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Incrementally passive infinite dimensional systems with a constrained state variable

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Abstract— **In this paper, we show that the passivity property of a linear infinite dimensional system, with respect to a given supply rate, is preserved in the presence of a saturating integrator, which restricts a one dimensional component of the state to a compact interval. The resulting nonlinear system is incrementally passive with the same supply rate. We give an application of our main result to a boundary controlled string equation, where the displacement of the string at some interior point is restricted to a compact interval.**

I. Introduction

In this article, we investigate infinite dimensional nonlinear systems obtained by saturating (i.e., restricting to an interval) one state variable of a passive linear infinite dimensional system. A system is called *passive* if at any moment of time, the rate of change of the storage function (intuitively, the stored energy of the system) cannot exceed the supply rate. The supply rate is usually a function of the input and the output of the system, while the storage function is a function of the state. Intuitively, one can think of passive systems as dynamical systems that do not have an internal source of energy, they can only dissipate energy internally, for instance by heat loss via the resistors in electrical circuits. The theory of passive systems has direct applications to electrical circuits, thermodynamics, interconnected or coupled systems and more, see for instance [23], [24]. The passivity of systems with a finite dimensional state space is covered in great detail in the textbook [14, Chapter 4] and the article [24], and it has led to the very fruitful concept of port–Hamiltonian system, introduced in [9]. For some applications of the passivity property, for instance, in output regulation and multi-agent synchronization, we refer to [3], [11], [12]. For passive infinite dimensional systems, we refer to [4], [11], [18], [19, Chapter 11] and [22].

Consider a system on the real Hilbert spaces X, U and Y, where X is the state space, U is the input space and Y is the output space. We define the *energy of the system* (or the storage function) at a time $t \geqslant 0$ to be $\|x(t)\|_X^2$, where $x(t) \in X$ is the state of the system. The system is called *passive* if it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_X^2 \leqslant S(u(t),y(t)) \qquad \forall\, t \geqslant 0, \tag{1}$$

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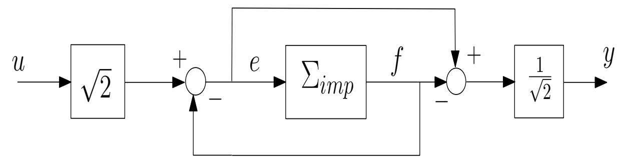

Fig. 1. The scattering passive system Σ_{sca} obtained from the impedance passive system Σ_{imp} via the external Cayley transformation.

where $u(t) \in U$ is the input to the system and $y(t) \in Y$ is the output of the system. The continuous function $S: U \times Y \to \mathbb{R}$, defined on the product space $U \times Y$, is called the *supply rate*.

There are various notions of passivity depending upon the supply rate S. For instance, a system Σ is called *impedance passive* if it satisfies (1) with

$$S(u(t),y(t)) = 2\langle u(t),y(t)\rangle.$$

Here it is assumed that Y is the dual of U and $\langle\cdot,\cdot\rangle$ denotes the duality pairing. In [14], systems that are passive with respect to the above supply rate, are called dissipative.

Another notion of passivity is the scattering passivity. A system Σ is called *scattering passive* if it satisfies (1) with

$$S(u(t),y(t)) = \|u(t)\|_U^2 - \|y(t)\|_Y^2\,.$$

In many cases, we can obtain a scattering passive system from an impedance passive system by the external Cayley transformation, which involves taking an output feedback along with a feedforward term, see Figure 1. Consider an impedance passive system Σ_{imp}, such that the input (also called the effort) to Σ_{imp} is $e(t) \in U$ and its output (also called the flow) is $f(t) \in U$ for $t \geqslant 0$. Then we can obtain a scattering passive system Σ_{sca} by the *external Cayley transformation*:

$$u(t) = \frac{e(t)+f(t)}{\sqrt{2}}, \quad y(t) = \frac{e(t)-f(t)}{\sqrt{2}}\,.$$

In this paper we consider a large class of linear time-invariant (LTI) possibly infinite dimensional systems, called system nodes, whose definition is recalled in Section II. Scattering passive system nodes are inherently well-posed systems. For more details about such passive LTI systems and the relationship between impedance passive systems and scattering passive systems, see [4], [15], [18], [20], [22]. Our aim in this article is to understand the effect of restricting one state variable of a linear system, on the passivity property of the system. We can model such constraints on state variables or on input functions by incorporating a saturating integrator in the dynamics of the system. The use of saturating integrators to satisfy input constraints in control problems for nonlinear finite dimensional systems is well-known, see for instance [8], [10]. Stability properties of infinite dimensional systems with saturated control inputs are studied in [1], [6] and [7]. In [1], set-point regulation of infinite dimensional systems with saturation of control input is investigated. In [6], nonlinear perturbations of a linear infinite dimensional systems are studied, where the nonlinear perturbation is the saturation function of the feedback term.

