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Dynamic Consensus under Weak Coupling: a case study of nonlinear oscillators

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***Abstract*— Dynamic consensus is a term coined in[1] [1] to denote the state of synchronization of complex networked systems. It covers the common paradigm of consensus in which case all the systems stabilize at a common *equilibrium* point. It is known that for certain networks (e.g., of homogeneous systems) dynamic consensus is achievable provided the interconnection gain is elevated. In this case, all the systems behave as *one* average dynamical system. In this paper we analyze the collective behavior of heterogeneous Stuart-Landau oscillators under weak coupling. We show that their behavior cannot be characterized by a single average system, but by a reduced-order network. We give a detailed characterization of the latter and establish a relation with the eigenvalues of the underlying Laplacian matrix, hence, with the network's topology.**

I. Introduction

Interconnected systems have garnered increasing interest across scientific communities over the years. One thoroughly studied problem pertains to the synchronization of nonlinear oscillators, at least since [2]; see also [3]–[5].

We study networks of Stuart Landau oscillators, which represent a generic model of nonlinear oscillators near a Hopf bifurcation [2], [6]. They are often used as a prototype model of different oscillatory systems, such as in LASERs [7], genetic networks [8], and neuronal networks [9], [10], *etc.* Networks of Stuart Landau's oscillators are very intriguing because of the rich behavior they may exhibit under different conditions pertaining to their parameters and the coupling strength, even in the case of "networks" of as few as *two* oscillators—see [2], let alone for networks of higher order (containing many more than two nodes).

One of the possible collective behaviors observed in networks of Stuart Landau oscillators includes total synchronization. In [1] we introduce the term "dynamic consensus" to denote the collective behavior consisting in all the oscillators achieving total synchronization with respect to an average oscillator [12]. Such behavior, however, is possible only for relatively high coupling—we specify farther below the meaning of "high".

If the coupling gain is relatively low, other phenomena, such as partial synchronization, (also known as 'clustering') may appear. That is, subgroups of oscillators synchronize among themselves, but with behaviors differing from one cluster to another, even if the nodes are not directly interconnected (remote synchronization)—see *e.g.*, [9] and [10]. As explained in [13] and [14], clusters typically appear for identical oscillators due to low coupling gain and symmetries in the interconnection graph. The spectral properties of the graph Laplacian may also strongly affect such emergent behavior. This particular behavior is also evoked in [15], where the authors analyze the *remote synchronization* in a star network of Stuart Landau oscillators, with mismatched parameters.

To ascertain the collective behavior of networked Stuart-Landau oscillators, [15] presents a bifurcation diagram by analyzing the behavior of the network of N units. The analysis applies, notably, to weakly-coupled oscillators. A different technique relies on model reduction. In [16], the authors present a reduced system comprising $N_R < N$ oscillators for a network of N nodes. However, the proposed reduced-order model therein is defined on the basis of a complex nonlinear coupling, even if, initially, the oscillators are interconnected via a diffusive (linear) coupling. The nonlinearities in the reduced-order model stymie the analysis of the overall collective behavior. In contrast, a model reduction of order one, as in [12], easens the analysis based on the properties of the network (invariant sets, bounds of the solutions, frequency of oscillations), but it is restricted to the case of high gain. In this article, for networks with weak coupling, we propose a moderately complex model of reduced order, but larger than one, which nevertheless involves linear interconnections, which favors the analysis.

The passage from a network of N agents to a reduced network has been treated before in the literature, but differs from our approach. For instance, in [17]–[19] the model reduction involves graph partition. In [20], a method based on eigenvalue assignment is derived for a structure-preserving model-reduction for linear multi-agent systems. Alternatively, balanced truncation is used in [21] and [22]. In this paper, we use the spectral properties of the network's Laplacian to provide a detailed characterization of the model's order and, most importantly, we show that the systems conforming the reduced network correspond to Stuart Landau oscillators themselves, interconnected via diffusive coupling. Our results apply to systems with certain heterogeneity but, as in [15], [16], they are restricted to network topologies satisfying specific structural properties. The problem in greater generality is, to the best of our knowledge, open.

