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Spatial group error and synchrony for tracking control of left-invariant kinematic systems on Lie groups

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***Abstract*— Trajectory tracking for underactuated systems has been heavily studied for several decades. When the system state-space is a Lie group, the group multiplication can be used to define a global error. In this paper, we consider the spatial, or right-invariant, group error in the design of a tracking controller for left-invariant systems. This choice of error is shown to exhibit synchrony; that is, the error kinematics depend linearly on the control input difference. In particular, if the actual system is driven by the desired control signal, then the error is constant. This property is used to propose a simple nonlinear tracking control scheme that is globally stable and locally exponentially stable for a class of persistently exciting trajectories for left-invariant systems on Lie-groups. We explore the example system of a mobile robot and show that in this particular case, the proposed control scheme is almost-globally asymptotically stable.**

I. Introduction

The proliferation of small autonomous systems in modern society underscores the requirement for effective tracking control algorithms for these systems. Importantly, the state-space of many of these systems is not naturally described by a vector space, but rather by a Lie group. This complicates the application of classical linear control techniques and necessitates the application of tools from geometric control theory. Of particular interest are (left-)invariant systems on Lie groups that model many of the systems of interest. The study of invariant control systems on Lie groups dates back to Brockett [2], [3], and Jurdjevic and Sussman [11], who studied their controllability and reachability. In [4], Brockett showed that underactuated drift-free nonlinear systems (including left-invariant systems on Lie groups) cannot be stabilised to a point by means of smooth, time-invariant feedback. This paper motivated a major research activity during the early 1990s into control and stabilisation of nonholonomic systems [13]. Leonard and Krishnaprasad [16] proposed time-varying control based on averaging trajectories. Morin and Samson [21] used transverse functions to achieve local practical stability. For fully actuated systems, Bullo and Murray [6] developed general PD type tracking controllers for systems on Lie-groups. This was extended in Maithripala *et al.* [17] and Cabecinhas *et al.* [7]. In [14], Lee *et al.* used differential flatness to derive an almost-globally stable tracking control for quadrotors posed on $\mathbf{SE}(3)$ with drift due to gravity.

A key example of left-invariant underactuated kinematic systems is the class of wheeled mobile vehicles. The kinematics of the archetypal wheeled mobile robot, the unicycle, can be posed as a left-invariant system with state $X \in \mathbf{SE}(2)$: the Lie group combining two-dimensional pose and configuration [19]. Due to its simplicity and the underlying nonholonomic structure, the unicycle model has received a great deal of attention [12], [19], [23], [10], [21], [22], [25], [18]. A consistent feature in the above references, even when not explicitly stated, is the use of the left-invariant or body error $X_d^{-1}X$ (where X is the actual state and X_d is some desired state to be tracked). Despite appearing algebraically very similar, the right-invariant or spatial error XX_d^{-1} has been eschewed historically (although it does receive some treatment in [6]). There are reasons for this: Bullo and Murray [5] point out that the spatial error introduces an additional arbitrary coordinate frame which is absent from the body error. Additionally, they point out that for rigid bodies the spatial error couples the rotational and positional error components, a property that is seen as undesirable.

In this paper, we analyse the spatial error for the control design of underactuated left-invariant systems on matrix Lie groups. We show that the spatial error dynamics exhibit synchrony; that is, the spatial error dynamics are stationary if the desired input is used to drive the real system. This is in contrast to the body error dynamics, where using the same input to drive the desired and real systems leads to a non-linear drift term in the error dynamics that must be compensated or dominated, a challenging design problem for underactuated systems. Exploiting synchrony of the spatial error, we propose a simple Lyapunov function and show that projecting the gradient of the Lyapunov function onto the actuated directions naturally leads to a globally stable gradient-based control design, even for underactuated systems. Moreover, we show that this control design yields local exponential stability of the system for a class of persistently exciting bounded desired trajectories. We explore the example of the mobile robot on $\mathbf{SE}(2)$ and prove almost-global asymptotic stability of the proposed control scheme. We use this example to provide physical insight into the difference between spatial and body group errors, and we provide numerical simulations that demonstrate the performance of the controller on an example trajectory.

II. Preliminaries

Let $\mathbf{G} \subset \mathbb{R}^{d\times d}$ denote an m-dimensional matrix Lie group and denote the identity element with I. The *Lie algebra*, $\mathfrak{g}$ of $\mathbf{G}$, is identified with the tangent space of $\mathbf{G}$ at identity, $\mathfrak{g} \simeq \mathrm{T}_I\mathbf{G} \subset \mathbb{R}^{d\times d}$. As $\mathfrak{g}$ is a finite-dimensional vector space, there exists a linear isomorphism $(\cdot)^\wedge : \mathbb{R}^m \to \mathfrak{g}$. Denote the

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inverse map by $(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^m$. Given a linear map $A : \mathfrak{g} \to \mathfrak{g}$, let $A^\vee : \mathbb{R}^m \to \mathbb{R}^m$ be the corresponding linear map on $\mathbb{R}^m$ defined by $A^\vee x = (A(x^\wedge))^\vee$. Similarly, given a linear map $C : \mathbb{R}^m \to \mathbb{R}^m$, let $C^\wedge : \mathfrak{g} \to \mathfrak{g}$ be the corresponding linear map on $\mathfrak{g}$ defined by $C^\wedge y = (C(y^\vee))^\wedge$. Given an element $X \in \mathbf{G}$, the adjoint operation $\mathrm{Ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{Ad}_X u = XuX^{-1}$. Given $u \in \mathfrak{g}$, the "little" adjoint map $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad}_u v = uv - vu$.

The Frobenius inner product is an inner product on $\mathbb{R}^{d\times d}$, defined by

$$\langle A, B\rangle_{\mathbb{F}} = \mathrm{tr}(A^\top B).$$

Note that

$$\langle A, BC\rangle_{\mathbb{F}} = \langle B^\top A, C\rangle_{\mathbb{F}}. \tag{1}$$

By the bilinearity of the Frobenius inner product, it holds that

$$\langle v^\wedge, w^\wedge\rangle_{\mathbb{F}} = \langle v, Sw\rangle, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^m$ and $S_{ij} = \langle e_i^\wedge, e_j^\wedge\rangle_{\mathbb{F}}$ (where $\{e_i\}$ is the standard basis) is symmetric positive definite.

