t) = 0$) if, locally in a neighbourhood $\mathbb{X}\times\mathbb{U} \subset \mathbb{R}^\chi\times\mathbb{R}^m$ of $x = 0$ and $u = 0$, continuously differentiable solutions of the system exist and are unique for every sufficiently differentiable input $u(t) \in \mathbb{U}$ and every initial condition $x_0 \in \mathbb{X}$ such that the initial condition x_0 is consistent. Hereafter, it is assumed that any system of the form (5)-(6) is regular.

In order to define the time-domain moments of the system (5)-(6) consider a signal generator given by the equations of the form

$$\dot{\omega}(t) = s\big(\omega(t)\big), \quad \omega(0) = \omega_0, \quad (7)$$

$$v(t) = \ell\big(\omega(t)\big), \quad (8)$$

with state $\omega(t) \in \mathbb{R}^\nu$, output $v(t) \in \mathbb{R}^m$, and smooth mappings $s : \mathbb{R}^\nu \to \mathbb{R}^\nu$ and $\ell : \mathbb{R}^\nu \to \mathbb{R}^m$ such that $s(0) = 0$ and $\ell(0) = 0$. It is assumed that (7)-(8) is observable and neutrally stable[2].

The time-domain moments are defined according to the interconnection of the systems (5)-(6) and (7)-(8) via the equation $u = v$. The resulting interconnected system has the generalized (differential-algebraic) state-space representation

$$\begin{bmatrix} \dot{\omega} \\ \overbrace{E(x)}^{\cdot} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{\partial E}{\partial x} \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} s(\omega) \\ f\big(x,\ell(\omega)\big) \end{bmatrix}, \quad (9)$$

$$y = h\big(x,\ell(\omega)\big). \quad (10)$$

It is assumed that the trajectory of the signal generator is such that $\ell\big(\omega(t)\big) \in \mathbb{U}$ with $u(t) = \ell\big(\omega(t)\big)$ sufficiently smooth so that, for a regular system of the form (5)-(6), the interconnected system (9)-(10) is an autonomous system for which there exists a unique solution for every consistent initial condition $(\omega_0^\top, x_0^\top)^\top$.

Associated to the interconnected system (9)-(10) is an invariant (when, locally, solutions are unique) manifold, $x = \pi(\omega)$, defined as the solution to the partial differential-algebraic equation (PDAE) with boundary condition

$$\frac{\partial E\big(\pi(\omega)\big)}{\partial \omega} s(\omega) = f\big(\pi(\omega),\ell(\omega)\big), \quad \pi(0) = 0. \quad (11)$$

Then the time-domain moment, $\eta : \mathbb{R}^\nu \to \mathbb{R}^p$, of the system (5)-(6) at (s,ℓ) is defined as

$$\eta(\omega) := h\big(\pi(\omega),\ell(\omega)\big).$$

[1]Note that in the system (1)-(2) we could instead consider a single mapping, $f(x,\dot{x})$, rather than the separate terms $D(x,\dot{x})\dot{x} + K(x,\dot{x})x$, and the results in the following sections would follow in a similar fashion. However, decomposing $f(\cdot)$ into the terms $D(\cdot)$ and $K(\cdot)$ allows for the distinction between "damping" phenomena and "stiffness" phenomena. Furthermore, if $f(0,0) = 0$, and if $f(\cdot)$ is differentiable in a neighbourhood of the origin, then via Hadamard's lemma there exist mappings $D(\cdot)$ and $K(\cdot)$ such $f(x,\dot{x}) = D(x,\dot{x})\dot{x} + K(x,\dot{x})x$, hence the two scenarios are equivalent.

[2]Neutral stability requires that the mapping $s(\cdot)$ admit a stable equilibrium point at $\omega = 0$ and that there exists a neighborhood of Poisson stable points around $\omega = 0$, see [9, Sec. 2.2].

In [10] it is shown that $\eta(\cdot)$ is the output of the interconnected system (9)-(10) evolving on the invariant manifold $\pi(\cdot)$, and if the manifold is locally attractive then the time-domain moment characterizes the steady-state output response of the interconnected system.

Given the moment at (s,ℓ) of the system (5)-(6), $\eta(\cdot)$, another system of the form (5)-(6) is said to *achieve moment matching* if it has the same time-domain moment at (s,ℓ). A family of systems having dimension ν and achieving moment matching can be constructed, as shown in [10, Theorem 1].

