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Data-driven reachability analysis of Lipschitz nonlinear systems via support vector data description

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Abstract— **This paper is concerned with data-driven reachability analysis of discrete-time nonlinear systems without any dynamical model. We use only a number of observations of trajectories of the system to estimate the actual reachable set. With the data set, using the Support Vector Data Description (SVDD) technique, we propose a sample-based approximation method to solve the reachability analysis problem, which can be considered as a one-class classification problem. Under the framework of scenario optimization, we then derive over-approximations of the reachable set in a probabilistic sense with Lipschitz continuity and other regularity conditions. Finally, we demonstrate the proposed method on a numerical example.**

I. Introduction

Real-world engineered systems are becoming more and more complex with the rapid evolution of Cyber-Physical Systems (CPS) technologies [1]. There is a need to check the correctness of such systems with computer-based techniques, taking into account the interplay between cyber and physical components [2]. This is particularly important for safety-critical applications. Many verification algorithms rely on reachability analysis, which aims to check properties of the system by computing the set of reachable states from a given initial set of configurations under all possible inputs, see, e.g, [3].

From a control-theoretic point of view, specifications of a dynamical system can be often described as a set of the state space and the properties of interest can be captured by propagation of sets under the framework of the so-called set-theoretic approach [4]. With this understanding, reachability analysis of dynamical systems boils down to the computation of set propagation. For special cases, set propagation can be computed using optimization techniques such as linear programming [5]–[7], linear matrix inequalities (LMI) [8], [9] and sum-of-squares (SOS) programming [8]–[10]. In general, it is numerically intractable to compute exact reachable sets [11]. For this reason, many algorithms and toolboxes have been developed to compute over-approximations of reachable sets (see, e.g., [12]). However, all the aforementioned works require an exact mathematical model of the system. In practice, identifying a dynamical model can be a challenging task, due to disturbances, uncertainties and hybrid behaviors arising from discrete actions in CPS [13], which restricts the applicability of model-based approaches.

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Modelling challenges in real-world systems have eventually motivated the research on data-driven reachability analysis, which computes or estimates reachable sets by only using a finite number of observations of trajectories. With a finite set of data, an important issue is that the accuracy of the computed set needs to be evaluated in order to infer the actual reachable set. A general way of obtaining over-approximations of reachable sets is to use the notions of coverings and packings of compact sets [14]. Indeed, when a covering of the reachable set is available, an over-approximation can be immediately constructed by taking a neighborhood of the covering with a proper covering radius. Such an idea is adopted in [15], [16]. However, to generate a covering with a small covering radius, the data set has to be dense, which means that it takes a large amount of data. In [17], interval-based approximations are also proposed with a guarantee on probabilistic correctness. As interval approximations can be conservative, ellipsoidal sets are used in [18] and probabilistic convex constraints satisfaction is derived based on chance-constrained results in the framework of scenario optimization [19]–[21]. The probabilistic guarantees in [17], [18] are all given in terms of ϵ-accurate reachable sets, which means that further developments are needed to achieve over-approximations. In addition, it has been reported in [22], [23] that it is in general conservative to use chance-constrained results as an intermediate step in the derivation of objective value function performance in scenario optimization. We show that the same reasoning is also applicable in the derivation of over-approximations of reachable sets since the sampled-based reachability analysis problem can be considered as a scenario program. Convex hulls are also considered in reachability anlaysis in [24] and probabilistic guarantees are derived using random set theory. While convex hulls allow to approximate complicated reachable sets as the number of samples increases, the method in [24] only provides asymptotic convergence results.

Data-driven reachability analysis is also closely related with estimation of the support of a distribution or one-class classification [25]–[27], in which generalization error bounds are obtained in the framework of statistical learning theory [28]. In principle, these bounds can be leveraged to provide probabilistic over-approximations of reachable sets. However, the results in these works can be conservative as they are developed in a very general setting without taking the structure of the underlying problem into account. It is well known in scenario optimization [19]–[21] that the convexity property can be utilized to derive tight bounds for random convex programs. In this paper, we use the Support Vector Data Description (SVDD) technique [29] to formulate the sampled-based reachability analysis problem. Inspired by scenario optimization, we formulate the reachability analysis problem as a convex optimization problem using the SVDD technique, in order to make use of the convexity property.

The rest of the paper is organized as follows. The next section gives some preliminaries on reachability analysis and the problem statement. Section III discusses the problem of learning compact sets and presents results on probabilistic over-approximations from the perspective of scenario optimization. In Section IV, based on the results in Section III, over-approximations of reachable sets are derived under some regularity conditions. Some simulation results are provided Section V.