The idea of saturating (or restricting) a one dimensional component of the state to a compact interval, is motivated from an engineering problem, the dampening of vibrations in a wind turbine tower. The vibrations in a wind turbine tower are sometimes dampened using a mechanical device called the tuned mass damper (TMD), which is essentially a mass, spring and damper system, and it is installed within the nacelle of the wind turbine tower. The idea is to dissipate the vibrational energy via frictional losses (due to the horizontal movement) of the TMD. However, the horizontal displacement of the TMD is restricted to a compact interval, depending upon the length of the nacelle. The coupled wind turbine and TMD system can be modelled as an Euler Bernoulli beam clamped at the bottom and free to move on the top. The free end is connect to the nacelle (with the TMD). The coupled system without the restriction on the horizontal displacement of the TMD, is scattering passive, see [16], [25]. We would like to know whether this system preserves its passivity property, on imposing this restriction. We shall answer this question and the question of well-posedness of such nonlinear systems, in the extended journal version of this article.

In this article, we answer the question of preservation of passivity for the class of system called the system nodes, with one state variable constrained to an interval. Our main result is given in Section III and we give an application of our main result to the string equation with boundary damping, in Section IV.

II. Passive infinite dimensional linear systems

We use standard notation from functional analysis. For any Hilbert space U and any interval J, $L^2(J;U)$ denotes the space of U-valued L^2 functions defined on J, $L^2_{\text{loc}}(J;U)$ is the space of functions such that for any bounded interval $J_0 \subset J$, their restriction to J_0 is in $L^2(J_0;U)$. For any open J, $\mathscr{H}^1(J;U)$ is the Sobolev space of functions in $L^2(J;U)$ that are integrals of functions in $L^2(J;U)$, while $\mathscr{H}^1_{\text{loc}}(J;U)$ is the space of functions that, when restricted to any open bounded interval $J_0 \subset J$, are in $\mathscr{H}^1(J_0;U)$. The space of n times continuously differentiable functions on J with values in U is denoted by $C^n(J;U)$. If $n=0$, we omit to write 0. We use the notation $\mathscr{L}(U,X)$ for the space of bounded linear operators from U to X, $\mathscr{D}(A)$ for the domain of an operator A and $\rho(A)$ for the resolvent set of A. We consider X, U and Y to be real Hilbert spaces.

On the state space X, the input space U and the output space Y, consider a linear system node Σ. *System nodes* are a class of infinite-dimensional linear systems introduced in [17] with very simple assumptions, that generalize the class of well-posed linear systems by stripping away the admissibility and well-posedness assumptions. We briefly recall some needed facts about system nodes and we refer to [17], [18], [20], [22] for more details.

A system node Σ is built around an operator semigroup $\mathbb{T}$ on X with the generator A. The state trajectories of Σ, which are in $C([0,\infty);X)$, evolve according to the differential equation

$$\dot{x}(t) = [A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{2}$$

where $A\&B : \mathscr{D}(A\&B) \to X$ is a closed linear (possibly unbounded) operator with $\mathscr{D}(A\&B)$ dense in $X \times U$. We have $Ax = A\&B\begin{bmatrix} x \\ 0 \end{bmatrix}$ for all $x \in \mathscr{D}(A)$. We introduce the space X_{-1}, which is the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|_X$, where $\beta \in \rho(A)$. This space is independent of the choice of β, since for different choices of β, the norms will be equivalent. The operator A has a unique extension that is bounded from X to X_{-1}, and we denote this extension by the same symbol A. Then there exists a unique operator $B \in \mathscr{L}(U, X_{-1})$ called the *control operator*, such that

$$[A\&B]\begin{bmatrix} x \\ u \end{bmatrix} = Ax + Bu, \quad \forall \begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{D}(A\&B). \tag{3}$$