Theorem 1: Let $\eta(\cdot)$ be the moment at (s,ℓ) of the system (5)-(6). Consider the system given by the equations

$$\overbrace{\overline{E}(\zeta)}^{\cdot} = \frac{\partial \overline{E}}{\partial \zeta} s(\zeta) - \overline{f}\big(\zeta, \ell(\zeta)\big) + \overline{f}(\zeta, u), \tag{12}$$

$$y_r = \eta(\zeta) - \overline{h}\big(\zeta, \ell(\zeta)\big) + \overline{h}(\zeta, u), \tag{13}$$

with state $\zeta(t) \in \mathbb{R}^\nu$, input $u(t) \in \mathbb{R}^m$, output $y_r(t) \in \mathbb{R}^p$, and any smooth mappings $\overline{E} : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $\overline{f} : \mathbb{R}^\nu \times \mathbb{R}^m \to \mathbb{R}^\nu$, and $\overline{h} : \mathbb{R}^\nu \times \mathbb{R}^m \to \mathbb{R}^p$ such that (12)-(13) is regular. Then the system (12)-(13) achieves moment matching at (s,ℓ).

In [10, Theorem 2], given an arbitrary interger $k \geq 0$, the authors parameterize a family of systems having dimension $\nu{+}k$ and achieving moment matching at (s,ℓ) by performing a dynamic extension on the interpolant of Theorem 1.

Theorem 2: Let $\eta(\cdot)$ be the moment at (s,ℓ) of the system (5)-(6). Let k be any integer such that[3] $k \geq 0$. Consider the system given by the equations

$$\begin{aligned} \begin{bmatrix} \overbrace{\overline{E}(\zeta)}^{\cdot} + \overbrace{Q(\zeta,\gamma)}^{\cdot} \\ \overbrace{P(\zeta,\gamma)}^{\cdot} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \overline{E}}{\partial \zeta} s(\zeta) - \overline{f}\big(\zeta, \ell(\zeta)\big) \\ N(\zeta,\gamma,u) \end{bmatrix} \\ &\quad + \begin{bmatrix} \overline{f}(\zeta,u) + Z(\zeta,\gamma,u) \\ 0 \end{bmatrix}, \end{aligned} \tag{14}$$

$$\begin{aligned} y_r = \eta(\zeta) &- \overline{h}\big(\zeta, \ell(\zeta)\big) \\ &+ \overline{h}(\zeta,u) + H(\zeta,\gamma,u), \end{aligned} \tag{15}$$

with states $\zeta(t) \in \mathbb{R}^\nu$ and $\gamma(t) \in \mathbb{R}^k$, input $u(t) \in \mathbb{R}^m$, output $y_r(t) \in \mathbb{R}^p$, and with any smooth functions $\overline{E} : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $Q : \mathbb{R}^\nu \times \mathbb{R}^k \to \mathbb{R}^\nu$, $P : \mathbb{R}^\nu \times \mathbb{R}^k \to \mathbb{R}^k$, $\overline{f} : \mathbb{R}^\nu \times \mathbb{R}^m \to \mathbb{R}^\nu$, $Z : \mathbb{R}^\nu \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^\nu$, $N : \mathbb{R}^\nu \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$, $\overline{h} : \mathbb{R}^\nu \times \mathbb{R}^m \to \mathbb{R}^p$, and $H : \mathbb{R}^\nu \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^p$ satisfying the conditions

$$\frac{\partial Q(\zeta,0)}{\partial \zeta} s(\zeta) = Z\big(\zeta, 0, \ell(\zeta)\big), \tag{16}$$

$$\frac{\partial P(\zeta,0)}{\partial \zeta} s(\zeta) = N\big(\zeta, 0, \ell(\zeta)\big), \tag{17}$$

$$H\big(\zeta, 0, \ell(\zeta)\big) = 0, \tag{18}$$

and such that the system (14), (15), (16), (17), (18) is regular. Then the system (14)-(18) achieves moment matching at (s,ℓ).

Finally, in [10, Theorem 3] the authors prove that, under mild conditions, for any order $\overline{n} = \nu + k \geq \nu$ the family of interpolants given in Theorem 2 parameterizes all systems of order $\overline{n}$ that achieve moment matching at (s,ℓ).

