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On complexity reduction in a variable terminal set setpoint-tracking MPC scheme

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***Abstract*— We applied and adapted the linear encodings from [1] to the feasible-reference tracking model predictive control (MPC) formulation from [2] to reduce its computational cost. The improvements come from avoiding to explicitly use the vertex-based representation of the variable terminal set in testing its inclusion in the constraint set. We considered both polytopic and zonotopic formulations. For the later we have also proposed a positive invariant (PI) zonotopic approximation of the maximal PI set.**

***Index Terms*— Model Predictive Control (MPC), Variable Terminal Set, Set inclusion, Zonotopic sets.**

I. Introduction

Model predictive control (MPC) is one of the most popular control techniques due to the many variations and overall robustness [3]. Its capacity to account for constraints and costs explain its popularity in control.

While MPC is inherently robust to both internal uncertainties and external disturbances, it was a source of some concern that no clear proofs of stability and feasibility were known. The recursive feasibility notion discussed at large in [4] provided the desired theoretical guarantees by introducing the ancillary elements of terminal set and control law. More recently, alternatives which define a safe minimum prediction horizon length have been presented [5]. Still, none of these approaches has proved entirely satisfactorily. The later has impractically large prediction horizons and the former forces the initial state to lie in the backward reachable set of the terminal set (the region of attraction), which may severely limit the problem's feasibility [6]. Consequently, significant effort has been put in the last decade to alter the standard formulations in the sense of enlarging the region of attraction, relax the terminal set conditions and/or reduce the complexities of computing the terminal set/control law. The last point is particularly relevant if we note that the terminal set is, in its generic form, control positive invariant (usually the implementations fall back to positive invariance characterizations by choosing a priori the terminal control law). Many variations exist in the literature, e.g., [2] proposes a variable terminal set, [7] considers a union of invariant sets, [8] proposes an online reconfiguration and [9] uses an implicit representation.

Both [2] and its expansion from [10] consider a variable terminal set which is an affine transformation of the maximal positive invariant set (MPI) of the dynamics. The additional degrees of freedom (offset term and scaling factor) ensure that the tracked reference corresponds to a pair of admissible terminal state and control law. The drawback of this approach is that, for each MPC call, two ancillary set inclusions have to be verified (to check admissibility of the terminal state and input, respectively). The standard test for polyhedral inclusion requires to check that all vertices of the inner one validate the constraints of the outer one. This may prove cumbersome for higher dimensions due to the (usually) exponential increase in vertices w.r.t. the number of inequalities which describe a set in half-space form [11]. On the other hand, a recent result [1], build upon [12], provides sufficient conditions for the polyhedral set containment problem employing only half-space descriptions. Motivated by these ideas, our contributions are:

i) use and adapt the sufficient containment formulations from [1] to avoid explicitly using the vertex representation of the polyhedral sets;
ii) propose a positive invariant set construction based on the scaled zonotope idea [13].

The rest of the paper is organized as follows. Section II introduces the set-point tracking MPC problem and describes the issues to be tackled. Section III adapts and simplifies the set containment results to the specifics of the MPC problem. The ideas are illustrated in Section IV over two examples. Section V draws the conclusions.

Notation. $O_{m\times n} \in \mathbb{R}^{m\times n}$ is the matrix with m rows and n columns whose entries are zero. Whenever $m = n$, we use the shorthand notation O_n. $I_n \in \mathbb{R}^{n\times n}$ is the matrix whose diagonal elements are one and zero otherwise. For an arbitrary matrix $G \in \mathbb{R}^{m\times n}$, G_i denotes its i-th column and $G_j^\top$ its j-th row. The Minkowski sum between two sets, X and Y, is defined as $X \oplus Y = \{x + y : \forall x \in X, \forall y \in Y\}$. $\mathbb{R}_+$ denotes the set of positive real numbers.