Given an inner product space W and a subspace $V \subseteq W$, let $V^\perp$ denote the orthogonal subspace, defined by

$$V^\perp = \{w \in W : \langle v, w\rangle = 0\ \forall v \in V\}.$$

Every vector $w \in W$ can be expressed as a sum $w = u + v$, with $v \in V$ and $u \in V^\perp$ ([1]). The subspace projection operator $\mathbb{P}_V$ is defined by $\mathbb{P}_V w = v$.

A matrix function $A : \mathbb{R} \to \mathbb{R}^{n\times m}$ is called *persistently exciting* if there exist real numbers $\varepsilon > 0$ and $T > 0$ such that

$$\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau \geq \varepsilon I_m, \tag{3}$$

for all $t \in \mathbb{R}$.

III. Problem Description

Let $\mathbf{G} \subset \mathbb{R}^{d\times d}$ be a m-dimensional matrix Lie group and $\mathfrak{g}$ be the associated Lie algebra. In this paper we consider the problem of tracking an underactuated left-invariant system on $\mathbf{G}$. That is, a system of the form

$$\dot{X} = XU = X(Bu)^\wedge, \tag{4}$$

where $X \in \mathbf{G}$ is the state, $u \in \mathbb{R}^\ell$ is the input and $B : \mathbb{R}^\ell \to \mathbb{R}^d$ is some constant rank-deficient matrix.

We address the problem of tracking a desired trajectory $X_d(t) : \mathbb{R} \to \mathbf{G}$. Such a trajectory satisfies

$$\dot{X}_d = X_d U_d = X_d(Bu_d)^\wedge, \tag{5}$$

for some $u_d(t) : \mathbb{R} \to \mathbb{R}^\ell$. The problem is to find a control $u(t) : \mathbb{R} \to \mathbb{R}^\ell$ so that the true system state $X(t)$ converges asymptotically to the desired system state $X_d(t)$.

IV. Lie Group Errors

The standard approach to constructive nonlinear trajectory tracking control does not directly try to control $X(t) \to X_d(t)$. Rather, the approach taken is to define an error $E(X(t), X_d(t))$ and study the problem of driving $E(X(t), X_d(t)) \to E_\star$ where $E_\star$ is some constant reference such that when $E = E_\star$ then $X = X_d$. On any Lie group $\mathbf{G}$ there are two natural Lie group errors that can be used for this role.

Body group error: The body (or left-invariant) group error

$$E_L \coloneqq X_d^{-1}X. \tag{6}$$

Spatial group error: The spatial (or right-invariant) group error

$$E_R \coloneqq XX_d^{-1}. \tag{7}$$

Proposition 4.1: Let $X_d(t) : \mathbb{R} \to \mathbf{G}$ be a trajectory satisfying (5) and let $X(t) : \mathbb{R} \to \mathbf{G}$ be a trajectory satisfying (4). The error dynamics of the body error (6) are given by

$$\dot{E}_L = -U_d E_L + E_L U.$$

Proof: By straightforward computation,

$$\begin{aligned}\dot{E}_L &= -X_d^{-1}\dot{X}_d X_d^{-1}X + X_d^{-1}\dot{X}\\ &= -U_d E_L + E_L U\end{aligned}$$

■

One notes here that in order to completely compensate for the exogenous dynamics $U_d E_L$, one would have to apply an input term of $U = \mathrm{Ad}_{E_L}^{-1} U_d$. This is not possible in general, as if the system is underactuated, this term may not be an admissable input (in other words, there may not be any u that satisfies $(Bu)^\wedge = \mathrm{Ad}_{E_L}^{-1} U_d$). On the other hand, consider the following proposition.

Proposition 4.2: Let $X_d(t) \in \mathbf{G}$ be a trajectory satisfying (5) and let $X(t) \in \mathbf{G}$ be a trajectory satisfying (4). Define the control difference $\tilde{u} \coloneqq u - u_d \in \mathbb{R}^\ell$ and note that $(B\tilde{u})^\wedge = U - U_d = \tilde{U} \in \mathfrak{g}$. Then the dynamics of E_R (7) are given by

$$\dot{E}_R = E_R \,\mathrm{Ad}_{X_d} \tilde{U} = E_R \,\mathrm{Ad}_{X_d}(B\tilde{u})^\wedge. \tag{8}$$

Proof: Taking the time derivative of (7):

$$\begin{aligned}\dot{E}_R &= \dot{X}X_d^{-1} - XX_d^{-1}\dot{X}_d X_d^{-1}\\ &= XUX_d^{-1} - XX_d^{-1}X_d U_d X_d^{-1}\\ &= XX_d^{-1}X_d(U - U_d)X_d^{-1}\\ &= E_R\,\mathrm{Ad}_{X_d}(B\tilde{u})^\wedge,\end{aligned}$$

as required.

■

In this case, by setting $U = U_d$ then $\tilde{U} = 0$. It follows that the feed-forward spatial error system is synchronous, that is, $\dot{E}_R = 0$ when $\tilde{U} = 0$. Studying the spatial error will lead to simpler error dynamics due to the synchrony property. However, the interpretation of the spatial error depends on the reference frame choice and is more complex than the body error. Although $E_R = I$ still guarantees that $X = X_d$, analysing convergence is more complex. The following

Lemma provides conditions under which the convergence of $E_R(t) \to I$ is equivalent to the convergence of $X(t) \to X_d(t)$.

Lemma 4.3: Let $X_d(t) : \mathbb{R} \to \mathbf{G}$ be a trajectory satisfying (5) and let $X(t) : \mathbb{R} \to \mathbf{G}$ be a trajectory satisfying (4). If $X_d(t)$ and $X_d(t)^{-1}$ are bounded then $E_R(t) \to I_d$ if and only if $X(t) \to X_d(t)$.