We introduce another space Z as follow:

$$Z = \mathscr{D}(A) + (\beta I - A)^{-1}BU. \tag{4}$$

This is a Hilbert space with the norm

$$\|z\|_Z^2 = \inf\left\{ \|(\beta I - A)x\|_X^2 + \|v\|_U^2 \;\middle|\; \begin{array}{c} x \in \mathscr{D}(A), v \in U, \\ z = x + (\beta I - A)^{-1}Bv \end{array} \right\}. \tag{5}$$

There exists an operator $C\&D : \mathscr{D}(A\&B) \to Y$ such that the output of Σ is

$$y(t) = [C\&D]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{6}$$

There exist operators $\overline{C} \in \mathscr{L}(Z,Y)$ and $D \in \mathscr{L}(U,X)$ such that

$$[C\&D]\begin{bmatrix} x \\ u \end{bmatrix} = \overline{C}x + Du, \quad \forall \begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{D}(A\&B). \tag{7}$$

Definition 2.1: Let Σ be a linear time invariant system on X, U and Y, defined by the differential equation (2) and the output equation (6). A triple (x,u,y) is called a *classical solution* of (2) and (6) on $[0,\infty)$ if:

(a) $x \in C^1([0,\infty);X)$,

(b) $u \in C([0,\infty);U)$, $y \in C([0,\infty);Y)$,

(c) $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathscr{D}(A\&B)$,

(d) (2) and (6) hold for all $t \geqslant 0$.

A triple (x,u,y) is called a *generalized solution* of (2) and (6) on $[0,\infty)$ if:

(e) $x \in C([0,\infty);X)$,

(f) $u \in L^2_{\rm loc}[0,\infty);U)$, $y \in L^2_{\rm loc}[0,\infty);Y)$,

(g) there exists a sequence (x_n, u_n, y_n) of solutions of (2) and (6) such that when restricting all the functions to an interval $[0,T]$ (where $T>0$), then $x_n \to x$ in $C([0,T];X)$, $u_n \to u$ in $L^2([0,T];U)$ and $y_n \to y$ in $L^2([0,T];Y)$. This must hold for any $T>0$.

A system node Σ admits classical solutions when the input is smooth enough and compatible with the initial state x_0. In fact, if $u \in C^2([0,\infty);U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathscr{D}(A\&B)$, then (2) has a unique classical solution, see [20, Proposition 3.3]. We refer to [19], [20], [22] for details about system nodes.

Remark 2.2: If $B \in \mathscr{L}(U;X)$, for instance in systems with input function acting pointwise over the entire spatial domain, then $[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = Ax(t) + Bu(t)$. We call the operator B bounded if $B \in \mathscr{L}(U;X)$ and unbounded otherwise.

In systems with input function acting on the boundary of the spatial domain, the operator B is usually unbounded, i.e., $B \in \mathscr{L}(U;X_{-1})$. In this case as well, it possible to split $[A\&B]$ as above, using the extension of operator A on X. However, it is often tedious to explicitly define the operator B and sometimes it is identified via its adjoint operator. Therefore, when a system node Σ is described by linear partial differential equations with boundary control, the equation (2) is often easier to represent as follow:

$$\dot{x}(t) = Lx(t), \quad Gx(t) = u(t), \tag{8}$$

where $L \in \mathscr{L}(Z,X)$ and $G \in \mathscr{L}(Z,U)$. Such systems are often called *boundary control systems* and the space Z is often called the *solution space*. The output equation will be the same, i.e., the output $y(t)$ is given by (6). Usually, L is a differential operator and G is a boundary trace operator. We have that $\mathrm{Ker}(G) = \mathscr{D}(A)$, G is onto, and L is an extension of A. We refer to [13] and [21, Chapter 10] for details about boundary control systems and the relation between (2) and (8). For boundary control systems,

$$[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = Lx(t) \qquad \forall \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathscr{D}(A\&B). \tag{9}$$

III. Passive Infinite Dimensional Systems with a Constrained State Variable

A possibly nonlinear time-invariant system $\Sigma^{\mathscr{S}}$ is called *incrementally passive* with respect to the supply rate $S: U \times Y \to \mathbb{R}$, if the solutions $x, \tilde{x} \in C^1([0,\infty);X)$ of $\Sigma^{\mathscr{S}}$ corresponding to the input functions $u, \tilde{u} \in C([0,\infty),U)$ respectively, and the corresponding output functions $y, \tilde{y} \in C([0,\infty);Y)$, satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - \tilde{x}(t)\|_X^2 \leqslant S(u(t) - \tilde{u}(t), y(t) - \tilde{y}(t)). \tag{10}$$

It is easy to check that for a system node Σ, the incremental passivity condition (10) is equivalent to (1).