II. Problem statement

Consider the discrete, linear, time-invariant system

$$x_{k+1} = Ax_k + Bu_k, \tag{1a}$$

$$y_k = Cx_k + Du_k, \tag{1b}$$

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with $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$. Furthermore, under the mild assumption of controllability, there exists a feedback law

$$u_k = \bar{u} - K(x_k - \bar{x}), \tag{2}$$

with $\bar{u} \in \mathbb{R}^m, \bar{x} \in \mathbb{R}^n$ chosen such as to respect the steady-point condition $\bar{x} = A\bar{x}+B\bar{u}$ and with $K \in \mathbb{R}^{n\times m}$ taken such that the closed loop dynamics

$$(x_{k+1} - \bar{x}) = (A - BK)(x_k - \bar{x}), \tag{3}$$

are asymptotically stable.

The goal is to steer output (1b) towards a reference profile r_k under input ($u_k \in \mathcal{U} \subset \mathbb{R}^m$) and state ($x_k \in \mathcal{X} \subset \mathbb{R}^n$) constraints. To do so, we consider the MPC set-point tracking formulation given in [2], [10]:

$$\begin{aligned}
\min_{\substack{u_k,\dots,u_{k+N_{\text{pred}}-1},\\ \lambda_k,\bar{x}_k,\bar{u}_k,\bar{r}_k}} & \sum_{i=0}^{N_{\text{pred}}-1} \left(\|x_{k+i} - \bar{x}_k\|_Q^2 + \|u_{k+i} - \bar{u}_k\|_R^2\right) \\
& + \left\|x_{k+N_{\text{pred}}} - \bar{x}_k\right\|_P^2 + \|\bar{r}_k - r_k\|_W^2 && \text{(4a)}\\
\text{s.t.}\quad & x_{k+i+1} = Ax_{k+i} + Bu_{k+i}, && \text{(4b)}\\
& x_{k+i} \in \mathcal{X},\ u_{k+i} \in \mathcal{U}, && \text{(4c)}\\
& x_{k+N_{\text{pred}}} \in \lambda_k\mathcal{T} \oplus \{\bar{x}_k\}, && \text{(4d)}\\
& \lambda_k\mathcal{T} \oplus \{\bar{x}_k\} \subseteq \mathcal{X}, && \text{(4e)}\\
& \bar{u}_k - K(x_{k+i} - \bar{x}_k) \in \mathcal{U},\ \forall\, x \in \lambda_k\mathcal{T} \oplus \{\bar{x}_k\}, && \text{(4f)}\\
& \begin{bmatrix} A - I & B \\ C & D \end{bmatrix}\begin{bmatrix} \bar{x}_k \\ \bar{u}_k \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{r}_k \end{bmatrix}, && \text{(4g)}\\
& \bar{u}_k \in \text{int}_\epsilon(\mathcal{U}),\ \bar{x}_k \in \text{int}_\epsilon(\mathcal{X}),\ i = 0 : N_{\text{pred}} - 1 && \text{(4h)}
\end{aligned}$$

Matrices Q, R, W are positive (semi-)definite and of appropriate dimensions. P is the result of the Lyapunov equation $(A - BK)^T P(A - BK) - P = -Q - K^T RK$, and N_{pred} is the prediction horizon.

The MPC from (4) differs from the standard 'MPC with terminal set' construction in several key aspects (see for further details [2] or the extensions from [10]):

- the set-point r_k might be infeasible and thus impossible to reach; thus a feasible $\bar{r}_k$ set-point results from imposing a penalty term in the cost and enforcing a steady-state condition in (4g); $\bar{r}_k$ is guaranteed to be the closest feasible (w.r.t. $\mathcal{U}, \mathcal{X}$) point to r_k, see [10, Theorem 2];
- to reduce conservatism, a variable (offset by steady-state $\bar{x}_k$ and scaled by scalar $\lambda_k \geq 0$) set, $\lambda_k\mathcal{T} \oplus \bar{x}_k$ is considered: terminal condition (4d), input (4f) and state (4f) restrictions ensure that closed-loop dynamics (3) are stable and admissible[1];
- inclusion (4e) ensures terminal set admissibility and (4h) avoid, via a small, positive scalar ϵ, that $\bar{x}_k, \bar{u}_k$ lie on the boundaries of $\mathcal{X},\mathcal{U}$.