Notation. The non-negative real number set and the non-negative integer set are denoted by $\mathbb{R}^+$ and $\mathbb{Z}^+$ respectively. For any $p \geq 1$, the ℓ_p norm of a vector $x \in \mathbb{R}^n$ is $\|x\|_p$ ($\|x\|$ is the ℓ_2 norm by default). For any $T \in \mathbb{Z}^+$, let $[T]$ denote the set $\{0, 1, \cdots, T\}$. Given any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$, $\mathbb{B}_n(x, r)$ and $\mathbb{S}_n(x, r)$ denote respectively the closed ball and the hypersphere with the center x and the radius r in $\mathbb{R}^n$. For convenience, let $\mathbb{B}_n$ and $\mathbb{S}_n$ be the unit closed ball and the unit hypersphere in $\mathbb{R}^n$. Let $\mu(S)$ denote the Lebesgue measure of S for any Lebesgue-measurable $S \subset \mathbb{R}^n$.

II. Preliminaries and problem statement

We consider discrete-time nonlinear systems in the form of

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)), \quad t \in \mathbb{Z}^+ \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector including uncertainties (or disturbances) and references, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is some continuous function. The input is subject to

$$\mathbf{u}(t) \in U, \quad t \in \mathbb{Z}^+ \tag{2}$$

where $U \subset \mathbb{R}^m$ is a compact set. Given an initial state x at $t = 0$ and an input signal $\mathbf{u} : \mathbb{Z}^+ \to \mathbb{R}^m$, let $\xi(t; x, \mathbf{u})$ denote the solution of (1). In this paper, we consider reachability analysis of System (1) from an initial set $X_0 \subset \mathbb{R}^n$.

The reachable sets of System (1) from X_0 can be defined as, $\forall t \in \mathbb{Z}^+$,

$$\mathcal{R}_t(X_0) := \{\xi(t; x, \mathbf{u}) : x \in X_0, \mathbf{u}(\ell) \in U, \forall \ell \in [t]\}. \tag{3}$$

with $\mathcal{R}_0(X_0) = X_0$. The reachable set over a horizon T is defined as

$$\mathcal{R}_{[T]}(X_0) := \cup_{t \in [T]} \mathcal{R}_t(X_0) \tag{4}$$

Our goal is to compute or estimate reachable sets defined above by only using a finite number of trajectories without knowing the dynamical model. We make the following assumptions.

Assumption 1. *$X_0 \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact.*

Assumption 2. *Given the horizon T, there exist positive constants L_x and L_u such that*

$$\|f(x_2, u_2) - f(x_1, u_1)\| \leq L_x \|x_2 - x_1\| + L_u \|u_2 - u_1\| \quad \forall (x_1, u_1), (x_2, u_2) \in \mathcal{R}_{[T]}(X_0) \times U. \tag{5}$$

III. Learning regular compact sets

In this section, we consider identification of compact sets that satisfy certain regularity conditions from data. Given a compact set $S \subset \mathbb{R}^n$, let $\omega_N := \{x_1, x_2 \cdots, x_N\}$ a data set sampled inside S, where $N \in \mathbb{Z}^+$ is the number of data points. From the data set ω_N, we attempt to compute an estimation of S. The problem of estimating sets can be considered as a one-class classification problem [25]–[27].

A. Support vector data description

We use the Support Vector Data Description (SVDD) technique to solve this set estimation problem. Let $\Phi : S \to \mathbb{R}^p$ be a feature map. In general, the objective of SVDD is to find the smallest hypersphere that encloses the majority of the data $\{\Phi(x) : x \in \omega_N\}$. In this paper, we are interested in over-approximations of the set S. From the perspective of scenario optimization [19]–[21], we formulate the following problem

$$(c(\omega_N), \gamma(\omega_N)) := \arg\min_{c,\gamma} \gamma \tag{6a}$$

$$\text{s.t.} \quad \|\Phi(x) - c\| \leq \gamma, \forall x \in \omega_N. \tag{6b}$$

The robust counterpart of (6) is the following problem

$$(c^*, \gamma^*) := \arg\min_{c,\gamma} \gamma \tag{7a}$$

$$\text{s.t.} \quad \|\Phi(x) - c\| \leq \gamma, \forall x \in S. \tag{7b}$$

The solution of this robust optimization problem provides the smallest hypersphere that encloses the set $\Phi(S)$.

The uniqueness of the solution of Problem (6) is guaranteed by the following lemma.