Proof: First, assume that $\lim_{t\to\infty} E_R(t) = I$. Let $\varepsilon > 0$ be given. Let C be an upper bound for $X_d(t)$, so that $\|X_d(t)\| < C$ for all t. There exists a $T > 0$ such that $\|E_R(t) - I\| < \frac{\varepsilon}{C}$ for $t > T$, and

$$\begin{aligned}\|X(t) - X_d(t)\| &\leq \|X(t)X_d(t)^{-1} - I\|\|X_d(t)\| \\ &\leq \|E_R(t) - I\|C \\ &< \varepsilon,\end{aligned}$$

so $X(t) \to X_d(t)$. The converse statement is proved by a similar argument. ∎

V. Constructive Lyapunov control on G

The synchrony property of the spatial error is key in the design of the tracking controller we propose here. Specifically, we choose a candidate Lyapunov function $\mathcal{L}(E_R)$. Then the correction term is generated by projecting the gradient of $\mathcal{L}(E_R)$ onto the actuation directions of the vehicle. Even if the vehicle is underactuated and the gradient projects onto the nonholonomic constraint directions (leading to a zero correction term), synchrony of the error ensures that the Lyapunov function never increases. In such situations, excitation of the reference trajectory ensures global asymptotic stability of the error dynamics.

We approach the controller design constructively: that is, we define a candidate Lyapunov function and use this to derive a control law. Recalling the definition of S (2) and defining $E := E_R$, define

$$A(t) := B^\top \mathrm{Ad}^{\vee\top}_{X_d(t)} S \qquad \in \mathbb{R}^{\ell\times d} \tag{9}$$

$$\eta(t) := \mathbb{P}^\vee_{\mathfrak{g}}(E(t)^\top E(t) - E(t)^\top) \qquad \in \mathbb{R}^d, \tag{10}$$

where $\mathbb{P}^\vee_{\mathfrak{g}}$ should be interpreted as the operator mapping $A \mapsto (\mathbb{P}_{\mathfrak{g}}(A))^\vee$.

Theorem 5.1: Let $X_d(t) : \mathbb{R} \to \mathbf{G}$ denote the desired system trajectory with bounded desired input $u_d : \mathbb{R} \to \mathbb{R}^\ell$, and let $X(t) : \mathbb{R} \to \mathbf{G}$ denote the true system state. Let $E = XX_d^{-1}$ denote the spatial error (7). Choose a gain $k > 0$ and define the control input delta $\tilde{u} : \mathbb{R} \to \mathbb{R}^\ell$ by

$$\tilde{u} = -kA(t)\eta(t), \tag{11}$$

where $A(t)$ and $\eta(t)$ are given by (9) and (10). Then, if $A(t)$ is persistently exciting (3), $E(t)$ converges to the equilibrium set

$$\mathcal{E}_0 = \{E \in \mathbf{G} : \mathbb{P}_{\mathfrak{g}}(E^\top E - E^\top) = 0\}.$$

Moreover, the identity is a locally uniformly exponentially stable equilibrium, $I \in \mathcal{E}_0$.

Proof: We begin the proof of the first claim by defining the candidate Lyapunov function

$$\mathcal{L}(E) = \frac{1}{2}\langle E - I, E - I\rangle_{\mathbb{F}}. \tag{12}$$

Differentiating yields

$$\begin{aligned}\dot{\mathcal{L}}(t) &= \langle E - I, \dot{E}\rangle_{\mathbb{F}} \\ &= \langle E - I, E\,\mathrm{Ad}_{X_d}(B\tilde{u})^\wedge\rangle_{\mathbb{F}} \\ &= \langle E^\top(E - I), \mathrm{Ad}_{X_d}(B\tilde{u})^\wedge\rangle_{\mathbb{F}} && (13)\\ &= \langle \mathbb{P}_{\mathfrak{g}}(E^\top E - E^\top), \mathrm{Ad}_{X_d}(B\tilde{u})^\wedge\rangle_{\mathbb{F}} && (14)\\ &= \langle S\mathbb{P}_{\mathfrak{g}}(E^\top E - E^\top)^\vee, \mathrm{Ad}^\vee_{X_d} B\tilde{u}\rangle && (15)\\ &= \langle B^\top \mathrm{Ad}^{\vee\top}_{X_d} S\mathbb{P}_{\mathfrak{g}}(E^\top E - E^\top)^\vee, \tilde{u}\rangle, && (16)\\ &= \langle A(t)\eta(t), \tilde{u}\rangle,\end{aligned}$$

where (13) and (16) follow from (1), (14) follows from the fact that the right-hand side of the inner product lies in $\mathfrak{g}$, so the inner product is unchanged if the left-hand side is projected onto $\mathfrak{g}$, and (15) follows from (2).

Substituting the control (11) yields

$$\dot{\mathcal{L}}(t) = -k\|A(t)\eta(t)\|^2. \tag{17}$$

This shows that $\mathcal{L}$ is indeed non-increasing and thus that the error dynamics are stable.

In order to characterise the equilibrium set of E, we will appeal to Barbalat's lemma [24]. Since $\mathcal{L}$ is non-increasing then $\mathcal{L}(t) \leq \mathcal{L}(0)$ for all t and $\mathcal{L}(t) \to c$ for some positive constant c. Writing $\mathcal{L}(t) = \frac{1}{2}\|E - I\|^2_{\mathbb{F}}$, this implies that E is also bounded.