In the next section we describe the problem formulation. In Section III we describe our model-reduction approach; in

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[1]The term dynamic consensus is also used in other instance in the literature, but with a completely different meaning—see *e.g.*, [11]

Section IV we present our main statements. Our theoretical findings are illustrated with numerical simulations in Section V and we provide some closing remarks in Section VI.

II. Problem formulation

Consider N interconnected Stuart-Landau oscillators,

$$\dot{x}_i = \alpha x_i - \omega_i y_i - x_i(x_i^2 + y_i^2) + u_{1i} \tag{1a}$$

$$\dot{y}_i = \omega_i x_i + \alpha y_i - y_i(x_i^2 + y_i^2) + u_{2i}, \tag{1b}$$

where α, ω_i, x_i, and $y_i \in \mathbb{R}$ for all $i \in \{1, 2, \cdots, N\}$, and x_i, y_i are Cartesian coordinates on the plane. Stuart Landau oscillators may also be modeled using complex state variables—see, *e.g.*, [6]. Relative to such models, x_i and y_i represent, respectively, the real and the imaginary parts of each oscillator's state.

We assume that the network units are connected via diffusive coupling over an undirected and connected graph. For the ith unit the coupling $u_i = [u_{1i}\, u_{2i}]^\top$ is given by

$$u_{1i} = -\bar{\gamma} \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{2a}$$

$$u_{2i} = -\bar{\gamma} \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{2b}$$

where the scalar $\bar{\gamma} > 0$ corresponds to the coupling gain. The weights of the interconnections amongst the nodes define the adjacency matrix, $\mathcal{A} = [a_{ij}]_{i,j \in \{1,2,\cdots,N\}}$, as well as the Laplacian matrix $L = \big[l_{ij}\big]$ where

$$l_{ij} = \begin{cases} -a_{ij} & \text{if} \quad i \neq j \\ \sum_{\substack{i=1 \\ i \neq j}}^{N} a_{ij} & \text{if} \quad i = j. \end{cases}$$

More precisely, we are interested in the possible synchronized behavior of the oscillators (1), under the effect of the inputs defined in (2). Two factors intervene. On one hand, the network's topology, which is defined by the coefficients a_{ij}, and, on the other hand, the magnitude of the coupling strength. For instance, for networks with an underlying undirected connected graph topology and with $\bar{\gamma} > 0$ sufficiently large, the networked systems' trajectories $z_i(t) = \big[x_i(t)\ y_i(t)\big]^\top$ converge, in a practical sense, to the solution of an averaged dynamical system,

$$\dot{z}_m = F_m(z_m, e), \tag{3}$$

where $z_m = \frac{1}{N} \sum_{k=1}^{N} z_k$ and e is a synchronization error defined as

$$e := z - \frac{1}{N} \big[\mathbf{1}_N \otimes I_2\big] \big[\mathbf{1}_N \otimes I_2\big]^\top z \iff e = \begin{bmatrix} z_1 - z_m \\ \vdots \\ z_N - z_m \end{bmatrix}. \tag{4}$$

That is, on the manifold $\{e = 0\}$, the trajectories are all synchronized with the averaged dynamical system $\dot{z}_m = F_m(z_m, 0)$—cf [1], [12].

This paper is devoted to analyzing networks whose coupling gain is not high enough to entail synchronization. It is known that in this case many possible behaviors may appear, even for small networks [2]. As we show in this paper, in the case of weakly coupled networks, the emergent dynamical system consists of a network itself, albeit of reduced order. Hence, with state $z_m = \big[z_{m1}^\top\, z_{m2}^\top \cdots z_{mN_R}^\top\big]^\top$, with z_{mi} corresponding to the state of one oscillator in the emergent network. Such a model, which generalizes the one based on (3)-(4), is described next.