Theorem 3: Let $\eta(\cdot)$ be the moment at (s,ℓ) of the system (5)-(6) and let k be any integer such that $k \geq 0$. Then the system (14)-(18) parameterizes all systems of order $\overline{n} = \nu{+}k$ achieving moment matching while possessing a solution for the associated PDAE with boundary condition (11) with full column rank Jacobian in a neighbourhood of the origin.

Consider now the goal of constructing a system achieving moment matching at (s,ℓ) while also satisfying additional desired properties. As a result of Theorem 3, if such a model exists then under mild conditions it must be realizable in the form (14)-(18).

III. Nonlinear Systems of Second-Order Equations Achieving Moment Matching

Given the time-domain moment, $\eta(\cdot)$, we begin by determining conditions for which a system of second-order equations of the form (1)-(2) achieves moment matching at (s,ℓ). The following theorem characterizes these conditions.

Theorem 4: Let $\eta(\cdot)$ be the moment at (s,ℓ) of the system (5)-(6). Then the system of second-order equations (1)-(2) achieves moment matching at (s,ℓ) if, and only if, there exists a mapping $\pi_1 : \mathbb{R}^\nu \to \mathbb{R}^n$ satisfying the PDAE with boundary condition

$$\begin{aligned} &\frac{\partial \left(M\left(\pi_1(\omega), \frac{\partial \pi_1}{\partial \omega} s(\omega)\right) \frac{\partial \pi_1}{\partial \omega} s(\omega) \right)}{\partial \omega} s(\omega) \\ &= -K\left(\pi_1(\omega), \frac{\partial \pi_1}{\partial \omega} s(\omega)\right) \pi_1(\omega) \\ &\quad - D\left(\pi_1(\omega), \frac{\partial \pi_1}{\partial \omega} s(\omega)\right) \frac{\partial \pi_1}{\partial \omega} s(\omega) \\ &\quad + B\left(\pi_1(\omega), \frac{\partial \pi_1}{\partial \omega} s(\omega)\right) \ell(\omega), \quad \pi_1(0) = 0, \end{aligned}$$

such that

$$C\left(\pi_1(\omega), \frac{\partial \pi_1}{\partial \omega} s(\omega), \ell(\omega)\right) = \eta(\omega).$$

Proof: Defining states $\zeta_1 := x$ and $\zeta_2 := \dot{x}$ yields the generalized state-space representation

$$\begin{aligned} \overbrace{\begin{bmatrix} I & 0 \\ 0 & M(\zeta_1,\zeta_2) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}}^{\cdot} &= \begin{bmatrix} 0 & I \\ -K(\zeta_1,\zeta_2) & -D(\zeta_1,\zeta_2) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ B(\zeta_1,\zeta_2) \end{bmatrix} u, \\ y &= C(\zeta_1,\zeta_2,u). \end{aligned}$$

[3]With some abuse of notation, setting $k = 0$ corresponds to the scenario in which the system (14)-(18) is exactly the interpolant of Theorem 1, *i.e.* by choosing $k = 0$ no new equations or variables are added.

Then the interconnection of the system (1)-(2) with the signal generator (7)-(8) induces the invariant manifold defined by the mappings $\pi_1 : \mathbb{R}^\nu \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^\nu \to \mathbb{R}^n$ satisfying the PDAE with boundary condition

$$\frac{\partial \begin{bmatrix} \pi_1(\omega) \\ M\big(\pi_1(\omega), \pi_2(\omega)\big)\pi_2(\omega) \end{bmatrix}}{\partial \omega} s(\omega) = \begin{bmatrix} \pi_2(\omega) \\ -K\big(\pi_1(\omega), \pi_2(\omega)\big)\pi_1(\omega) \end{bmatrix} - \begin{bmatrix} 0 \\ D\big(\pi_1(\omega), \pi_2(\omega)\big)\pi_2(\omega) \end{bmatrix} + \begin{bmatrix} 0 \\ B\big(\pi_1(\omega), \pi_2(\omega)\big)\ell(\omega) \end{bmatrix},$$

$$\pi_1(0) = 0, \quad \pi_2(0) = 0.$$

It follows that the time-domain moment of the system (1)-(2) at (s, ℓ) is $C\big(\pi_1(\omega), \pi_2(\omega), \ell(\omega)\big)$. Both necessity and sufficiency of the conditions follow now from this PDAE and the definitions of time-domain moment at (s, ℓ). ■