Taking the second derivative of $\mathcal{L}$ and using the Cauchy-Schwarz inequality and submultiplicativity of matrix norms one has

$$\begin{aligned}\|\ddot{\mathcal{L}}(t)\| &= 2k\langle A(t)\eta(t), \dot{A}(t)\eta(t) + A(t)\dot{\eta}(t)\rangle \\ &\leq 2k\|A(t)\eta(t)\|\|\dot{A}(t)\eta(t) + A(t)\dot{\eta}(t)\| \\ &\leq 2k\|A(t)\|\|\eta(t)\|\left(\|\dot{A}(t)\|\|\eta(t)\| + \|A(t)\|\|\dot{\eta}(t)\|\right).\end{aligned}$$

Thus, in order to show that $\ddot{\mathcal{L}}(t)$ is bounded, it suffices to show that each of $A(t), \eta(t), \dot{A}(t)$ and $\dot{\eta}(t)$ are bounded. The matrix $A(t)$ and vector $\eta(t)$ are linear functions of a bounded variable and so are bounded. We have $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{X_d(t)} = \mathrm{Ad}_{X_d(t)}\mathrm{ad}_{U_d(t)}$, so $\dot{A}(t) = B^\top \mathrm{ad}^{\vee\top}_{U_d(t)}\mathrm{Ad}^{\vee\top}_{X_d(t)} S$, which is bounded if $U_d(t)$ is bounded. Additionally,

$$\begin{aligned}\dot{\eta} &= \mathbb{P}_{\mathfrak{g}}(\dot{E}^\top E + E^\top \dot{E} - \dot{E}^\top)^\vee \\ &= \mathbb{P}_{\mathfrak{g}}((E\,\mathrm{Ad}_{X_d}\tilde{U})^\top E + E^\top(E\,\mathrm{Ad}_{X_d}\tilde{U}) \\ &\qquad - (E\,\mathrm{Ad}_{X_d}\tilde{U})^\top)^\vee \\ &= \mathbb{P}_{\mathfrak{g}}(\tilde{U}^\top \mathrm{Ad}^\top_{X_d} E^\top E + E^\top(E\,\mathrm{Ad}_{X_d}\tilde{U}) \\ &\qquad - \tilde{U}^\top \mathrm{Ad}^\top_{X_d} E^\top)^\vee\end{aligned}$$

The terms $\tilde{U}(t) = (A(t)\eta(t))^\wedge, \mathrm{Ad}_{X_d(t)}$ and $E(t)$ are all bounded, so $\dot{\eta}$ is bounded. Thus, by Barbalat's lemma [24], $\dot{\mathcal{L}}(t) \to 0$. This also implies that $\tilde{u} = A(t)\eta(t)$ approaches zero, and so $\dot{E} \to 0$ and $\dot{\eta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_{\mathfrak{g}}(E(t)^\top E(t) - E(t)^\top)^\vee \to 0$.

Since $A(t)$ is persistently exciting, it now follows that $\eta(t) \coloneqq \mathbb{P}_{\mathfrak{g}}(E(t)^\top E(t) - E(t)^\top)^\vee \to 0$ by Lemma 9.1. Thus $E(t)$ approaches the equilibrium set $\mathcal{E}_0$.

To prove the second claim, let $\sqrt{S}$ denote the principal square root of S and define the local coordinates $\mu = \sqrt{S}\log(E)^\vee$. Then

$$\begin{aligned}\dot{\mu}^\wedge &= \sqrt{S}^\wedge \mathrm{D}\log(E)\dot{E} \\ &= \sqrt{S}^\wedge \mathrm{D}\log(E) E\, \mathrm{Ad}_{X_d}(B\tilde{u})^\wedge\end{aligned}$$

Substituting (11) for $\tilde{u}$, linearising around identity and using the approximation $E \approx I + (\sqrt{S}^{-1}\mu)^\wedge$, we find that

$$\begin{aligned}\dot{\mu}^\wedge &\approx -k\sqrt{S}^\wedge \mathrm{Ad}_{X_d} B^\wedge (A(t)\sqrt{S}^{-1}\mu)^\wedge \\ &= -k(\sqrt{S}^{-1}A(t)^\top A(t)\sqrt{S}^{-1}\mu)^\wedge,\end{aligned}$$

where we have ignored higher-order terms. Thus, the corresponding linearised system in $\mathbb{R}^n$ is given by

$$\dot{\mu} \approx -k\sqrt{S}^{-1}A(t)^\top A(t)\sqrt{S}^{-1}\mu. \tag{18}$$

By assumption, $A(t)$ is persistently exciting, so there exists some lower bound $\varepsilon > 0$ such that

$$\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau \geq \varepsilon I.$$

Then

$$\int_t^{t+T} k\sqrt{S}^{-1}A(t)^\top A(t)\sqrt{S}^{-1}\mathrm{d}\tau \geq k\varepsilon S^{-1} I \geq \|\lambda\| k\varepsilon I,$$

where λ is the smallest eigenvalue of S^{-1}. Additionally, the matrix $k\sqrt{S}^{-1}A(t)^\top A(t)\sqrt{S}^{-1}$ is positive semi-definite and symmetric, so by Proposition 9.2, the linearised system (18) is uniformly exponentially stable. ■

VI. Example: The Special Euclidean Group $\mathbf{SE}(2)$

The Special Euclidean Group $\mathbf{SE}(2)$ is a 3-dimensional matrix Lie group

$$\mathbf{SE}(2) = \left\{ \begin{pmatrix} R & p \\ 0_{1\times 2} & 1 \end{pmatrix} : R \in \mathbf{SO}(2), p \in \mathbb{R}^2 \right\} \subset \mathbb{R}^{3\times 3}.$$

The associated Lie algebra $\mathfrak{se}(2)$ is defined by the set

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} \Omega \mathbf{1}^\times & v \\ 0_{1\times 2} & 0 \end{pmatrix} : \Omega \in \mathbb{R}, v \in \mathbb{R}^2 \right\},$$

where

$$\mathbf{1}^\times \coloneqq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The algebra $\mathfrak{se}(2)$ is isomorphic to $\mathbb{R}^3$ and we define the wedge map $(\cdot)^\wedge : \mathbb{R}^3 \to \mathfrak{se}(2)$ to be

$$\begin{pmatrix} \Omega \\ v_x \\ v_y \end{pmatrix}^\wedge = \begin{pmatrix} 0 & -\Omega & v_x \\ \Omega & 0 & v_y \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that, for any $u_1^\wedge, u_2^\wedge \in \mathfrak{se}(2)$, one has

$$\langle u_1^\wedge, u_2^\wedge \rangle_{\mathbb{F}} = \langle u_1, S u_2 \rangle,$$

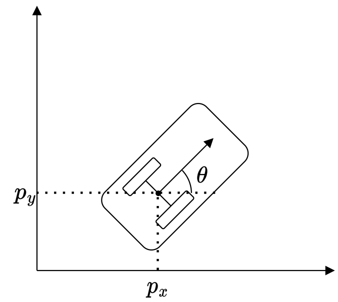

Fig. 1. Mobile robot model

where $S = \mathrm{diag}(2, 1, 1)$.