Lemma 1. *For any $N \in \mathbb{Z}^+$ and $\omega_N \subset S$, Problem* (6) *has a unique solution.*

Proof: With the feature map Φ, Problem (6) can be interpreted as the smallest bounding sphere problem and there exist several proofs, see, e.g., Proposition 3.9 in [30]. To be self-contained, we present a proof from an elementary geometric argument. Suppose there exist two different solutions (c_1, γ) and (c_2, γ) with $c_1 \neq c_2$. Then, the set $\Phi(\omega_N)$ is enclosed by both $\mathbb{B}_p(c_1, \gamma)$ and $\mathbb{B}_p(c_2, \gamma_2)$, which means that $\Phi(\omega_N) \subseteq \mathbb{B}_p(c_1, \gamma) \cap \mathbb{B}_p(c_2, \gamma)$. Let γ' be the distance from $\frac{c_1+c_2}{2}$ to the boundary of $\mathbb{B}_p(c_1, \gamma) \cap \mathbb{B}_p(c_2, \gamma)$. By definition, $\mathbb{B}_p(c_1, \gamma) \cap \mathbb{B}_p(c_2, \gamma) \subset \mathbb{B}_p(\frac{c_1+c_2}{2}, \gamma')$. Since $c_1 \neq c_2$, $\gamma' < \gamma$. This means that $\Phi(\omega_N)$ can be enclosed by a smaller hypersphere $\mathbb{B}_p(\frac{c_1+c_2}{2}, \gamma')$, which violates the optimality. Thus, we conclude that $c_1 = c_2$. □

For notational convenience, given any $c \in \mathbb{R}^p$ and $\gamma \in \mathbb{R}^+$, let

$$\mathcal{S}(c, \gamma) := \{x \in \mathbb{R}^n : \|\Phi(x) - c\| \leq \gamma\}. \tag{8}$$

We then present an important result which provides the connection between Problem (6) and Problem (7).

Lemma 2. *Suppose the set $S \subset \mathbb{R}^n$ is compact and the feature map $\Phi : S \to \mathbb{R}^p$ is continuous. For any $N \in \mathbb{Z}^+$ and $\omega_N \subset S$, let $(c(\omega_N), \gamma(\omega_N))$ be the solution of Problem* (6). *Then, there exists a set $\bar{\omega}_{p+1} \subset S$ of $p+1$ elements such that $S \subseteq \mathcal{S}(c(\bar{\omega}_{p+1}), \gamma(\bar{\omega}_{p+1}))$.*

Lemma 2 suggests that, to enclose the set $\Phi(S)$, we only need to enclose $p+1$ points in $\Phi(S)$. This result serves as a key tool in the derivation of over-approximations of the set S in the sequel. With the results above, we also show that the solution of the robust problem (7) is unique, as stated in the following proposition.

Proposition 1. *Consider Problem* (7) *with a compact set $S \subset \mathbb{R}^n$ and a continuous map $\Phi : S \to \mathbb{R}^p$. The solution (c^*, γ^*) of Problem* (7) *is unique.*

B. Probabilistic set over-approximations

We now discuss formal guarantees on the sampled solution in (6). Let $(S, \mathcal{B}(S), \mathbb{P}_S)$ be a probability space where $\mathcal{B}(S)$ is the Borel σ-algebra of S and $\mathbb{P}_S : \mathcal{B}(S) \to [0,1]$ is a probability measure. The following Lipschitz continuity condition is needed.

Assumption 3. *The function $\Phi : S \to \mathbb{R}^p$ is Lipschitz continuous with constant L, i.e., $\forall x_1, x_2 \in S$,*

$$\|\Phi(x_2) - \Phi(x_1)\| \leq L\|x_2 - x_1\|. \tag{9}$$

Following [31], we also impose a regularity condition on the set S.

Assumption 4. *For the probability space $(S, \mathcal{B}(S), \mathbb{P}_S)$, there exists a $\bar{r} \in \mathbb{R}^+$ and $\delta \in \mathbb{R}^+$ such that*

$$\forall x \in S, \forall r \leq \bar{r}, \mathbb{P}_S\{\mathbb{B}_n(x,r) \cap S\} \geq \delta\mu(\mathbb{B}_n(x,r)) \tag{10}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^n$.

For any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$, it is shown in [32] that

$$\mu(\mathbb{B}_n(x,r)) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n \tag{11}$$

where $\Gamma(\cdot)$ is the gamma function defined as

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \tag{12}$$

With the discussions above, we derive a probabilistic set over-approximation for the set S from the solution in (6) as stated in the following theorem.