III. Model description

The first step to describe the reduced-order model is to define the order N_R, *i.e.*, the number of nodes that constitute the reduced-order network. To that end, akin to [1] we define the synchronization error. Then, let $z_i = [x_i\ y_i]^\top$,

$$z := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}, \qquad F(z) := \begin{bmatrix} f(x_1, y_1) \\ f(x_2, y_2) \\ \vdots \\ f(x_N, y_N) \end{bmatrix}, \tag{5}$$

where $f(x_i, y_i) = \big[-x_i(x_i^2 + y_i^2) \quad -y_i(x_i^2 + y_i^2)\big]^\top$. With this notation, the diffusive coupling inputs u_i, defined in (2), can be re-written in the compact form $u = -\bar{\gamma}\big[L \otimes I_2\big] z$. Hence, the network dynamics become

$$\dot{z} = F(z) + \mathcal{M} z - \bar{\gamma}\big[L \otimes I_2\big] z, \tag{6}$$

where $\mathcal{M} \in \mathbb{R}^{2N \times 2N}$ corresponds to the block diagonal matrix

$$\mathcal{M} := \begin{bmatrix} \mathcal{M}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{M}_2 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \mathcal{M}_N \end{bmatrix},$$

with $\mathcal{M}_i \in \mathbb{R}^{2 \times 2}$ such that

$$\mathcal{M}_i = \begin{bmatrix} \alpha & -\omega_i \\ \omega_i & \alpha \end{bmatrix}. \tag{7}$$

Characterizing the collective emergent behavior and multi-agent synchronization for heterogeneous systems interconnected over generic graphs remains an open problem. In what follows, focus on networked systems with underlying graphs satisfying the following hypotheses.

Assumption 1: The eigenvalues $\lambda_i(L)$ of the Laplacian L and their associated eigenvectors v_i are such that:

$$\lambda_1(L) = 0 < \lambda_2(L) < \lambda_3(L) \leq \cdots \leq \lambda_N(L),$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

•

We stress that many networks with weighted links possess these properties. This class of networks contains, for example, weighted small-world networks and weighted grid networks—see Fig 1 in the following page.

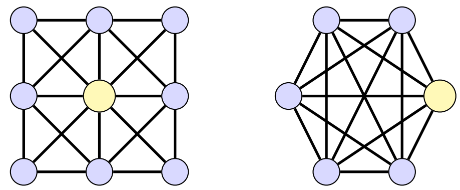

Fig. 1. Example of topologies satisfying hypothesis 1 when the inner parameters of the yellow unit and its coupling gain differ from the remaining nodes' parameters. The left network corresponds to a grid topology, while the one on the right corresponds to the small-world topology.

Next, we pose a hypothesis on the units' individual parameters.

Assumption 2: For $i \in \{2, 3, \cdots, N\}$, $\mathcal{M}_i = \mathcal{M}_2$. •

Assumption 2 allows for only one system in the network to be different from the others. Although this may appear restrictive, it is fairly common in the context of problems concerning "parameters mismatch". In other words, one system in the network has completely different parameters from the rest of the network, and the effect of this discrepancy is studied —see, *e.g.* [23], [24], [25]. The fact that a part of the network is homogeneous is helpful in characterizing emergent dynamics; it has also been used in [15] for bifurcation analysis in a star network.

A. Reduced order network

The definition of the order of the reduced network N_R is intrinsically related to the spectral properties of the linear part of (6). To analyze this relationship, as it may become clearer later, it is convenient to rewrite (6) as

$$\dot{z} = F(z) + \mathcal{M}z - \gamma\big[\mathcal{L} \otimes I_2\big]z, \tag{8}$$

where γ and $\mathcal{L}$ are scaled as follows:

$$\mathcal{L} = \frac{2}{\lambda_2(L)}L, \quad \gamma = \frac{\lambda_2(L)}{2}\bar{\gamma}. \tag{9}$$

Then, Eq. (6) becomes

$$\dot{z} = F(z) + \gamma\tilde{\mathcal{L}}z, \tag{10}$$

where $\tilde{\mathcal{L}} := \big[-\big[\mathcal{L} \otimes I_2\big] + \frac{1}{\gamma}\mathcal{M}\big]$.