By fixing the dimension of (1)-(2) to be $n = \nu$ and assigning the solution $\pi_1(\omega) = \omega$ in Theorem 4, the conditions on the mappings $M(\cdot)$, $D(\cdot)$, $K(\cdot)$, $B(\cdot)$, and $C(\cdot)$ for (1)-(2) to achieve moment matching at (s, ℓ) become

$$\frac{\partial \big(M\big(\omega, s(\omega)\big)s(\omega)\big)}{\partial \omega} s(\omega) = -D\big(\omega, s(\omega)\big)s(\omega) - K\big(\omega, s(\omega)\big)\omega + B\big(\omega, s(\omega)\big)\ell(\omega),$$

and

$$C\big(\omega, s(\omega), \ell(\omega)\big) = \eta(\omega).$$

Hence, by construction, a system of nonlinear second-order equations achieving moment matching at (s, ℓ) is given by

$$\overbrace{M(\zeta, \dot{\zeta})\dot{\zeta}}^{\cdot} = -D(\zeta, \dot{\zeta})\dot{\zeta} + \Big(K\big(\zeta, s(\zeta)\big) - K(\zeta, \dot{\zeta})\Big)\zeta + \left(D\big(\zeta, s(\zeta)\big) + \frac{\partial \big(M\big(\zeta, s(\zeta)\big)s(\zeta)\big)}{\partial \zeta}\right) s(\zeta) - B\big(\zeta, s(\zeta)\big)\ell(\zeta) + B(\zeta, \dot{\zeta})u, \quad (19)$$

$$y = \eta(\zeta) - C\big(\zeta, s(\zeta), \ell(\zeta)\big) + C(\zeta, \dot{\zeta}, u), \quad (20)$$

with state $\zeta(t) \in \mathbb{R}^\nu$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and where $M(\cdot)$, $D(\cdot)$, $K(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are any smooth mappings such that the system is regular. This can be proven by putting the system (19)-(20) into a generalized state-space form and determining the time-domain moment at (s, ℓ), or alternatively by extracting this family of systems from the parameterization given in [10, Theorem 2] as shown in the following theorem.

Theorem 5: Let $\eta(\cdot)$ be the moment at (s, ℓ) of the system (5)-(6). Then the system (19)-(20) achieves moment matching at (s, ℓ) for any smooth mappings $M(\cdot)$, $D(\cdot)$, $K(\cdot)$, $B(\cdot)$, and $C(\cdot)$ such that the system is regular.

Proof: Consider the parameterized family of systems in Theorem 2 for $k = \nu$. Let $\overline{E}(\cdot)$, $\overline{f}(\cdot)$, and $\overline{h}(\cdot)$ be arbitrary smooth mappings and select

$$H(\zeta, \gamma, u) := C\big(\zeta, \gamma + s(\zeta), u\big) - C\big(\zeta, s(\zeta), \ell(\zeta)\big) + \overline{h}\big(\zeta, \ell(\zeta)\big) - \overline{h}(\zeta, u),$$

so that $H\big(\zeta, 0, \ell(\zeta)\big) = 0$, select

$$P(\zeta, \gamma) := M\big(\zeta, \gamma + s(\zeta)\big)\big(\gamma + s(\zeta)\big),$$

and

$$N(\zeta, \gamma, u) := \Big(K\big(\zeta, s(\zeta)\big) - K\big(\zeta, \gamma + s(\zeta)\big)\Big)\zeta - D\big(\zeta, \gamma + s(\zeta)\big)\big(\gamma + s(\zeta)\big) + D\big(\zeta, s(\zeta)\big)s(\zeta) + \frac{\partial \big(M\big(\zeta, s(\zeta)\big)s(\zeta)\big)}{\partial \zeta} s(\zeta) - B\big(\zeta, s(\zeta)\big)\ell(\zeta) + B\big(\zeta, \gamma + s(\zeta)\big)u,$$

so that

$$N\big(\zeta, 0, \ell(\zeta)\big) = \frac{\partial \big(M\big(\zeta, s(\zeta)\big)s(\zeta)\big)}{\partial \zeta} s(\zeta) = \frac{\partial P(\zeta, 0)}{\partial \zeta} s(\zeta),$$