We denote by X_c a one dimensional subspace of the Hilbert space X, such that $X_c = \{\lambda \varphi_1 \,|\, \lambda \in \mathbb{R}\}$, where $\varphi_1 \in X$ with $\|\varphi_1\|_X = 1$. On X we define an operator $\widetilde{P}_1 : X \to X_c$ by

$$\widetilde{P}_1 x = (P_1 x)\,\varphi_1, \tag{11}$$

where $P_1 x = \langle x, \varphi_1 \rangle_X$. Thus, $\widetilde{P}_1$ is the orthogonal projection onto X_c and $I - \widetilde{P}_1$ is the orthogonal projection onto $X_c^{\perp}$.

Let $x(t) \in X$, for $t \geq 0$. We denote:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} P_1 x(t) \\ (I - \widetilde{P}_1)x(t) \end{bmatrix}, \tag{12}$$

so that we have $x(t) = x_1(t)\varphi_1 + x_2(t)$. By a slight abuse of notation, we will write that:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{13}$$

Using the notation (13), the differential equation (2) can be rewritten as

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} P_1\,[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ (I - \widetilde{P}_1)\,[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{bmatrix}. \tag{14}$$

Consider two real constants $w_{\min}$ and $w_{\max}$ such that $w_{\min} < w_{\max}$. We define the function $\mathscr{S}$ as follows:

$$\mathscr{S}(w,g) = \begin{cases} \max\{g,0\} & w \leqslant w_{\min} \\ g & w \in (w_{\min}, w_{\max}) \\ \min\{g,0\} & w \geqslant w_{\max} \end{cases}. \tag{15}$$

The *saturating integrator* with input g and output w is a system described by

$$\dot{w}(t) = \mathscr{S}(w,g).$$

Thus, for $w(t) \in (w_{\min}, w_{\max})$, this system behaves like a usual integrator. For the existence of solutions for $g \in L^2_{\rm loc}([0,\infty);\mathbb{R})$ and other properties we refer to [8]. If the initial state $w(0) \in [w_{\min}, w_{\max}]$, then $w(t)$ will remain in $[w_{\min}, w_{\max}]$ for all $t \geqslant 0$. To constrain x_1 to the interval $[w_{\min}, w_{\max}]$, we replace the differential equation of x_1 given in the first row of (14) with

$$\dot{x}_1(t) = \mathscr{S}\left(x_1, P_1[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}\right). \tag{16}$$

We are interested in the following question: If $x_1(t)$ is constrained within a compact interval using a saturating integrator, then does the resultant nonlinear system preserve the passivity property (10)? The following theorem shows that indeed, the system with the evolution of x_1 given by (16), preserves the passivity property (10).

Theorem 3.1: Let Σ be a system node defined by (2) and (6), such that Σ is passive with respect to the supply rate S, as in (1). Then the system $\Sigma^{\mathscr{S}}$ evolving according to the differential equation

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \mathscr{S}\left(x_1(t), P_1[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}\right) \\ (I - \widetilde{P}_1)[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{bmatrix} \qquad \forall\, t \geqslant 0, \tag{17}$$

with $x_1(0) \in [w_{\min}, w_{\max}]$, and with the output $y(t)$ given by (6), is incrementally passive with the respect to the same supply rate S. In other words, for any time $t \geqslant 0$, the classical solutions of (17), $x(t), \tilde{x}(t) \in X$ corresponding to the inputs $u, \tilde{u} \in C([0,\infty);U)$ respectively, and the corresponding outputs $y, \tilde{y} \in C([0,\infty);Y)$ of $\Sigma^{\mathscr{S}}$, satisfy (10).

Remark 3.2: The definition of classical solutions for (17) is similar to the one from Definition 2.1, with $x \in \mathscr{H}^1_{\mathrm{loc}}((0,\infty);X)$ (in property (a)) and with (2) replaced with (17) (in property (d)). A challenging question would be to prove the existence and uniqueness of classical solutions of (17) for suitably smooth inputs and initial states. Even more challenging would be a well-posedness theorem for $\Sigma^{\mathscr{S}}$. This is beyond the scope of this paper.