For the sake of argument, let us momentarily disregard the nonlinear terms in (10), *i.e.*, $F(z)$. For the system $\dot{z} = \gamma\tilde{\mathcal{L}}z$, the eigenvalues with positive real parts in $\tilde{\mathcal{L}}$ generate unstable modes. In contrast, those with negative real parts generate stable ones. That is, the solution to $\dot{z} = \gamma\tilde{\mathcal{L}}z$ takes the form

$$\begin{aligned} z(t) = &\big[v_1 v_{l1}^\top \otimes I_2\big]z(t) + \big[v_2 v_{l2}^\top \otimes I_2\big]z(t) \\ &+ \cdots + \big[v_{N_R} v_{lN_R}^\top \otimes I_2\big]z(t) + e(t), \end{aligned}$$

where v_k and v_{lk}, for all $k \in \{1, 2, \ldots, M\}$, are respectively the right and the left eigenvectors of $\tilde{\mathcal{L}}$ associated with the M pairs of positive real part eigenvalues of $\tilde{\mathcal{L}}$. On the other hand, $e(t)$ contains the contributions to the solution generated by the stable modes. As $e(t) \to 0$, only the contributions of the unstable modes remain. The number of pairs of positive real part eigenvalues M defines therefore the order N_R of the reduced order network, and the unstable modes determine the asymptotic behavior of the network.

Now, to determine N_R we observe that, under Assumption 1, $\mathcal{L}$ has a unique zero eigenvalue and admits the decomposition

$$\mathcal{L} = V \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2(\mathcal{L}) & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3(\mathcal{L}) & & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_N(\mathcal{L}) \end{bmatrix} V^{-1},$$

where V is the matrix whose columns contain the eigenvectors associated with the eigenvalues of $\mathcal{L}$.

Claim 1: Under Assumptions 1 and 2,

$$\begin{aligned} &\big[V^{-1} \otimes I_2\big]\mathcal{M}\big[V \otimes I_2\big] = \\ &\begin{bmatrix} \frac{\mathcal{M}_1+\mathcal{M}_2}{2} & \frac{\mathcal{M}_1-\mathcal{M}_2}{2} & 0 & \cdots & 0 \\ \frac{\mathcal{M}_1-\mathcal{M}_2}{2} & \frac{\mathcal{M}_1+\mathcal{M}_2}{2} & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{M}_2 & & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \mathcal{M}_2 \end{bmatrix}, \end{aligned} \tag{11}$$

where the matrices $\mathcal{M}_i$ are defined in (7). □

Then given the form of $\tilde{\mathcal{L}}$ we obtain

$$[V^{-1} \otimes I_2]\tilde{\mathcal{L}}[V \otimes I_2] = \text{blockdiag}\big[\,\Omega_i\,\big] \tag{12}$$

where

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} \frac{\mathcal{M}_1+\mathcal{M}_2}{2\gamma} & \frac{\mathcal{M}_1-\mathcal{M}_2}{2\gamma} \\ \frac{\mathcal{M}_1-\mathcal{M}_2}{2\gamma} & \frac{\mathcal{M}_1+\mathcal{M}_2}{2\gamma} - \lambda_2(\mathcal{L})I_2 \end{bmatrix}, \\ \Omega_i &= \frac{1}{\gamma}\mathcal{M}_2 - \lambda_i(\mathcal{L})I_2, \end{aligned}$$

for all $i \in \{3, 4, \cdots, N\}$. Note that $\Omega_1 \in \mathbb{R}^{4\times4}$, $\Omega_i \in \mathbb{R}^{2\times2}$.

The eigenvalues associated to the blocks $\Omega_3, \Omega_4, \cdots, \Omega_N$, are given by

$$\lambda_i(\tilde{\mathcal{L}}) \;= \frac{\alpha}{\gamma} - \lambda_i(\mathcal{L}) \pm i\frac{\omega_2}{\gamma},$$

where $i \in \{5, 6, \cdots, 2N\}$. We see that the number of eigenvalues with positive real parts for these blocks is inversely proportional to the value of γ. Taking into account the change of scale (9), we have $\lambda_2(\mathcal{L}) = 2$. On the other hand, the four eigenvalues associated with the block Ω_1 are given by

$$\lambda_i(\mathcal{L}) = \frac{\alpha}{\gamma} - 1 \pm \frac{\big[4\gamma^2 - 2\omega_1^2 - 2\omega_2^2 \pm 2\tilde{\omega}\big[\bar{\omega}^2 - 4\gamma^2\big]^{\frac{1}{2}}\big]^{\frac{1}{2}}}{\gamma}$$

for $i \in \{1, 2, 3, 4\}$ and $\tilde{\omega} = \omega_1 + \omega_2$—cf [15]. The sign of the real part of those eigenvalues depends on γ and also on $\bar{\omega} = \omega_1 - \omega_2$.