Let $\mathbb{P}_{\mathfrak{se}(2)} : \mathbb{R}^{3\times 3} \to \mathfrak{se}(2)$ denote the $\mathfrak{se}(2)$ projection operator. Then, for any $M_1 \in \mathbb{R}^{2\times 2}$, $m_2, m_3 \in \mathbb{R}^2$, and $m_4 \in \mathbb{R}$, one has

$$\mathbb{P}_{\mathfrak{se}(2)} \begin{pmatrix} M_1 & m_2 \\ m_3^\top & m_4 \end{pmatrix} = \begin{pmatrix} \frac{M_1 - M_1^\top}{2} & m_2 \\ 0 & 0 \end{pmatrix}.$$

To provide intuition and motivate a physical example with underactuation, consider the standard mobile robot (Fig. 1) with position $p = (p_x, p_y) \in \mathbb{R}^2$, heading angle $\theta \in \mathrm{S}^1 \simeq [-\pi, \pi)$, and forward and angular inputs $v, \Omega \in \mathbb{R}$, respectively. The system states of a mobile robot evolve according to the kinematics

$$\dot{p}_x = v\cos(\theta) \tag{19a}$$

$$\dot{p}_y = v\sin(\theta) \tag{19b}$$

$$\dot{\theta} = \Omega. \tag{19c}$$

The system state of a mobile robot can be represented in the matrix Lie group $\mathbf{SE}(2)$ by

$$X = \begin{pmatrix} R(\theta) & p \\ 0_{1\times 2} & 1 \end{pmatrix} \tag{20}$$

Using this representation, the system dynamics may be expressed as left-invariant dynamics on the group,

$$\dot{X} = XU, \qquad U = \begin{pmatrix} \Omega \mathbf{1}^\times & v\mathbf{e}_1 \\ 0_{1\times 2} & 0 \end{pmatrix} \in \mathfrak{se}(2). \tag{21}$$

Additionally, by defining

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} \Omega \\ v \end{pmatrix}, \tag{22}$$

one has $U = (Bu)^\wedge$ and

$$\dot{X} = X(Bu)^\wedge.$$

A. Interpretation of Lie group errors

In $\mathbf{SE}(2)$, the Lie group errors (6) and (7) can be computed to be given by

$$E_L = \begin{pmatrix} R_d^\top R & R_d^\top (x - x_d) \\ 0_{1\times 2} & 1 \end{pmatrix}$$

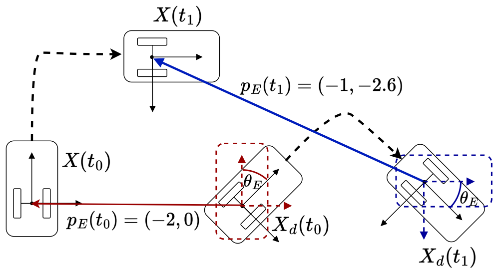

Fig. 2. Visualisation of the body, or left-invariant, error for correlated motion; that is where both vehicles have the same inputs. The vehicle motion from poses $X(t_0)$ and $X_d(t_0)$ to poses $X(t_1)$ and $X_d(t_1)$ is generated by the same input, dotted black arrows. The relative *body* transformation between X_d and X in both cases involves an anti-clockwise rotation of 45 degrees. However, at t_0 the translation is (-2, 0) while at t_1 the translation is (-1, -2.6) demonstrating that the error is not preserved.

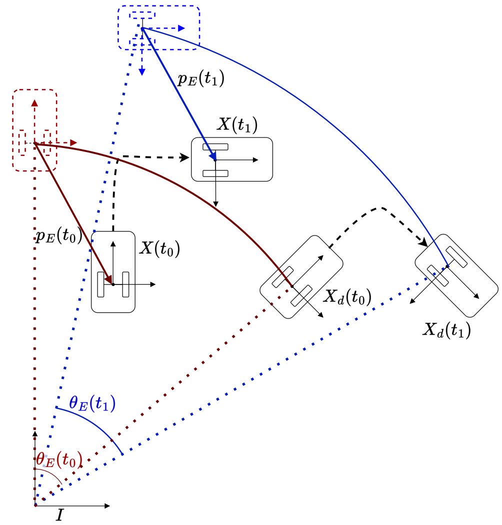

Fig. 3. Visualisation of the spatial (or right-invariant) error for correlated motion. The vehicle motion from poses $X(t_0)$ and $X_d(t_0)$ to poses $X(t_1)$ and $X_d(t_1)$ is generated by the same input, dotted black arrows. The same *spatial* transformation applies in both cases, $X(t_0)$ and $X(t_1)$ are rotated around the reference origin by 45 degrees and then translated (in the reference frame) in the $(1, -1)$ direction.

$$E_R = \begin{pmatrix} RR_d^\top & x - RR_d^\top x_d \\ 0_{1\times 2} & 1 \end{pmatrix} \tag{23}$$

Clearly, if either error is identity ($E = I_3 \in \mathbf{SE}(2)$) then $X = X_d$. However, the physical transformations encoded by the errors are quite different as seen in Figures 2 and 3. For the body transformation the rotation and translation are decoupled since the rotation is undertaken around the body reference. Conversely, in the spatial transformation, the rotation is undertaken around the origin of the reference frame and moves the frame.

The body error has been the natural choice for tracking control design for several reasons. It is the group error formulation taught in most text books and corresponds to the coordinate change formula that most roboticists use to understand rigid body transformations. It is also the natural error to encode a rigid body transformation from the perspective of the robot itself. In contrast, the spatial error representation is less commonly used in mainstream robotics and depends on the reference frame. As such, it is not intrinsic to the motion of the vehicles and the justification for considering the spatial group error comes from studying the error dynamics.