Proof. Consider two classical solutions $x, \tilde{x} \in \mathscr{H}^1_{\mathrm{loc}}([0,\infty);X)$ of the system node Σ, corresponding to the two inputs $u, \tilde{u}$ and initial states $x(0), \tilde{x}(0)$ satisfying $\begin{bmatrix} x(0) \\ u(0) \end{bmatrix}, \begin{bmatrix} \tilde{x}(0) \\ \tilde{u}(0) \end{bmatrix} \in \mathscr{D}(A\&B)$. For $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} z \\ v \end{bmatrix} \in \mathscr{D}(A\&B)$ and $\begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix} = \begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix} \in \mathscr{D}(A\&B)$, the left-hand side of the incremental passivity condition (10) for the system node Σ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2 = 2(z_1-\tilde{z}_1)\left(P_1[A\&B]\begin{bmatrix} z \\ v \end{bmatrix} - P_1[A\&B]\begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}\right) + 2\left\langle z_2-\tilde{z}_2, (I-P_1)[A\&B]\begin{bmatrix} z \\ v \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}\right\rangle. \tag{18}$$

In (18), we have used the splitting of the states as in (14). It follows from (10) and $y(t) = [C\&D]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $\tilde{y}(t) = [C\&D]\begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix}$, that

$$\begin{aligned} &2(z_1-\tilde{z}_1)\left(P_1[A\&B]\begin{bmatrix} z \\ v \end{bmatrix} - P_1[A\&B]\begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}\right) \\ &+2\left\langle z_2-\tilde{z}_2, (I-P_1)[A\&B]\begin{bmatrix} z \\ v \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}\right\rangle \\ &\leqslant S\left(v-\tilde{v}, [C\&D]\begin{bmatrix} z \\ v \end{bmatrix} - [C\&D]\begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}\right). \end{aligned} \tag{19}$$

Suppose that the initial conditions of the first state component $x_1(0), \tilde{x}_1(0) \in [w_{\min}, w_{\max}]$. Then the state components $x_1(t), \tilde{x}_1(t)$ evolve according to (16), and are constrained within the interval $[w_{\min}, w_{\max}]$. Thus, the constrained system $\Sigma^{\mathscr{S}}$ evolves according to (17). For $\Sigma^{\mathscr{S}}$, the left-hand side of the incremental passivity condition (10) at initial time $t=0$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\|x_2(t)-\tilde{x}_2(t)\|_X^2\Big|_{t=0} + 2\Big\langle x_1(0)-\tilde{x}_1(0), \mathscr{S}\left(x_1(0), P_1[A\&B]\begin{bmatrix} x(0) \\ u(0) \end{bmatrix}\right) - \mathscr{S}\left(\tilde{x}_1(0), P_1[A\&B]\begin{bmatrix} \tilde{x}(0) \\ \tilde{u}(0) \end{bmatrix}\right)\Big\rangle. \tag{20}$$

Denote $\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} \eta \\ \varphi \end{bmatrix}$, $\begin{bmatrix} \tilde{x}(0) \\ \tilde{u}(0) \end{bmatrix} = \begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}$. Here, $\begin{bmatrix} \eta \\ \varphi \end{bmatrix}, \begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} \in \mathscr{D}(A\&B)$.

Case 1. *Consider $x_1(0) = w_{\min}$ and $\tilde{x}_1(0) \in (w_{\min}, w_{\max}]$.* If

$$P_1[A\&B]\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} > 0,$$

then according to (15), we have that

$$\mathscr{S}\left(\eta_1, P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}\right) = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}.$$

Thus, using that $\eta_1 - \tilde{\eta}_1 < 0$, we obtain from (20) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2\Big|_{t=0} \\ &\leqslant 2(\eta_1-\tilde{\eta}_1)\left(P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right) \\ &\quad +2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle. \end{aligned}$$

The expression on the right-hand side above is the same as the left-hand side of (19), with $\begin{bmatrix} \eta \\ \varphi \end{bmatrix} = \begin{bmatrix} z \\ v \end{bmatrix}$ and with $\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} = \begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}$. Hence, by (19), $\Sigma^{\mathscr{S}}$ satisfies the incremental passivity condition (10) in this case.