Note that the form of these eigenvalues is identical to those obtained in [15, equation 20]. It is therefore possible to use the results in the latter to determine the evolution of the sign of these eigenvalues. We summarize the cases pertaining to the number of eigenvalues with positive real parts (hence to the order N_R) in Fig 2 below. Since the real part of the eigenvalues also depends on $\bar{\omega} = \omega_1 - \omega_2$, two cases emerge depending on the value of this variable.

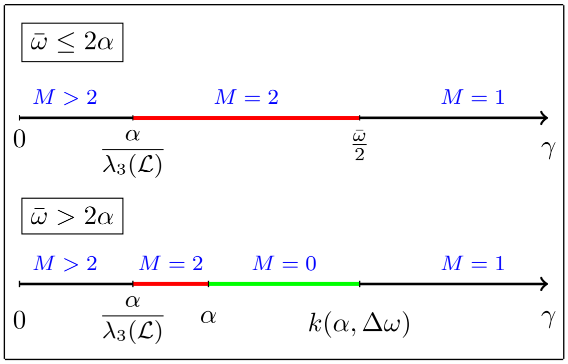

Fig. 2. Variations of M, the number of positive real part eigenvalues of the matrix $\tilde{\mathcal{L}}$ depending on the value of γ. Intervals in red are those where $M = 2$ and $k(\alpha, \bar{\omega}) := \frac{4\alpha^2+|\bar{\omega}|^2}{8\alpha}$.

We gather these two cases in the same figure by presenting the evolution of M as a function of the gain γ.

The case of $M = 0$ corresponds to global asymptotic stability of the linear part $\dot{z} = \gamma\tilde{\mathcal{L}}z$. Conversely, $M = 1$ brings us back to the known result of [12] on the practical stability of the set $\{z_1 = z_2 = \cdots = z_N\}$. In this paper, we focus on the case in which, given the relatively low values of the coupling gain γ, we have $M > 1$. It would therefore make sense for such a network to be modeled by a reduced network of $N_R = 2$ oscillators, since it can also represent the two behaviors explained above ($M = 0$ and $M = 1$).

Remark 1: For the specific case of a star-topology network, in [15] the authors give an analysis of the sign eigenvalues of the block Ω_1 and identify the threshold $k(\alpha, \bar{\omega})$, between the cases of zero and one eigenvalue with positive real part, for $\bar{\omega} > 2\alpha$ and between the cases of two and one such eigenvalue, for $\bar{\omega} \leq 2\alpha$. •

B. Synchronization error characterization

We now unfold a natural definition of the synchronization errors e. Let $V = [V_1 \ V_2]$, where $V_1 \in \mathbb{R}^{N\times N_R}$ gathers the eigenvectors associated to the N_R positive real part eigenvalues in Eq. (12) and $V_2 \in \mathbb{R}^{N\times(N-N_R)}$ the remaining $N - N_R$ eigenvectors of $\mathcal{L}$. Then,

$$V^{-1} = \begin{bmatrix} V_1^\dagger \\ V_2^\dagger \end{bmatrix}.$$

Then, we use V_1 and V_2 these two matrices to introduce the new coordinate $\bar{z} = \left[V^{-1} \otimes I_2\right]z$. Then, using the partition

$$\bar{z} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} := \begin{bmatrix} \left[V_1^\dagger \otimes I_2\right]z \\ \left[V_2^\dagger \otimes I_2\right]z \end{bmatrix}, \tag{13}$$

with $\xi_1 \in \mathbb{R}^{2N_R}$, $\xi_2 \in \mathbb{R}^{2(N-N_R)}$. Using $VV^{-1} = V_1V_1^\dagger + V_2V_2^\dagger = I_N$ we deduce the relation

$$\left[V_2 \otimes I_2\right]\xi_2 = z - \left[V_1 \otimes I_2\right]\xi_1, \tag{14}$$

which is useful to define the synchronization errors $e := \left[V_2 \otimes I_2\right]\xi_2$, as

$$e = z - \left[V_1 \otimes I_2\right]\xi_1.$$

That is, $e = 0$ if and only if $\xi_2 = 0$ and $z = \left[V_1 \otimes I_2\right]\xi_1$. Consequently, for $N_R = 2$, we have $V_1 = [v_1 \, v_2]$ and $V_1^\dagger = [v_{l1}^\top \, v_{l2}^\top]$, which makes e take the form

$$e = z - \left[v_1 \otimes I_2\right]\left[v_{l1}^\top \otimes I_2\right]z - \left[v_2 \otimes I_2\right]\left[v_{l2}^\top \otimes I_2\right]z. \tag{15}$$