Theorem 1. *Consider the compact set $S \subset \mathbb{R}^n$, the feature map $\Phi : S \to \mathbb{R}^p$ and the probability space $(S, \mathcal{B}(S), \mathbb{P}_S)$. Suppose Assumptions 3 & 4 hold. For any $N \in \mathbb{Z}^+$ with $N \geq p+1$, let ω_N be N independent and identically distributed (i.i.d.) samples drawn according to the probability measure $\mathbb{P}_S$ and $(c(\omega_N), \gamma(\omega_N))$ be given as in* (6). *For any $\epsilon \in [0, \min\{1/(p+1), \frac{\delta\pi^{n/2}\bar{r}^n}{\Gamma(\frac{n}{2}+1)}\}]$, with probability no smaller than $1 - \Phi(\epsilon; p+1, N)$,*

$$S \subset \mathcal{S}\left(c(\omega_N), \overline{\Psi}(\gamma(\omega_N) + L\zeta(\epsilon; n, \delta)), \gamma(\omega_N)\right), \tag{13}$$

where $\bar{r}$ and δ are from Assumption 4, (c^*, γ^*) is defined as in (7),

$$\zeta(\epsilon; n, \delta) := \sqrt[n]{\frac{\Gamma(\frac{n}{2}+1)\epsilon}{\delta\pi^{\frac{n}{2}}}}, \tag{14}$$

$$\overline{\Psi}(\gamma_1, \gamma_2) := \gamma_1 + \sqrt{\frac{\gamma_1^2 - \gamma_2^2}{2}}, \text{ and} \tag{15}$$

$$\Phi(\epsilon; p+1, N) := \sum_{i=1}^{p+1} (-1)^{i-1} \binom{p+1}{i} (1 - i\epsilon)^N. \tag{16}$$

C. Sublevel sets with shape conditions

In the rest of this section, we discuss sufficient conditions to fulfill Assumption 4 when S is a sublevel set of some function. Suppose that the set S can be expressed as a sublevel set of a function $v : \mathbb{R}^n \to \mathbb{R}$ as

$$S := \{x : v(x) \leq 0\} \tag{17}$$

We assume that the function $v(\cdot)$ satisfies some smoothness and boundedness conditions.

Assumption 5. *The function $v : \mathbb{R}^n \to \mathbb{R}$ is differentiable in S, and there exists L_v such that*

$$v(x_2) \leq v(x_1) + (\nabla v(x_1))^\top (x_2 - x_1) + \frac{1}{2} L_v \|x_2 - x_1\|^2, \quad \forall x_1, x_2 \in S \tag{18}$$

In addition, there exists D_v such that

$$\|\nabla v(x)\| \leq D_v, \forall x \in S. \tag{19}$$

In addition, to ensure the regularity condition in Assumption 4, we also need the following boundary condition.

Assumption 6. *There exist positive constants α_v and β_v such that $\{x \in S : v(x) > -\alpha_v, \|\nabla v(x)\| \leq \beta_v\} = \emptyset$.*

Assumption 5 is a standard smoothness and boundedness condition. Assumption 6 essentially means that for any point x near the boundary of S, $\|\nabla v(x)\|$ is strictly larger than some positive constant, which is illustrated in Figure 1. With this assumption, we exclude isolated points or regions. Under these assumptions, we show that Assumption 4 is satisfied, as stated in the following proposition.

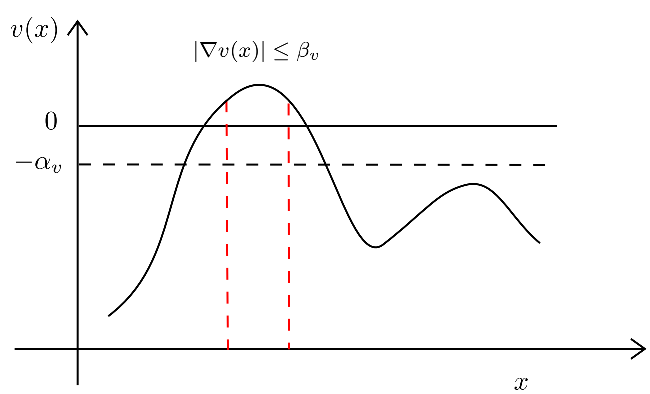

Fig. 1. Illustration of the boundary condition: The segment between the two red dashed lines denotes the set $\{x : |\nabla v(x)| \leq \beta_v\}$.