Now suppose that

$$P_1[A\&B]\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} < 0.$$

Then according to (15), we get:

$$\mathscr{S}\left(\eta_1, P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}\right) = 0.$$

Thus, from (20) we have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2\Big|_{t=0} = -2(\eta_1-\tilde{\eta}_1)P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} \\ &\quad +2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle \\ &\quad \leqslant \underbrace{\underbrace{2(\eta_1-\tilde{\eta}_1)}_{<0}P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}}_{<0} - 2(\eta_1-\tilde{\eta}_1)P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} \\ &\quad +2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle. \end{aligned}$$

The right-hand side of the above inequality is the left-hand side of (19), with $\begin{bmatrix} \eta \\ \varphi \end{bmatrix} = \begin{bmatrix} z \\ v \end{bmatrix}$ and with $\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} = \begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix}$. Thus, again $\Sigma^{\mathscr{S}}$ satisfies (10) in this case.

Case 2. *Consider $x_1(0) = w_{\max}$ and $\tilde{x}_1(0) \in [w_{\min}, w_{\max})$.* If

$$P_1[A\&B]\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} < 0,$$

then according to (15), we have:

$$\mathscr{S}\left(\eta_1, P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}\right) = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}.$$

Thus, using that $\eta_1 - \tilde{\eta}_1 > 0$, from (20) we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2\Big|_{t=0} \\ &\leqslant 2(\eta_1-\tilde{\eta}_1)\left(P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right) \\ &\quad +2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle. \end{aligned}$$

The expression on the right-hand side above is the same as the left-hand side of (19). Hence, by (19), (10) holds.

Now suppose that

$$P_1[A\&B]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} > 0,$$

then

$$\mathscr{S}\left(\eta_1, P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}\right) = 0.$$

Thus, for $\Sigma^{\mathscr{S}}$ we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-\tilde{x}(t)\|_X^2\Big|_{t=0} = -2(\eta_1-\tilde{\eta}_1)P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}$$
$$+2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle$$
$$\leqslant \underbrace{2(\eta_1-\tilde{\eta}_1)}_{>0}\underbrace{P_1[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix}}_{>0} -2(\eta_1-\tilde{\eta}_1)P_1[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}$$
$$+2\left\langle \eta_2-\tilde{\eta}_2, (I-P_1)[A\&B]\begin{bmatrix} \eta \\ \varphi \end{bmatrix} - (I-P_1)[A\&B]\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix}\right\rangle.$$

The right-hand side of the above inequality is the left-hand side of (19), so that again $\Sigma^{\mathscr{S}}$ satisfies (10).

Case 3. When $x_1(0), \tilde{x}_1(0) \in (w_{\min}, w_{\max})$, then the conclusion (10) is trivial, as in this case $\Sigma^{\mathscr{S}}$ will behave like the linear system Σ.

We have shown that if the linear system node Σ is passive with respect to the supply rate $S(u(t), y(t))$ then, the nonlinear system $\Sigma^{\mathscr{S}}$ satisfies (10) at the initial time $t=0$. If the passivity condition for $\Sigma^{\mathscr{S}}$ is satisfied at $t=0$ then, it also holds for any $t \geqslant 0$. This is due to the time invariance property of the nonlinear system. Therefore, the above proof will hold for $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix} \in \mathscr{D}(A\&B)$, by taking the derivative (with respect to time) of $\|x(t)-\tilde{x}(t)\|$ at some initial time $t=t_0$. Thus, we have proved that $\Sigma^{\mathscr{S}}$ is incrementally passive with respect to the supply rate S. $\square$

IV. String equation with restricted displacement

In this section, we give an application of our main result, Theorem 3.1, to the string equation on the interval $J=[0,\pi]$. The boundary input is applied at the left end ($\xi=0$), while the other end ($\xi=\pi$) is fixed, see Figure. 2.

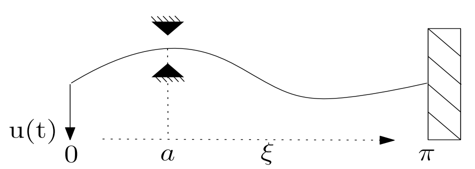

Fig. 2. Vibrating string system Σ, with obstacles that limit the displacement in one point ($\xi=a$).