and select

$$Q(\zeta, \gamma) := \zeta - \overline{E}(\zeta),$$

and

$$Z(\zeta, \gamma, u) := \gamma + s(\zeta) - \frac{\partial \overline{E}}{\partial \zeta} s(\zeta) + \overline{f}\big(\zeta, \ell(\zeta)\big) - \overline{f}(\zeta, u),$$

so that

$$Z\big(\zeta, 0, \ell(\zeta)\big) = s(\zeta) - \frac{\partial \overline{E}}{\partial \zeta} s(\zeta) = \frac{\partial Q(\zeta, 0)}{\partial \zeta} s(\zeta).$$

Having met the conditions (16)-(18), substituting these mappings into the model (14)-(15) yields the system

$$\dot{\zeta} = \gamma + s(\zeta),$$

$$\overbrace{M\big(\zeta, \gamma + s(\zeta)\big)\big(\gamma + s(\zeta)\big)}^{\cdot} = -D\big(\zeta, \gamma + s(\zeta)\big)\big(\gamma + s(\zeta)\big) + \Big(K\big(\zeta, s(\zeta)\big) - K\big(\zeta, \gamma + s(\zeta)\big)\Big)\zeta + \left(D\big(\zeta, s(\zeta)\big) + \frac{\partial \big(M\big(\zeta, s(\zeta)\big)s(\zeta)\big)}{\partial \zeta}\right) s(\zeta) - B\big(\zeta, s(\zeta)\big)\ell(\zeta) + B\big(\zeta, \gamma + s(\zeta)\big)u,$$

$$y = \eta(\zeta) - C\big(\zeta, s(\zeta), \ell(\zeta)\big) + C\big(\zeta, \gamma + s(\zeta), u\big),$$

and substituting the first equation into the second yields the representation (19)-(20). Hence, the system (14)-(18) parameterizes the system (19)-(20) which proves the result. ■

IV. The Case of Linear Systems

The results of Section III can easily be specialized to the linear setting. Suppose that the signal generator (7)-(8) is linear, so $s(\omega) := S\omega$ and $\ell(\omega) := L\omega$ with $S \in \mathbb{R}^{\nu\times\nu}$ and $L \in \mathbb{R}^{m\times\nu}$. Considering the model (19)-(20), by defining $M(\zeta,\dot\zeta) := M$, $K(\zeta,\dot\zeta) := K$, $D(\zeta,\dot\zeta) := D$, and $B(\zeta,\dot\zeta) := B$, where $M \in \mathbb{R}^{\nu\times\nu}$, $K \in \mathbb{R}^{\nu\times\nu}$, $D \in \mathbb{R}^{\nu\times\nu}$, and $B \in \mathbb{R}^{\nu\times m}$ are any matrices such that the system is regular, one obtains the model achieving moment matching at $(S\omega, L\omega)$

$$M\ddot\zeta = (K-K)\zeta - D\dot\zeta + DS\zeta + MS^2\zeta - BL\zeta + Bu,$$

$$y = \eta(\zeta) + C(\zeta,\dot\zeta,u) - C(\zeta,S\zeta,L\zeta),$$

or

$$Bu = M\ddot\zeta + D\dot\zeta + (BL - DS - MS^2)\zeta, \quad (21)$$

$$y = \eta(\zeta) + C(\zeta,\dot\zeta,u) - C(\zeta,S\zeta,L\zeta), \quad (22)$$

where the matrices M, D, B, and the mapping $C(\cdot)$, are free parameters. If, in addition, the time-domain moment is linear, so $\eta(\omega) = W\omega$, $W \in \mathbb{R}^{p\times\nu}$, and the mapping $C(\cdot)$ is selected to be linear, so $C(\zeta,\dot\zeta,u) = C_1\zeta + C_2\dot\zeta + \overline{D}u$, $C_1 \in \mathbb{R}^{p\times\nu}$, $C_2 \in \mathbb{R}^{p\times\nu}$, $\overline{D} \in \mathbb{R}^{p\times m}$, then

$$Bu = M\ddot\zeta + D\dot\zeta + (BL - DS - MS^2)\zeta,$$

$$y = (W - C_2S - \overline{D}L)\zeta + C_2\dot\zeta + \overline{D}u,$$

where the matrices C_2 and $\overline{D}$ are additional free parameters. This discussion is formalized in the following corollary which follows directly from Theorem 5.