Proposition 2. *Consider a compact set S, expressed as in* (17) *for some function $v : \mathbb{R}^n \to \mathbb{R}$. Let $(S, \mathcal{B}(S), \mathbb{P}_S)$ be the uniform probability space. Suppose Assumptions 5 & 6 hold. Let*

$$\bar{r}_v := \min\{\sqrt{\frac{2\alpha_v}{L} + \frac{D_v^2}{L^2}} - \frac{D_v}{L}, \frac{2}{L}\beta_v\}, \tag{20}$$

$$\delta_v := \min\{\frac{1}{\mu(S)}, \frac{2\Gamma(\frac{n}{2}+1)}{\pi^{1/2}\Gamma(\frac{n+1}{2})}\Theta(n)\}, \tag{21}$$

where $\mu(S)$ is the Lebesgue measure of S and

$$\Theta(n) := \begin{cases} \frac{2}{2^{2n}} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}-k} \binom{n}{k} \frac{1}{n-2k}, & n \text{ is odd;} \\ \frac{1}{2^{2n}} \binom{n}{\frac{n}{2}}, & n \text{ is even.} \end{cases} \tag{22}$$

Then, it holds that, $\forall x \in S, \forall r \leq \bar{r}_v$,

$$\mathbb{P}_S\{\mathbb{B}_n(x,r) \cap S\} \geq \delta_v \mu(\mathbb{B}_n(x,r)). \tag{23}$$

where $\mu(\mathbb{B}_n(x,r))$ is the Lebesgue measure of $\mathbb{B}_n(x,r)$.

IV. Data-driven reachability analysis

Based on the results in the previous section, we discuss data-driven computation of reachable sets of System (1) with the initial set X_0 and the input set U. Suppose there exist $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ such that

$$X_0 := \{x \in \mathbb{R}^n : h(x) \leq 0\}, \tag{24}$$

$$U := \{u \in \mathbb{R}^m : g(u) \leq 0\}. \tag{25}$$

Similarly, we need a smoothness and boundedness condition for the both functions.

Assumption 7. *The functions $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are differentiable in X_0 and U, and there exist L_h and L_g such that*

$$h(x_2) \leq h(x_1) + (\nabla h(x_1))^\top (x_2 - x_1) + \frac{1}{2}L_h\|x_2 - x_1\|^2, \quad \forall x_1, x_2 \in X_0, \tag{26}$$

$$g(u_2) \leq g(u_1) + (\nabla g(u_1))^\top (u_2 - u_1) + \frac{1}{2}L_g\|u_2 - u_1\|^2, \quad \forall u_1, u_2 \in U. \tag{27}$$

In addition, there exist D_h and D_g such that

$$\|\nabla h(x)\| \leq D_h, \forall x \in X_0, \tag{28}$$

$$\|\nabla g(u)\| \leq D_g, \forall u \in U. \tag{29}$$

This assumption is not restrictive in practice. When it does not hold, we can replace X_0 and U with their over-approximations in the form of (24) and (25) respectively. We also impose a similar boundary condition as in Assumption 6 on X_0 and U.

Assumption 8. *There exist positive constants $\alpha_h, \alpha_g, \beta_h$ and β_g such that*

$$\{x \in X_0 : h(x) > -\alpha_h, \|\nabla h(x)\| \leq \beta_h\} = \emptyset, \tag{30}$$

$$\{u \in U : g(u) > -\alpha_g, \|\nabla g(u)\| \leq \beta_g\} = \emptyset. \tag{31}$$

Let the horizon be T. Given a set of initial states $\omega_N^x := \{x_1, x_2, \cdots, x_N\}$ and a set of input signals $\omega_N^u := \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_N\}$ of horizon T, we define

$$\omega_N(t) := \{\xi(t; x_i, \mathbf{u}_i) : i = 1, 2, \cdots, N\}, \forall t \in [T]. \tag{32}$$

with $\omega_N(0) = \omega_N^x$. Consider the uniform probability spaces $(X_0, \mathcal{B}(X_0), \mathbb{P}_{X_0})$ and $(U, \mathcal{B}(U), \mathbb{P}_U)$, let the initial states and inputs be sampled independently from the uniform probability measures $\mathbb{P}_{X_0}$ and $\mathbb{P}_U$. With a feature map $\Phi : \mathbb{R}^n \to \mathbb{R}^p$, we again use the SVDD method to approximate the reachable set $\mathcal{R}_T(X_0)$. Using the definition in (8), the obtained reachable set can be expressed as $\mathcal{S}(c(\omega_N(T)), \gamma(\omega_N(T)))$. Following the arguments in the previous section, we provide probabilistic guarantees on the computed reachable set.

We first show a Lipschitz continuity result on the solution of System (1), which is needed for the derivation of over-approximations of $\mathcal{R}_T(X_0)$.