Consider applying the reference input $U = U_d$ as a feed-forward compensation. Then the body group error evolves according to $\dot{E} = E_L U_d - U_d E_L$. The evolution of the error term is visualised in Figure 2 and it is clear that the system is not synchronous, that is, the error is not preserved under feed-forward control. In the context of control design, this non-zero term $\dot{E} = E_L U_d - U_d E_L$ in the error dynamics presents as a drift term that must be compensated by the control action. If the system were fully actuated, then the exogenous dynamics component $U_d E$ can be compensated by setting

$$U = \mathrm{Ad}_{E_L^{-1}} U_d + E_L^{-1}\Delta,$$

where Δ is the additional input that drives $E_L \to I_3$. For example, see [8], [14]. However, if the system is underactuated, as with the mobile robot, the term $\mathrm{Ad}_{E_L^{-1}} U_d$ may not lie in the actuated directions. In this case, the exogenous dynamics cannot be directly compensated and the control must be used to dominate the effects of the drift in the Lyapunov analysis. There are various well established control algorithms that take this approach in the literature, such as [12] or [15].

On the other hand, consider the choice of the spatial or right-invariant error $E_R := X X_d^{-1}$.

Setting $U = U_d$, then $\tilde{U} = 0$. It follows that the feed-forward system is synchronous, that is, $\dot{E}_R = 0$. This is visualised in Figure 3. Note that since the error $\dot{E}_R = 0$ then $E_R(t_1) = E_R(t_0)$. It is also clear from Figure 3 that constant spatial error does not correspond to a "constant local distance" between the desired trajectory and vehicle trajectory. Studying the spatial error leads to simpler error dynamics but introduces a more complex interpretation of the meaning of the error, and in particular, introduces dependence on the reference frame choice.

B. Mobile robot control input and Lyapunov analysis

For this example system, the control input (11) can be expressed in components as

$$\tilde{\Omega} = -2k\sin(\theta_E) - kp_d^\top \mathbf{1}^\times R_E^\top p_E \tag{24a}$$
$$\tilde{v} = -ke_1^\top R_d^\top R_E^\top p_E. \tag{24b}$$

The term $A(t) = B^\top \mathrm{Ad}^{\vee\top}_{X_d(t)} S$ is given in components by

$$A(t) = \begin{pmatrix} 2 & p_d(t)^\top \mathbf{1}^\times \\ 0 & e_1^\top R_d(t)^\top \end{pmatrix}. \tag{25}$$

We have the following stability result for the mobile robot.

Proposition 6.1: Let $X_d(t) : \mathbb{R} \to \mathbf{SE}(2)$ denote a desired system trajectory of (21) with bounded input $u_d(t)$, and let $X(t) : \mathbb{R} \to \mathbf{SE}(2)$ denote the true system state. Let $E = XX_d^{-1}$ denote the error (23). Let the input $\tilde{u}$ be given by (24). If the term (25) is persistently exciting, then the system $\dot{E}$ is almost globally asymptotically stable to $(\theta_E, p_E) = (0, 0)$.

Proof: By Theorem 5.1, it follows that $E(t)$ converges to the equilibrium set $\mathcal{E}_0 = \{E \in \mathbf{SE}(2) : \mathbb{P}_{\mathfrak{se}(2)}(E^\top E - E^\top)^\vee = 0\}$. Using (23), the expression $\mathbb{P}_{\mathfrak{se}(2)}(E^\top E - E^\top)^\vee$ can be expressed in components as $(\sin\theta_E, R_E^\top p_E)$, so the equilibrium set $\{\mathbb{P}_{\mathfrak{se}(2)}(E^\top E - E^\top)^\vee = 0\}$ is given by $\{(\theta_E, p_E) : \sin\theta_E = 0, R_E^\top p_E = 0\}$. In other words, the set is given by the two points

$$\mathcal{E}_0 = \{(\theta_E, p_E) = (0, 0), (\theta_E, p_E) = (\pi, 0)\}.$$

The point $(\theta_E, p_E) = (0, 0)$ is shown to be locally exponentially stable by Theorem 5.1. To see that the point $(\pi, 0)$ is an unstable equilibrium point, it suffices to show that every neighbourhood of $(\pi, 0)$ contains a point q with $\mathcal{L}(q) < \mathcal{L}(\pi, 0)$. Every neighborhood of $(\pi, 0)$ contains a point $(\pi - \varepsilon, 0)$ for small enough $\varepsilon > 0$. In general, $\mathcal{L}(\theta, p) = 2(1 - \cos(\theta)) + \frac{1}{2}\|p\|^2$, so $\mathcal{L}(\pi, 0) = 4$. At a point $(\pi - \varepsilon, 0)$, $\mathcal{L}(\pi - \varepsilon, 0) = 2(1 - \cos(\pi - \varepsilon)) = 2(1 + \varepsilon) < 4$. As an intermediate corollary of (17), $\dot{L}$ is shown to be non-positive, so the equilibrium $(\pi, 0)$ is unstable. As a result, the system $\dot{E}$ is almost globally asymptotically stable to $(\theta_E, p_E) = (0, 0)$. ∎

VII. Simulation

In order to empirically verify that the example controller works, the system is implemented in simulation for an elliptical trajectory. In general, an elliptical trajectory has the form

$$p_d(t) = \begin{pmatrix} a\cos(ht) \\ b\sin(ht) \end{pmatrix}, \tag{26}$$

implying that $\Omega_d(t) = h, \theta_d(t) = ht$.

A. Excitation of elliptical trajectories

For an elliptical trajectory of the form (26), direct computation shows that

$$\begin{aligned} &\int_t^{t+\frac{2\pi}{h}} S^\top \operatorname{Ad}^\vee_{X_d(\tau)} BB^\top \operatorname{Ad}^{\vee\top}_{X_d(\tau)} S \mathrm{d}\tau \\ &\qquad = \begin{pmatrix} \frac{8\pi}{h} & 0 & 0 \\ 0 & \frac{(b^2+1)\pi}{h} & 0 \\ 0 & 0 & \frac{(a^2+1)\pi}{h} \end{pmatrix} \\ &\qquad \geq \varepsilon I, \end{aligned}$$

where ε is any positive number satisfying $\varepsilon \leq \frac{\pi}{h}\min\{8, a^2 + 1, b^2 + 1\}$. Therefore, the ellipse is persistently exciting and can be stabilised with the proposed controller.