Corollary 1: Let $\eta(\cdot)$ be the time-domain moment of the system (5)-(6) at $(S\omega, L\omega)$. Let M, D, and B be any matrices such that the system (21)-(22) is regular, and let $C(\cdot)$ be any smooth mapping. Then the system (21)-(22) achieves moment matching at $(S\omega, L\omega)$.

As a result of Corollary 1, if the signal generator (7)-(8) is a linear system then the system of second-order equations achieving moment matching at $(S\omega, L\omega)$ in Theorem 5 can always be constructed to be linear in the dynamics given by (19), just as in the first-order equations scenario of [6].

V. Example

In [17] a model for a planar elbow manipulator with a remotely driven link is given in the form of a system of second-order equations. After performing a suitable change of coordinates so that $(q,\dot q) = (0,0)$ is a stable equilibrium corresponding to the straight-downward configuration of the manipulator, the model, denoted HOM, is given by the equations

$$\begin{bmatrix} m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c2}\cos(q_2-q_1) \\ m_2\ell_1\ell_{c2}\cos(q_2-q_1) & m_2\ell_{c2}^2 + I_2 \end{bmatrix}\begin{bmatrix}\ddot q_1 \\ \ddot q_2\end{bmatrix} + \begin{bmatrix} d_1\dot q_1 - m_2\ell_1\ell_{c2}\sin(q_2-q_1)\dot q_2^2 \\ m_2\ell_1\ell_{c2}\sin(q_2-q_1)\dot q_1^2 + d_2\dot q_2 \end{bmatrix} + \begin{bmatrix} (m_1\ell_{c1} + m_2\ell_1)g\sin(q_1) \\ m_2\ell_{c2}g\sin(q_2) \end{bmatrix} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix},$$

where, for $i = 1, 2$, q_i denotes the angle in radians between the ith link and the downward configuration, τ_i denotes the torque input to the ith joint, m_i denotes the mass of the ith link, I_i denotes the moment of inertia of the ith link, ℓ_i denotes the length of the ith link, ℓ_{ci} denotes the distance from the previous joint to the centre of mass of the ith link, g is the gravitational constant, and where we have added the constants $d_i > 0$, $i = 1, 2$, so that the equilibrium point is locally asymptotically stable.

Selecting an output mapping $y_{HOM} = (q_1,\ q_2)^\top$, restricting the system to be single-input by setting $\tau_1 := B_1u$, $\tau_2 := B_2u$, $B_1 \in \mathbb{R}$, $B_2 \in \mathbb{R}$, $u(t) \in \mathbb{R}$, and choosing a linear signal generator with $S = 0$ and $L = 1$, then by putting the model into a first-order form the time-domain moment of the system at $(S\omega, L\omega)$ is analytically determined to be

$$\eta(\omega) = \begin{bmatrix} \sin^{-1}\left(\frac{B_1\omega}{(m_1\ell_{c1}+m_2\ell_1)g}\right) \\ \sin^{-1}\left(\frac{B_2\omega}{m_2\ell_{c2}g}\right)\end{bmatrix}.$$

Consider now the parameters $m_1 = 3$, $m_2 = 2$, $\ell_1 = 1$, $\ell_{c1} = 0.5$, $\ell_2 = 1$, $\ell_{c2} = 0.5$, $I_1 = 1$, $I_2 = 0.5$, $g = 9.8$, $d_1 = 1$, $d_2 = 1$, $B_1 = 0.5$, $B_2 = -1$. We construct two systems achieving moment matching at $(S\omega, L\omega)$. Consider a first reduced order model, denoted ROM1, which has linear state dynamics

$$u = \ddot z + \dot z + z, \quad y_{ROM1} = \eta(z),$$

with states $z(t) \in \mathbb{R}$ and $\dot z(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y_{ROM1}(t) \in \mathbb{R}^2$, constructed from (21)-(22) by selecting the parameters $M = 1$, $D = 1$, and $B = 1$, and selecting the mapping $C(z,\dot z,u) = 0$. Consider also a second reduced order model, denoted ROM2, which has a nonlinear stiffness property

$$\big(1 + 0.5(\sqrt{\gamma} + \dot\gamma)\big)u = \ddot\gamma + \dot\gamma + (\gamma + 0.5\sqrt{\gamma}^3),$$
$$y_{ROM2} = \eta(\gamma),$$

with states $\gamma(t) \in \mathbb{R}$ and $\dot\gamma(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y_{ROM2}(t) \in \mathbb{R}^2$, constructed from (19)-(20) by setting $M(\gamma,\dot\gamma) := 1$, $D(\gamma,\dot\gamma) := 1$, $K(\gamma,\dot\gamma) = 0$, $B(\gamma,\dot\gamma) := 1 + 0.5(\sqrt{\gamma} + \dot\gamma)$, and $C(\gamma,\dot\gamma,u) = 0$.