Lemma 3. *Consider System* (1) *with the solution $\xi(t; x, \mathbf{u})$ for any initial state x at $t = 0$ and any input signal $\mathbf{u} : \mathbb{Z}^+ \to \mathbb{R}^m$. Suppose Assumption 2 holds. For any $t \in [T-1]$, it holds that*

$$\|\xi(t+1; x_2, \mathbf{u}_2) - \xi(t+1; x_1, \mathbf{u}_1)\| \leq L_x^{t+1}\|x_2 - x_1\| + \sum_{\ell=0}^{t} L_x^{t-\ell} L_u \|\mathbf{u}_2(\ell) - \mathbf{u}_1(\ell)\|, \tag{33}$$
$$\forall x_1, x_2 \in X_0, \forall \mathbf{u}_1(\ell), \mathbf{u}_2(\ell) \in U, \forall \ell \in [T-1].$$

Following the definitions in Proposition 2, we define:

$$\bar{r}_h := \min\{\sqrt{\frac{2\alpha_h}{L} + \frac{D_h^2}{L^2}} - \frac{D_h}{L}, \frac{2}{L}\beta_h\}, \tag{34}$$

$$\delta_h := \min\{\frac{1}{\mu(X_0)}, \frac{2\Gamma(\frac{n}{2}+1)}{\pi^{1/2}\Gamma(\frac{n+1}{2})}\Theta(n)\}, \tag{35}$$

$$\bar{r}_g := \min\{\sqrt{\frac{2\alpha_g}{L} + \frac{D_g^2}{L^2}} - \frac{D_g}{L}, \frac{2}{L}\beta_g\}, \tag{36}$$

$$\delta_g := \min\{\frac{1}{\mu(U)}, \frac{2\Gamma(\frac{m}{2}+1)}{\pi^{1/2}\Gamma(\frac{m+1}{2})}\Theta(m)\} \tag{37}$$

where $\Theta(\cdot)$ is given in (22) and $\mu(\cdot)$ is the Lebesgue measure. By slight abuse of notation, we use $\mu(\cdot)$ to denote the Lebesgue measure in both $\mathbb{R}^n$ and $\mathbb{R}^m$.

With this result, we are able to derive probabilistic over-approximations of the reachable set $\mathcal{R}_T(X_0)$, as stated in the following theorem.

Theorem 2. *Consider System* (1) *with the solution $\xi(t; x, \mathbf{u})$ for any initial state x at $t = 0$ and any input signal $\mathbf{u} : \mathbb{Z}^+ \to \mathbb{R}^m$, the initial state set X_0 and the input set U. Suppose Assumptions 1, 2, 7 & 8 hold and the feature map $\Phi : \mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz continuous with constant L in $\mathcal{R}_T(X_0)$. Given $N \in \mathbb{Z}^+$ with $N \geq \max\{p+1, \ln(\frac{1}{p})/\ln(1-(\min\{\bar{\epsilon}_x, \bar{\epsilon}_u\})^{T+1})\}$, let ω_N^x and ω_N^u be N independent and identically distributed (i.i.d.) initial states and input signals drawn according to the uniform probability measures $\mathbb{P}_{X_0}$ and $\mathbb{P}_U$ respectively. We define $\omega_N(t)$ as in* (32) *for any*

$t \in [T]$. *Then, for any* $\epsilon \in [0, \min\{1/(p+1), \epsilon_x, \epsilon_u\}]$, *with probability no smaller than* $1 - \Phi(\epsilon^{T+1}; p+1, N)$,

$$\mathcal{R}_T(X_0) \subset \mathcal{S}\left(c(\omega_N(T)), \overline{\Psi}(\gamma(\omega_N(T)) + L\eta(T,\epsilon)), \gamma(\omega_N(T))\right) \quad (38)$$

where $\epsilon_x := \frac{\delta_h \pi^{n/2} \bar{r}_h^n}{\Gamma(\frac{n}{2}+1)}, \epsilon_u := \frac{\delta_g \pi^{m/2} \bar{r}_g^m}{\Gamma(\frac{m}{2}+1)}$, $\overline{\Psi}(\cdot,\cdot)$ *is given in* (15) *and*

$$\eta(t,\epsilon) := L_x^t \sqrt[n]{\frac{\Gamma(\frac{n}{2}+1)\epsilon}{\delta_h \pi^{\frac{n}{2}}}} + \sum_{\ell=0}^{t-1} L_x^\ell L_u \sqrt[m]{\frac{\Gamma(\frac{m}{2}+1)\epsilon}{\delta_g \pi^{\frac{m}{2}}}}, \forall t \geq 1, \quad (39)$$

with $\eta(0,\epsilon) := \sqrt[n]{\frac{\Gamma(\frac{n}{2}+1)\epsilon}{\delta_h \pi^{\frac{n}{2}}}}$.