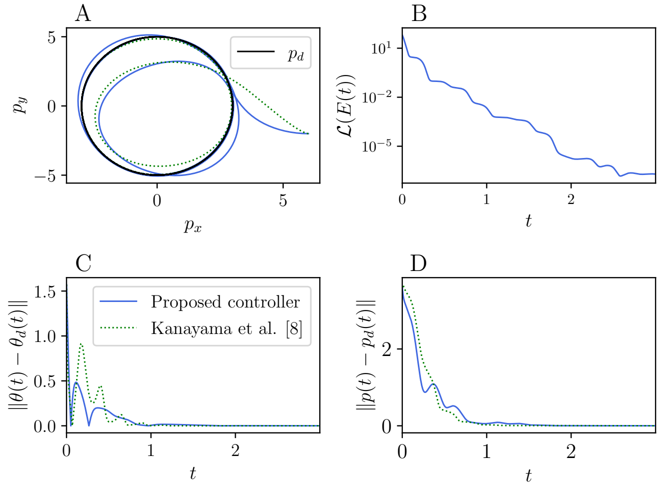

Fig. 4. Results with ellipse origin at (0, 0). (a) Desired (black) and actual position trajectories for the proposed controller and classic controller from literature. (b) Lyapunov function vs time for the proposed controller. (c) Heading error vs time for the proposed controller and another controller from literature. (d) Position error vs time for the proposed controller and another controller from literature.

B. Results

The specific trajectory to be tracked is given by:

$$p_d(t) = \begin{pmatrix} 3\cos(\frac{2\pi}{5}t) \\ 5\sin(\frac{2\pi}{5}t) \end{pmatrix}.$$

The corresponding inputs v_d, Ω_d can be recovered by

$$v_d(t) = \|\dot{p}_d\| = \sqrt{9\cos^2(\frac{2\pi}{5}t) + 25\sin^2(\frac{2\pi}{5}t)},$$
$$\Omega_d(t) = \frac{2\pi}{5}.$$

The simulation is run in two configurations: firstly, with the origin of the ellipse at $(0, 0)$, and then with the origin at $(3, 3)$. In both cases, the initial system state is perturbed to $p(0) = p_d(0) + (3.0, -2.0), \theta(0) = \theta_d(0) + \frac{\pi}{2}$, so that the initial relative error is the same. The simulation is repeated for the controller proposed in [12] in order to obtain comparative results. The simulation results are shown in Figure 4 for the choice of a $(0, 0)$ origin and in Figure 5 for the choice of a $(3, 3)$ origin.

C. Discussion

The simple simulations provided (Figure 4, Figure 5) verify the proposed control design. In both cases, the Lyapunov function is globally non-increasing and is linear in logarithmic coordinates, showing the local exponential stability. When the ellipse is centered on $(0, 0)$ (Figure 4), we note that the performance of the proposed controller is similar to a classic controller reported in the literature ([12]), although this is dependent on gain tuning. It is interesting to note that although the spatial error may appear unconventional (Figure 3), the convergence of the trajectory appears quite natural when the ellipse is centered on $(0, 0)$. Comparing Figure 4 to the results with an ellipse centered on $(3, 3)$ (Figure 5), the effect of the dependence of the spatial error

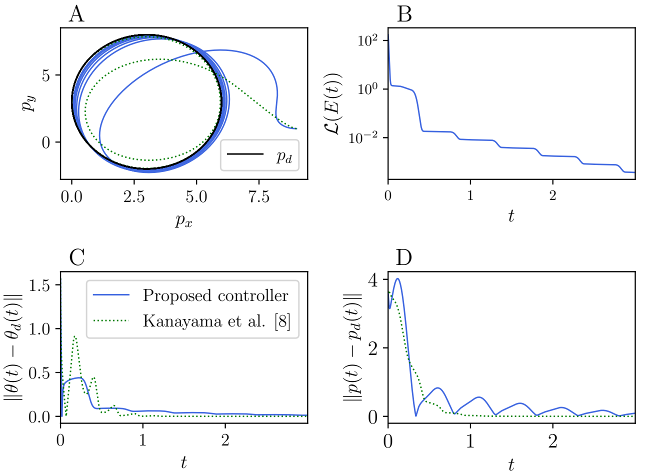

Fig. 5. Results with ellipse origin at (3, 3). (a) Desired (black) and actual position trajectories for the proposed controller and classic controller from literature. (b) Lyapunov function vs time for the proposed controller. (c) Heading error vs time for the proposed controller and another controller from literature. (d) Position error vs time for the proposed controller and another controller from literature.

on the inertial frame becomes evident. Here, our proposed controller takes a longer time to converge in both heading and position, and follows a noticeably different trajectory (although the performance is still comparable with the other controller). In contrast, the classic controller from literature, which uses a body error, follows the exact same trajectory. For our proposed controller, the fact that the trajectory depends on the choice of origin with respect to which the error is computed opens an interesting research direction in how to choose a good origin for a particular trajectory tracking problem.

VIII. Conclusion

The choice of error for systems on Lie groups is not merely a matter of taste, but results in qualitatively different error systems. The properties of the emergent error dynamics are closely tied to the physical meaning of the error choice. We demonstrated that the choice of a spatial error leads to a simple gradient-based controller design. We have shown that this control scheme yields a locally exponentially stable controller for a class of persistently exciting trajectories. We then used the example of the mobile robot to provide physical interpretations of the body and spatial, or left- and right-invariant errors for a rigid body with an $\mathbf{SE}(2)$ symmetry. We have proved the almost-global asymptotic convergence of the proposed control scheme for the mobile robot with persistently exciting trajectories. Finally, we have verified the effectiveness of this control in simulation.

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IX. Appendix

Lemma 9.1: Let $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ be bounded functions. If $A(t)C(t) \to 0$ and $\dot{C}(t) \to 0$ as $t \to \infty$, and $A(t)$ is persistently exciting, then $C(t) \to 0$.