The model HOM, and the reduced order models ROM1 and ROM2, are simulated for a piecewise continuous input signal with a discontinuous change every 20 seconds. The time histories in Figure 1 indicate that moment matching at $(S\omega, L\omega)$ is achieved.

VI. Conclusion

This paper has studied the problem of constructing nonlinear systems of second-order equations achieving moment matching. This goal has been accomplished by selecting the free mappings in a family of parameterized systems achieving moment matching in such a way that the resulting state-space model can be manipulated into a second-order form. In an equivalent state-space form, the resulting interpolant has twice the number of states as the minimal first-order interpolant, however, this leaves the designer with many additional free parameters in the model and allows for the

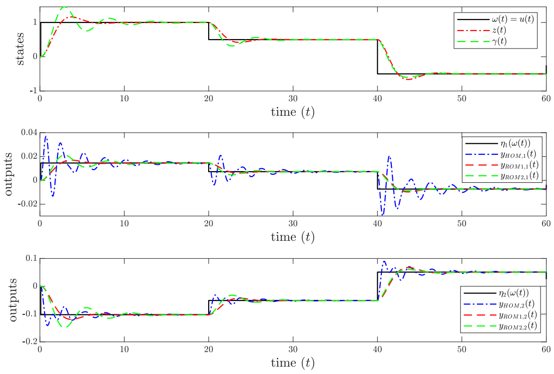

Fig. 1. State and output responses of the high order model (HOM) and of the reduced order models ROM1 and ROM2.

construction of second-order equations achieving moment matching without any additional conditions on the signal generator or the time-domain moment; given an arbitrary signal generator and associated time-domain moment, a system of second-order equations achieving moment matching can always be constructed. The results of the paper are demonstrated by determining the time-domain moment for a two-link robotic manipulator and constructing second-order equations achieving moment matching.

Future research should consider determining conditions for which the parameters of second-order moment matching systems can be associated to physical systems. For example, given a high-dimensional mechanical system one could construct a lower-dimensional mechanical system which produces the same steady-state response for selected input signals, potentially allowing one to build simpler systems achieving desired specifications. Furthermore, the problem of moment matching for nonlinear network systems on graphs should be considered.

REFERENCES

[1] J. M. A. Scherpen, "Balancing for nonlinear systems," *Systems & Control Letters*, vol. 21, no. 2, pp. 143–153, 1993.

[2] J. M. A. Scherpen, "H_∞ balancing for nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 7, pp. 645–668, 1996.

[3] W. S. Gray and J. Mesko, "General input balancing and model reduction for linear and nonlinear systems," in *European Control Conference (ECC)*, pp. 2862–2867, IEEE, 1997.

[4] S. Kung and D. Lin, "Optimal Hankel-norm model reductions: Multivariable systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 832–852, 1981.

[5] M. G. Safonov, R. Y. Chiang, and D. J. N. Limebeer, "Optimal Hankel model reduction for nonminimal systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 4, pp. 496–502, 1990.

[6] A. Astolfi, "Model reduction by moment matching for linear and nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2321–2336, 2010.

[7] T. C. Ionescu, "Two-sided time-domain moment matching for linear systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2632–2637, 2015.

[8] P. Schulze, T. C. Ionescu, and J. M. A. Scherpen, "Families of moment matching-based reduced order models for linear descriptor systems," in *European Control Conference (ECC)*, pp. 1964–1969, IEEE, 2016.

[9] G. Scarciotti and A. Astolfi, "Nonlinear model reduction by moment matching," *Foundations and Trends® in Systems and Control*, vol. 4, no. 3-4, pp. 224–409, 2017.