The result in Theorem 2 can be adapted to approximate the reachable set over horizon T, defined as $\mathcal{R}_{[T]}$ in (4), by using the trajectory data over horizon T.

Corollary 1. *Suppose the conditions in Theorem 2 hold. Then, for any* $\epsilon \in [0, \min\{1/(p+1), \epsilon_x, \epsilon_u\}]$, *with probability no smaller than* $1 - \Phi(\epsilon^{T+1}; p+1, N)$,

$$\mathcal{R}_{[T]}(X_0) \subset \mathcal{S}\left(c(\omega_N([T])), \overline{\Psi}(\gamma(\omega_N([T])) + L\eta_{[T]}(\epsilon)), \gamma(\omega_N([T]))\right) \quad (40)$$

where

$$\eta_{[T]}(\epsilon) := \max_{t\in[T]} \eta(t,\epsilon), \quad \omega_N([T]) := \bigcup_{t\in[T]} \omega_N(t). \quad (41)$$

V. Simulation

We consider the following disturbed nonlinear system (called the LaSalle system [33]):

$$\mathbf{x}_1(t+1) = -\frac{\mathbf{x}_2(t)}{1+\mathbf{x}_1^2(t)},$$
$$\mathbf{x}_2(t+1) = \frac{0.9\mathbf{x}_1(t)}{1+\mathbf{x}_2^2(t)} + \mathbf{u}(t), \quad t \in \mathbb{Z}^+$$

where $\mathbf{u}(t) \in \mathbb{R}$ is a disturbance varying in the set $U = [-0.1, 0.1]$. The nominal system of this example is globally asymptotically stable at the origin. It can also be verified that $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$ is a robust invariant set of the disturbed system, i.e., for any $x \in X$ and $u \in U$, $x^+ \in X$. We consider the reachability analysis problem with the initial set $X_0 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. Note that X_0 and U can be expressed as sublevel sets of the functions $h(x) = \|x\|^2 - 1$ and $g(u) = u^2 - 0.01$ respectively. We simulate $N = 10000$ trajectories over a horizon of 10 and the initial conditions are generated randomly and uniformly in the unit box X_0.

We first take a feature map of monomials with the maximal degree being 5, given by $T(x) = (x^{[1]}, x^{[2]}, \cdots, x^{[5]})^\top$, where $x^{[d]} \in \mathbb{R}^{\binom{n+d-1}{d}}$ denote the d-lift of x which consists of all possible monomials of degree d, indexed by all the possible exponents α of degree d, $x_\alpha^{[d]} = \sqrt{\alpha!}x^\alpha$ where $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = d$ and $\alpha!$ denotes the multinomial coefficient $\alpha! := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. By calculation, the dimension of the feature map is $p = 21$. For comparison, we also take 21 radial basis functions (RBF) in the form of

$$T(x) = \begin{pmatrix} \|x - s_1\|^2 \log(\|x - s_1\|) \\ \vdots \\ \|x - s_{21}\|^2 \log(\|x - s_{21}\|) \end{pmatrix}$$

where s_i is randomly selected with the uniform distribution on X. By solving Problem (6), the results are given in Figure 2. For the case with mononials, four support points (which are the points that define the solution of Problem (6)) are detected; for the RBF case, there are 3 support points. By computing the constants in Assumptions 2, 7 & 8, over-approximations can be also computed using Theorem 2.

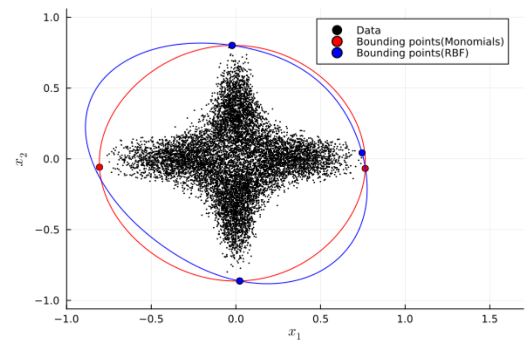

Fig. 2. Reachable set computation for the LaSalle system: The red and blue curves denote the approximations using monomials and radial basis functions.