This definition of e covers the cases treated previously in the literature where the emerging dynamic is assimilated to a single system and in which case the synchronization error is defined as in (4). Explicitly, (15) yields

$$e = z - \left(\begin{bmatrix} 1 & \mathbf{0}_K^\top \\ \mathbf{0}_K & \frac{1}{K}\mathbf{1}_K\mathbf{1}_K^\top \end{bmatrix} \otimes I_2\right)z,$$

where $\mathbf{1}_K$ is a vector of ones of size K and $\mathbf{0}_K$ a vector of size $K = N - 1$ where all entries are equal to zero. With this definition of e, we see that on $\{e = 0\}$, z_1 is unchanged and, for each $i \in \{2, 3, \cdots, N\}$, z_i converges to the average of the latter. Defining $z_R = \left[z_1^\top \ z_2^\top\right]^\top$ as the state of the reduced order network, on $\{e = 0\}$, we have

$$z_R = \left[W^\top \otimes I_2\right]z, \quad z = \left[Q \otimes I_2\right]z_R \tag{16}$$

where

$$W^\top := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{K} & \frac{1}{K} & \cdots & \frac{1}{K} \end{bmatrix} \quad Q^\top := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{bmatrix}.$$

Thus, on the synchronization manifold $\{e = 0\}$, we have $z_2 = z_3 = \cdots = z_N = \frac{1}{K}\sum_{i=2}^N z_i$. We see that when the coupling gain is not sufficiently high, two dynamics described by the state $z_R = \left[z_1^\top \ z_2^\top\right]^\top = \left[z_1^\top \ \frac{1}{K}\sum_{i=2}^N z_i^\top\right]^\top$ persist asymptotically.

We are ready to present our main statements regarding the dynamics of the synchronization errors e and those of the reduced-order network, with state z_R.

IV. Main Result

Since the partial synchronization studied holds when the set $\{e = 0\}$ is globally asymptotically stable, our first result concerns the analysis of the stability of this set.

Proposition 1: For a network of systems with dynamics (1) in closed loop with (2), under Assumptions 1 and 2, the set $\{e = 0\}$, where e is defined in (15), is globally exponentially stable, for any $\gamma > \gamma_m = \frac{\alpha}{\lambda_3(\mathcal{L})}$. □

Our second, and main statement pertains to the dimension and nature of a reduced-order network that exists for values of the coupling gain satisfying $\gamma > \gamma_m$.

Proposition 2 (Main result): Consider a network of N Stuart-Landau oscillators with dynamics (1), in closed loop with (2) and under Assumptions 1 and 2. Then, on the synchronization manifold $\{e = 0\}$, if $\gamma > \gamma_m = \frac{\alpha}{\lambda_3(\mathcal{L})}$, there exists a network of reduced order $N_R = 2$, whose nodes are dynamical systems of the form

$$\dot{x}_i = \alpha x_i - \omega_i y_i - x_i(x_i^2 + y_i^2) - \gamma\sum_{j=1}^{2}(x_i - x_j) \tag{17a}$$

$$\dot{y}_i = \omega_i x_i + \alpha y_i - y_i(x_i^2 + y_i^2) - \gamma\sum_{j=1}^{2}(y_i - y_j), \tag{17b}$$

where $i \in \{1, 2\}$. □

This reduction of the network of N units to a network of $N_R = 2$ units has the benefit of providing a characterization of different behaviors that the original network may exhibit.