[10] J. D. Simard, A. Moreschini, and A. Astolfi, "Parameterization of all moment matching interpolants," in *2023 European Control Conference (ECC)*, pp. 1–6, IEEE, 2023.

[11] A. J. Mayo and A. C. Antoulas, "A framework for the solution of the generalized realization problem," *Linear algebra and its applications*, vol. 425, no. 2-3, pp. 634–662, 2007.

[12] A. C. Antoulas, S. Lefteriu, and A. C. Ionita, "A tutorial introduction to the Loewner framework for model reduction," *Model Reduction and Approximation: Theory and Algorithms*, vol. 15, p. 335, 2017.

[13] I. V. Gosea and A. C. Antoulas, "Data-driven model order reduction of quadratic-bilinear systems," *Numerical Linear Algebra with Applications*, vol. 25, no. 6, p. e2200, 2018.

[14] J. D. Simard and A. Astolfi, "Nonlinear model reduction in the Loewner framework," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 5711–5726, 2021.

[15] J. D. Simard and A. Astolfi, "Loewner functions and model order reduction for nonlinear input-affine descriptor systems," in *60th IEEE Conference on Decision and Control (CDC)*, pp. 6887–6894, 2021.

[16] B. Van Der Pol and J. Van Der Mark, "The heartbeat considered as a relaxation oscillation, and an electrical model of the heart," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 6, no. 38, pp. 763–775, 1928.

[17] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot modeling and control*, vol. 3. Wiley New York, 2006.

[18] A. Eichler, J. Moser, J. Chaste, M. Zdrojek, I. Wilson-Rae, and A. Bachtold, "Nonlinear damping in mechanical resonators made from carbon nanotubes and graphene," *Nature nanotechnology*, vol. 6, no. 6, pp. 339–342, 2011.

[19] S. J. Elliott, M. G. Tehrani, and R. S. Langley, "Nonlinear damping and quasi-linear modelling," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 373, 2015.

[20] D. G. Meyer and S. Srinivasan, "Balancing and model reduction for second-order form linear systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 11, pp. 1632–1644, 1996.

[21] Y. Chahlaoui, D. Lemonnier, A. Vandendorpe, and P. Van Dooren, "Second-order balanced truncation," *Linear Algebra and its Applications*, vol. 415, no. 2, pp. 373–384, 2006.

[22] T. Reis and T. Stykel, "Balanced truncation model reduction of second-order systems," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 14, no. 5, pp. 391–406, 2008.

[23] C. Hartmann, V.-M. Vulcanov, and C. Schütte, "Balanced truncation of linear second-order systems: a hamiltonian approach," *Multiscale Modeling & Simulation*, vol. 8, no. 4, pp. 1348–1367, 2010.

[24] Z. Bai and Y. Su, "Dimension reduction of large-scale second-order dynamical systems via a second-order Arnoldi method," *SIAM Journal on Scientific Computing*, vol. 26, no. 5, pp. 1692–1709, 2005.

[25] B. Salimbahrami and B. Lohmann, "Order reduction of large scale second-order systems using Krylov subspace methods," *Linear Algebra and its Applications*, vol. 415, no. 2, pp. 385–405, 2006.

[26] C. A. Beattie and S. Gugercin, "Krylov-based model reduction of second-order systems with proportional damping," in *Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05*, pp. 2278–2283, IEEE, 2005.

[27] X. Cheng, Y. Kawano, and J. M. A. Scherpen, "Reduction of second-order network systems with structure preservation," *IEEE Transactions on Automatic Control*, vol. 62, pp. 5026 – 5038, 2017.

[28] J. D. Simard and A. Astolfi, "On the construction and parameterization of interpolants in the Loewner framework," *Automatica*, 2023. accepted.

[29] J. D. Simard, X. Cheng, and A. Moreschini, "Interpolants with second-order structure in the Loewner framework," in *22nd World Congress of the International Federation of Automatic Control (IFAC)*, 2023.

[30] G. L. Baker and J. A. Blackburn, *The pendulum: a case study in physics*. OUP Oxford, 2008.

[31] W. Weaver Jr., S. P. Timoshenko, and D. H. Young, *Vibration problems in engineering*. John Wiley & Sons, 1991.

[32] D. S. Bernstein and S. P. Bhat, "Lyapunov stability, semistability, and asymptotic stability of matrix second-order systems," 1995.