VI. Conclusions

We have proposed a data-driven reachable set computation method for Lipschitz nonlinear systems. Our method does not require any mathematical model of the system. We use the SVDD technique to formulate the reachability anlaysis problem as a robust convex program, which allows to use the convexity property to analyze the probabilistic correctness of our method. We have found that, unlike the statistical learning literature where the covering number of a compact set is often used to derive generalization error bounds, for the reachable set computation problem in this paper, we only need to consider the covering of a few critical points. A numerical example is taken to illustrate the proposed method.

Appendix

Proof of Proposition 2

Consider any $\bar{x} \in S$, by definition, $v(\bar{x}) \leq 0$. From Assumption 5, for any $x \in S$,

$$v(x) \leq v(\bar{x}) + (\nabla v(\bar{x}))^\top (x - \bar{x}) + \frac{1}{2}L_v\|x - \bar{x}\|^2. \quad (42)$$

Case I: We first consider the case when $v(\bar{x}) \leq -\alpha_v$. Note that $\|\nabla v(\bar{x})\| \leq D_v$. By some manipulations, it can be verified that the right-hand side of (42) is smaller than or equal to 0 when $\|x - \bar{x}\| \leq \sqrt{\frac{2\alpha_v}{L} + \frac{D_v^2}{L^2}} - \frac{D_v}{L}$. Thus,

$$\mathbb{B}_n(\bar{x}, \sqrt{\frac{2\alpha_v}{L} + \frac{D_v^2}{L^2}} - \frac{D_v}{L}) \subset S.$$

This means that, for any $r \leq \sqrt{\frac{2\alpha_v}{L} + \frac{D_v^2}{L^2}} - \frac{D_v}{L}$,

$$\mathbb{P}_S\{\mathbb{B}_n(\bar{x}, r) \cap S\} \geq \frac{1}{\mu(S)}\mu(\mathbb{B}_n(\bar{x}, r)).$$

Case II: We then consider the case when $v(\bar{x}) > -\alpha_v$. From Assumption 6, $\|\nabla v(\bar{x})\| > \beta_v$. To ensure that the right-hand side of (42) is nonpositive, we consider the points in the set $\mathcal{T}_{\bar{x}} := \{x \in \mathbb{R}^n : (\nabla v(\bar{x}))^\top (x - \bar{x}) + \frac{1}{2}L_v\|x - \bar{x}\|^2 \leq 0\}$. In addition, we consider the ball $\mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})$. The Lebesgue measure of $\mathcal{T}_{\bar{x}} \cap \mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})$ can be expressed as

$$\int_0^{\pi/2} V_{n-1}\left(\frac{\|\nabla v(\bar{x}))\|}{L}\sin(2\theta)\right)\frac{2\|\nabla v(\bar{x}))\|}{L}\sin(2\theta)d\theta$$

where $V_{n-1}\left(\frac{\|\nabla v(\bar{x}))\|}{L}\sin(2\theta)\right)$ denote the volume of the hypersphere with radius $\frac{\|\nabla v(\bar{x}))\|}{L}\sin(2\theta)$ in $\mathbb{R}^{n-1}$ (an explicit expression can be found in [32]). By some manipulations,

$$\begin{aligned}
&\mu\left(\mathcal{T}_{\bar{x}} \cap \mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})\right) \\
=&\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})}(\frac{\|\nabla v(\bar{x}))\|}{L})^n \int_0^{\pi/2}(\sin(2\theta))^n d\theta \\
=&\frac{2^{n+1}\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})}(\frac{\|\nabla v(\bar{x}))\|}{L})^n \Theta(n)
\end{aligned}$$

Note that

$$\mu\left(\mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})\right) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}(\frac{2\|\nabla v(\bar{x}))\|}{L})^n.$$

Hence,

$$\frac{\mu\left(\mathcal{T}_{\bar{x}} \cap \mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})\right)}{\mu\left(\mathbb{B}_n(\bar{x}, \frac{2\|\nabla v(\bar{x}))\|}{L})\right)} = \frac{2\Gamma(\frac{n}{2}+1)}{\pi^{1/2}\Gamma(\frac{n+1}{2})}\Theta(n).$$

This, together with the fact that $\|\nabla v(\bar{x}))\| > \beta_v$, implies that for any $r \leq \frac{2\beta_v}{L}$,

$$\frac{\mu\left(\mathcal{T}_{\bar{x}} \cap \mathbb{B}_n(\bar{x}, r)\right)}{\mu\left(\mathbb{B}_n(\bar{x}, r)\right)} \geq \frac{2\Gamma(\frac{n}{2}+1)}{\pi^{1/2}\Gamma(\frac{n+1}{2})}\Theta(n).$$

Combining **Case I** and **Case II** yields the result. □

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