Proposition 3: Consider a network of N Stuart-Landau oscillators with dynamics (1), in closed loop with (2) and satisfying Assumptions 1 and 2. Let $\bar{\omega} = \omega_1 - \omega_2$, $k(\alpha, \bar{\omega}) := \frac{\alpha^2 + |\bar{\omega}|^2}{8\alpha}$, and $\gamma > \gamma_m = \frac{\alpha}{\lambda_3(\mathcal{L})}$. Then,

if $\bar{\omega} > 2\alpha$,

(i) the origin $\{z = 0\}$ is globally exponentially stable, for all $\gamma \in \left]\alpha, k(\alpha, \bar{\omega})\right[$;

(ii) if $\gamma \leq \alpha$, the network shows two trajectories with a phase drift, and,

(iii) if $\gamma \geq k(\alpha, \bar{\omega})$, the network shows two trajectories with a phase lock.

If, otherwise, $\bar{\omega} \leq 2\alpha$,

(iv) the network shows two trajectories with a phase drift for all $\gamma < \frac{\bar{\omega}}{2}$ and

(v) the network shows two trajectories with a phase lock for all $\gamma \geq \frac{\bar{\omega}}{2}$. □

V. Simulation Results

To illustrate our theoretical findings, we present some numerical simulation results. We consider a network of $N = 5$ Stuart Landau oscillators distributed over a grid network—see Fig 3 below.

$$\mathcal{L} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

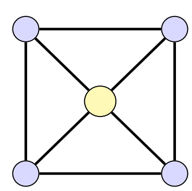

Fig. 3. Graph topology considered in the simulations

For $\mathcal{L}$ above, the eigenvalues satisfy Assumption 2; indeed, we have

$$\lambda_1(\mathcal{L}) = 0,\ \lambda_2(\mathcal{L}) = 2,\ \lambda_3(\mathcal{L}) = \lambda_4(\mathcal{L}) = 3,\ \lambda_5(\mathcal{L}) = 5;$$

idem for the corresponding eigenvectors v_1 and v_2. Furthermore, for all $i \in \{2, 3, \cdots, N\}$, the dynamics of the system corresponds to that given in Eq. (1), with $\alpha = 1$, $\omega_2 = 2$.

As stated in Proposition 3, two possible cases emerge depending on the value of $\bar{\omega} = \omega_1 - \omega_2$. To highlight the possible behaviors, we set two values for ω_1, first satisfying $\bar{\omega} \leq 2\alpha$, then meeting $\bar{\omega} > 2\alpha$.

The third eigenvalue of $\mathcal{L}$ remains important to set the lower bound γ_m in Proposition 1 of the coupling gain γ, here $\lambda_3(\mathcal{L}) = 3$. Therefore, $\gamma_m = \frac{\alpha}{\lambda_3(\mathcal{L})} = \frac{1}{3}$.

Figure 4 presents the evolution of the error $e(t)$ for $\bar{\omega} = 1$ and $\bar{\omega} = 4$ with $\gamma = 0.8$, in the first case, $\omega_1 = 3$, in the second one $\omega_1 = 6$. Nevertheless, in both cases, the synchronization error converges to the origin $\{e = 0\}$, which is a globally asymptotically stable set as seen in Proposition 1.

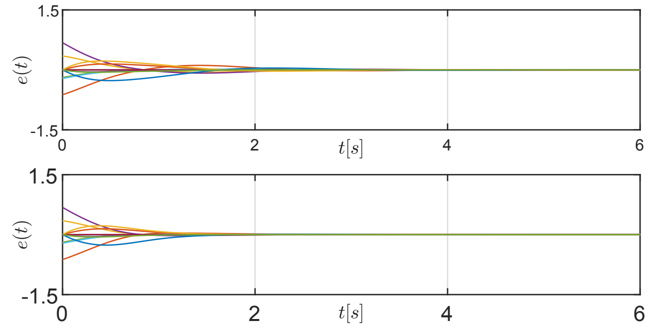

Fig. 4. Evolution of the synchronization error $e(t)$ for $\gamma = 0,8$. In the upper plot $\bar{\omega} = 1$, in the lower plot $\bar{\omega} = 4$.

This remark is in line with what is observed in Figures 5 and 6, which present the behavior of $x(t)$ for each of the two cases, for different values of the coupling gain γ. Furthermore, we remind that in these figures, the dashed black curves represent the behavior of the reduced model; after a short transition interval, all the network units synchronize their behavior with the behavior of the reduced network. On the other hand, the higher the coupling gain γ, the faster the synchronization of the initial network behavior with the reduced network behavior.