The partial differential and boundary equations representing the vibrating string system Σ are

$$\begin{cases} \frac{\partial^2}{\partial t^2}w(\xi,t) = \frac{\partial^2}{\partial \xi^2}w(x,t), \quad w(\pi,t)=0, \\ w(\xi,0)=w_0(\xi), \quad \frac{\partial}{\partial t}w(\xi,0)=z_0(\xi), \\ -\frac{\partial}{\partial \xi}w(0,t)+b^2\frac{\partial}{\partial t}w(0,t) = \sqrt{2}bu(t), \\ -\frac{\partial}{\partial \xi}w(0,t)-b^2\frac{\partial}{\partial t}w(0,t) = \sqrt{2}by(t), \end{cases} \tag{21}$$

where $\xi \in J$ and $t \geqslant 0$. Here $b \neq 0$ is a constant. The functions w_0 and z_0 are the initial state of the system. We denote by $x(t) = \begin{bmatrix} w(\cdot,t) \\ \frac{\partial}{\partial t}w(\cdot,t) \end{bmatrix}$ the state of the above linear system Σ. Denote

$$\mathscr{H}_r^1(0,\pi) = \{w \in \mathscr{H}^1(0,\pi) \mid w(\pi)=0\}.$$

Then the natural state space is $X = \mathscr{H}_r^1(0,\pi)\times L^2[0,\pi]$, and on X we define the norm as follow: For $\begin{bmatrix} f \\ g \end{bmatrix} \in X$

$$\left\|\begin{bmatrix} f \\ g \end{bmatrix}\right\|_X^2 = \int_0^\pi \left|\frac{\mathrm{d}f}{\mathrm{d}\xi}(\xi)\right|^2 \mathrm{d}\xi + \int_0^\pi |g(\xi)|^2 \mathrm{d}\xi. \tag{22}$$

On X we can define the operator A as follows:

$$A\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} & 0 \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{D}(A), \tag{23}$$

$$\mathscr{D}(A) = \left\{\begin{bmatrix} f \\ g \end{bmatrix} \in \begin{matrix} \mathscr{H}^2(0,\pi)\cap\mathscr{H}_r^1(0,\pi) \\ \times \\ \mathscr{H}_r^1(0,\pi) \end{matrix} \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}\xi}(0) = b^2 g(0)\right\}.$$

Clearly, $\mathscr{D}(A)$ is dense in X, with the norm defined by (22).

For the above system Σ, the space Z from (4) is

$$Z = \left[\mathscr{H}^2(0,\pi)\cap\mathscr{H}_r^1(0,\pi)\right]\times\mathscr{H}_r^1(0,\pi). \tag{24}$$

On the space Z, we can define the operators $L \in \mathscr{L}(Z,X)$ and $G \in \mathscr{L}(Z,U)$ as follows:

$$L\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}f \end{bmatrix},$$

$$G\begin{bmatrix} f \\ g \end{bmatrix} = -\frac{1}{\sqrt{2}b}\frac{\mathrm{d}f}{\mathrm{d}\xi}(0) + \frac{b}{\sqrt{2}}g(0) \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in Z,$$

then $\mathrm{Ker}(G) = \mathscr{D}(A)$. The first three lines of the boundary controlled string equation (21) can be reformulated like (8), where $x(t) = \begin{bmatrix} w(\cdot,t) \\ \frac{\partial}{\partial t}w(\cdot,t) \end{bmatrix} \in Z$ and $u(t) \in \mathbb{R}$. The operator A is the restriction of L to the space $\mathscr{D}(A)$.

The output equation of the vibrating string system Σ is

$$y(t) = \bar{C}x(t) + Du(t) = -\frac{b}{\sqrt{2}}\frac{\partial w}{\partial t}(0,t) - \frac{1}{\sqrt{2}b}\frac{\partial w}{\partial \xi}(0,t), \tag{25}$$

where $\bar{C} = \begin{bmatrix} 0 & -\sqrt{2}b\delta_0^* \end{bmatrix}$ and $D=I$. Here δ_0^* is the operator of point evaluation at 0. The output $y(t)$ is observed at the left end of the string (i.e., at $\xi=0$). On the input and the output space $U=Y=\mathbb{R}$, the supply rate $S:\mathbb{R}^2\to\mathbb{R}$ is defined as follows:

$$S(u(t),y(t)) = |u(t)|^2 - |y(t)|^2. \tag{26}$$

On substituting $u(t)$ and $y(t)$ from the last two equations in (21), we obtain that

$$S(u(t),y(t)) = -\frac{\partial w}{\partial t}(0,t)\frac{\partial w}{\partial \xi}(0,t) \qquad \forall t \geqslant 0. \tag{27}$$