Proof: Let $T > 0$ be such that the persistent excitation condition holds. We first show that $\lim_{t\to\infty} C(t)^\top \int_t^{t+T} A(\tau)^\top A(\tau)C(\tau)\mathrm{d}\tau = 0$. Let $L = \sup_t \|C(t)\|$ and $M = \sup_t \|A(t)\|$. Given $\varepsilon > 0$, let $T' > 0$ be large enough such that $\|A(t)C(t)\| < \frac{\varepsilon}{LMT}$ for $t > T'$. Then for $t > T'$, it holds that

$$\begin{aligned}\|C(t)^\top &\int_t^{t+T} A(\tau)^\top A(\tau)C(\tau)\mathrm{d}\tau\| \\ &\leq \|C(\tau)^\top\| \int_t^{t+T} \|A(\tau)^\top\|\|A(\tau)C(\tau)\|\mathrm{d}\tau \\ &< L\int_t^{t+T} M\frac{\varepsilon}{LMT} = \varepsilon,\end{aligned}$$

so $C(t)^\top \int_t^{t+T} A(\tau)^\top A(\tau)C(\tau)\mathrm{d}\tau \to 0$. On the other hand, by noting that for $\tau \in [t, t+T]$, $C(\tau) = C(t) + \int_t^\tau \dot{C}(s)\mathrm{d}s$, it also holds that

$$\begin{aligned}C(t)^\top &\int_t^{t+T} A(\tau)^\top A(\tau)C(\tau)\mathrm{d}\tau \\ &= C(t)^\top \left(\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau\right) C(t) \\ &+ C(t)^\top \int_t^{t+T} A(\tau)^\top A(\tau) \int_t^\tau \dot{C}(s)\mathrm{d}s\mathrm{d}\tau.\end{aligned}$$

The term $C(t)^\top \int_t^{t+T} A(\tau)^\top A(\tau) \int_t^\tau \dot{C}(s)\mathrm{d}s\mathrm{d}\tau$ can also be shown to approach 0 as $t \to \infty$: let L and M be defined as before and given $\varepsilon > 0$, let $T' > 0$ be large enough such that $\|\dot{C}(t)\| < \frac{2\varepsilon}{LM^2T^2}$ for $t > T'$. Then for $t > T'$, it holds that

$$\begin{aligned}\|C(t)^\top &\int_t^{t+T} A(\tau)^\top A(\tau) \int_t^\tau \dot{C}(s)\mathrm{d}s\mathrm{d}\tau\| \\ &< L\int_t^{t+T} M^2 \int_t^\tau \frac{2\varepsilon}{LM^2T^2}\mathrm{d}s\mathrm{d}\tau \\ &= LM^2\frac{2\varepsilon}{LM^2T^2}\int_t^{t+T}(\tau - t)\mathrm{d}\tau \\ &= \varepsilon,\end{aligned}$$

so $\int_t^{t+T} A(\tau)^\top A(\tau) \int_t^\tau \dot{C}(s)\mathrm{d}s\mathrm{d}\tau \to 0$. Thus, the remaining term of the integral, $C(t)^\top \left(\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau\right) C(t)$, must also approach zero as $t \to \infty$. But because $A(t)$ is exciting, the integral $\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau$ is lower bounded by αI, for some $\alpha > 0$, for all t. So

$$0 \leq \alpha\|C(t)\|^2 \leq C(t)^\top \left(\int_t^{t+T} A(\tau)^\top A(\tau)\mathrm{d}\tau\right) C(t)$$

for all t and hence $C(t) \to 0$. ■

Proposition 9.2: Consider the linear time-varying system $\dot{x} = -A(t)x$, with $A(t) \in \mathbb{R}^{n\times n}$ symmetric and positive semi-definite. Assume that there exist a $T > 0$ and $\varepsilon > 0$ such that for all t,

$$\int_t^{t+T} A(\tau)\mathrm{d}\tau \geq \varepsilon I. \tag{27}$$

Then $\dot{x} = -A(t)x$ is uniformly exponentially stable at $x = 0$.

Proof: We aim to show that the assumed conditions of symmetric positive semi-definite $A(t)$ and (27) imply the conditions required of [20] (Theorem 1). This proof reproduces an argument used in [26] (Proposition 4.6). By the Cauchy-Schwarz inequality, the fact that $\int_t^{t+T} A(\tau)\mathrm{d}\tau$ is lower-bounded implies that for any unit vector $y \in \mathbb{R}^n$, the integral $\int_t^{t+T} \|A(\tau)y\|\mathrm{d}\tau$ is also lower bounded, as

$$\begin{aligned}\int_t^{t+T} \|A(\tau)y\|\mathrm{d}\tau &= \int_t^{t+T} \|y\|\|A(\tau)y\|\mathrm{d}\tau \\ &\geq \int_t^{t+T} y^\top A(\tau)y\mathrm{d}\tau \\ &= y^\top \int_t^{t+T} A(\tau)\mathrm{d}\tau y \\ &\geq y^\top(\varepsilon I)y \\ &= \varepsilon.\end{aligned}$$

Given $t \geq t_0 > 0$, define $N = \lfloor \frac{t-t_0}{T} \rfloor$ and consider subdividing the interval $[t_0, t]$ into N intervals of length T and a remainder $[t_0 + NT, t]$:

$$\begin{aligned}[t_0, t] = [t_0, t_0 + T] \cup [t_0 + T, t_0 + 2T] \cup ... \\ [t_0 + (N-1)T, t_0 + NT] \cup [t_0 + NT, t].\end{aligned}$$

Then

$$\begin{aligned}\int_{t_0}^t \|A(\tau)y\|\mathrm{d}\tau &= \sum_{i=1}^N \int_{t_0+(i-1)T}^{t_0+iT} \|A(\tau)y\|\mathrm{d}\tau \\ &+ \int_{t_0+NT}^t \|A(\tau)y\|\mathrm{d}\tau \\ &\geq N\varepsilon = \frac{\varepsilon}{T}\left\lfloor \frac{t-t_0}{T} \right\rfloor T \\ &\geq \frac{\varepsilon}{T}((t-t_0) - T) = \frac{\varepsilon}{T}(t-t_0) - \varepsilon.\end{aligned}$$

This is the condition required by [20] (Theorem 1) and so the system (27) is uniformly asymptotically stable. Then by [9] (Theorem III.2.1), the system is also uniformly exponentially stable. ■