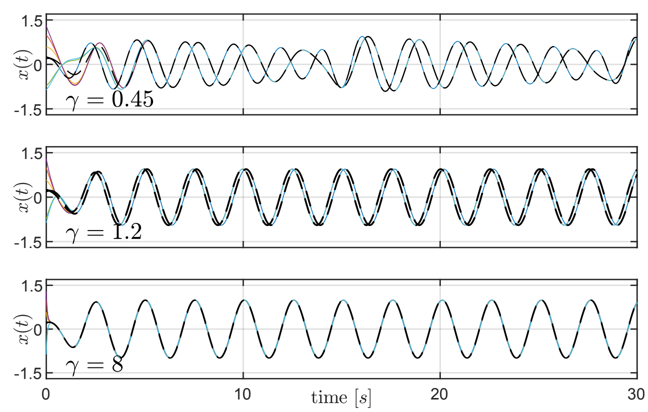

Fig. 5. Numerical results for $\bar{\omega} = 1$, the dashed black curves stand for the trajectories of the reduced order network. After a short transient, the behavior of the initial network is identical to the behavior of the reduced order network, which explains the superposition of the curves

Referring to Figure 2, for the cases that $\bar{\omega} > 2\alpha$, we see the three behaviors mentioned in Proposition 3. For γ between $\frac{\alpha}{\lambda_3(\mathcal{L})}$ and α, we see that the network is well represented by a reduced order network of $N_R = 2$ oscillators. There are no eigenvalues with positive real parts for γ between α and $k(\alpha, \bar{\omega})$. This means that all the oscillators achieve a consensus at the origin $\{x = 0\}$. Finally, with high values of γ, the emergent behavior of the network is best represented by a single oscillator, as mentioned in [12]. The same remarks apply to the correspondence between Figures 2 and 5 for $\bar{\omega} \leq 2\alpha$. This feature underlines the richness of the behavior exhibited by such an oscillator network for $\gamma > \gamma_m$.

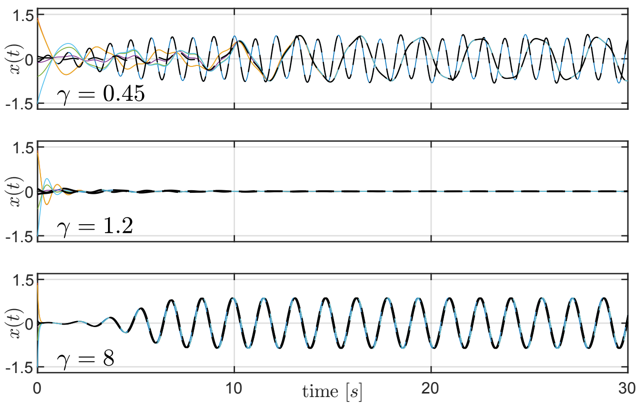

Fig. 6. Numerical results for $\bar{\omega} = 4$, the dashed black curves stand for the trajectories of the reduced order network. After a short transient, the behavior of the initial network is identical to the behavior of the reduced order network, which explains the superposition of the curves.

In Figures 5 and 6, for the interval between $\frac{\alpha}{\lambda_3(\mathcal{L})}$ and $k(\alpha, \bar{\omega})$ we have a phase drift, a phase lock or even a convergence to the origin as explained in Proposition 3, which makes the behavior better represented by two oscillators. The trajectories present a phase lock when the coupling gain is more significant than $k(\alpha, \bar{\omega})$. Therefore as explained in [12] and shown in Figures 5 and 6 we can represent the behavior with only one oscillator to which the convergence error is practically stable. On the other side, besides confirming the result concerning the order of the system, these figures also allow us to observe the results of Proposition 3.

VI. Conclusion

This paper presents an approach to model reduction for a network of nonlinear oscillators that allows modeling N units by a network of only $N_R < N$ units. In the future, we plan to go further in this direction by considering this method for more general networks of nonlinear systems. On the other side, since two oscillators networks of this type have been extensively analyzed in the literature [2], this reduction would also allow us to explore the different behaviors the initial network could adopt based on the reduced model.

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