Then along a classical solution,

$$\frac{1}{2}\frac{\mathrm{d}\|x(t)\|^2}{\mathrm{d}t} = \langle \dot{x}(t), x(t)\rangle_X = \left\langle \begin{bmatrix} \frac{\partial}{\partial t}w(\xi,t) \\ \frac{\partial^2}{\partial \xi^2}w(\xi,t) \end{bmatrix}, \begin{bmatrix} w(\xi,t) \\ \frac{\partial}{\partial t}w(\xi,t) \end{bmatrix}\right\rangle$$
$$= \int_0^\pi \frac{\partial^2 w(\xi,t)}{\partial t\partial\xi}\cdot\frac{\partial w(\xi,t)}{\partial\xi}\mathrm{d}\xi + \int_0^\pi \frac{\partial^2 w(\xi,t)}{\partial\xi^2}\cdot\frac{\partial w(\xi,t)}{\partial t}\mathrm{d}\xi.$$

Using integration by parts, it immediately follows that

$$\frac{1}{2}\frac{\mathrm{d}\|x(t)\|^2}{\mathrm{d}t} = -\frac{\partial w}{\partial t}(0,t)\frac{\partial w}{\partial \xi}(0,t).$$

Thus, from (27), the vibrating string system node Σ is scattering passive. In fact, the string system Σ is scattering energy preserving, i.e., Σ satisfies (1) with an equality.

Let $a \in (0,\pi)$. Consider the continuous linear functional P_1 on the space X, defined by the point evaluation of the first function in $\begin{bmatrix} f & g \end{bmatrix}^\top$, at the point a, that is to say:

$$P_1 \begin{bmatrix} f \\ g \end{bmatrix} = f(a) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

By the Riesz representation theorem, there exists a unique function $\bar{\varphi}_1 \in \mathscr{H}_r^1(0,\pi)$ such that:

$$P_1 \begin{bmatrix} f \\ g \end{bmatrix} = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \bar{\varphi}_1 \\ 0 \end{bmatrix} \right\rangle \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

Denote $\varphi_1 = \begin{bmatrix} \bar{\varphi}_1 \\ 0 \end{bmatrix}$. It is easy to find that

$$\bar{\varphi}_1 = \begin{cases} \pi - a & \text{for} \quad \xi \in [0,a), \\ \pi - \xi & \text{for} \quad \xi \in [a,\pi]. \end{cases}$$

As given after (9), we define $X_c = \{\lambda \varphi_1 \,|\, \lambda \in \mathbb{R}\}$. On X, the orthogonal projection operator $\widetilde{P}_1 : X \to X_c$ is defined by (11). Thus, we can partition the differential equation $\dot{x}(t) = Lx(t)$ as follows:

$$\dot{x}(t) = \begin{bmatrix} P_1 Lx(t) \\ (I - \widetilde{P}_1)Lx(t) \end{bmatrix}. \tag{28}$$

Here the first component of the vector $x(t)$, denoted by $x_1(t)$, is $w(a,t)$. Now suppose that $x_1(t)$ is constrained to be in the interval $[w_{\min}, w_{\max}]$. Then the above differential equation (28) can be modified to incorporate this constraint. Thus, we obtain the new (nonlinear) system $\Sigma^{\mathscr{S}}$ described by the differential equation

$$\dot{x}(t) = \begin{bmatrix} \mathscr{S}(x_1(t), P_1 Lx(t)) \\ (I - \widetilde{P}_1)Lx(t) \end{bmatrix},$$

and the output equation (25). On invoking our Theorem 3.1, we obtain that for $t \geqslant 0$, the solutions $x(t), \tilde{x}(t) \in X$ of $\Sigma^{\mathscr{S}}$, corresponding to the inputs $u(t), \tilde{u}(t) \in U$ respectively, and the corresponding outputs $y(t), \tilde{y}(t) \in U$, satisfy (10), where S is given by (26) so that

$$S((u(t) - \tilde{u}(t), y(t) - \tilde{y}(t)) = |u(t) - \tilde{u}(t)|^2 - |y(t) - \tilde{y}(t)|^2.$$

Therefore, $\Sigma^{\mathscr{S}}$ is incrementally scattering passive with respect to the supply rate S.

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