Remark 1. *The standard point-tracking MPC works under two contradictory impulses. On one hand, to reduce conservatism, the terminal set should be as large as possible (i.e., chosen as the maximal positive invariant (MPI) set [14]). On the other hand, tracking a reference set-point may bring the state close to the boundary of its admissible set (and outside the terminal set). Thus, [2], [10], pre-compute the terminal set $\mathcal{T}$ as the MPI set for dynamics* (3)*, see Sec. III-B for further constructive details. Then, $\lambda_k\mathcal{T}$ is also PI under* (3) *which implies in turn that $\lambda_k\mathcal{T} \oplus \{\bar{x}_k\}$ contains x_k at all times.* ♦

Remark 2. *While r_k is called a 'reference' we do not make the assumption to know its evolution along the prediction horizon. The value received at the current simulation step (the k-th) is used as set-point to be tracked by the terminal predicted output ($y_{k+N_{pred}}$) via suitable choices of $\bar{x}_k, \bar{u}_k$ in* (4g) *and penalties in the cost* (4a).♦

The weakness of (4) comes from enforcing the set-inclusions[2] (4e) and (4f):

$$\lambda_k\mathcal{T} \oplus \{\bar{x}_k\} \subseteq \mathcal{X}, \tag{5a}$$
$$-\lambda_k K\mathcal{T} \oplus \{\bar{u}_k\} \subseteq \mathcal{U}. \tag{5b}$$

The terms on both sides of the inclusion operator are usually polyhedral sets, given in half-space form (as an intersection of inequalities). Checking the inclusions requires to enumerate the vertices defining the polyhedra on the left (to ensure that each of them checks the constraints of the one on the right). Specifically, taking $\mathcal{X} = \{x : H_\mathcal{X} x \leq h_\mathcal{X}\}$, $\mathcal{U} = \{u : H_\mathcal{U} u \leq h_\mathcal{U}\}$ and $\mathcal{T} = \{x = \sum_j \alpha_j v_j,\ \sum_j \alpha_j = 1, \alpha_j \geq 0, \forall j\}$ allows to reformulate (5) into:

$$H_\mathcal{X}\left(\lambda_k v_j + \bar{x}_k\right) \leq h_\mathcal{X}, \tag{6a}$$
$$H_\mathcal{U}\left(\bar{u}_k - \lambda_k K v_j\right) \leq h_\mathcal{U}, \quad \forall j. \tag{6b}$$

Usually, the number of vertices is exponential w.r.t. the number of inequalities from the half-space representation [11] and enumerating them may prove non-trivial, especially for large state dimensions. In any case, even having them, we would still have to add many inequalities in the MPC problem (all of these from (6) into (4)). Our idea, and the bulk of the remainder of the paper, has two parts:

i) make use of the sufficient containment formulations (polytopic and zonotopic forms) given in [1] to avoid explicitly using the vertex representation;
ii) exploit zonotopic formulations for sets $\mathcal{U}, \mathcal{X}, \mathcal{T}$ to further reduce the complexity of the representation.

III. Main idea

Consider the notion of an *AH polytope* [1], $\mathbb{Z}$, defined as the affine transformation of a polytope given by its half-space representation $\mathbb{P}_z = \{z \in \mathbb{R}^{n_z} : H_z z \leq h_z\} \subset \mathbb{R}^{n_z}$:

$$\mathbb{Z} = \{\bar{z}\} \oplus Z\mathbb{P}_z, \tag{7}$$

where $Z \subset \mathbb{R}^{n\times n_z}$, $\bar{z} \in \mathbb{R}^n$. Based on this definition we recall two of the results from [1] which provide sufficient conditions for set inclusion verification.

[1]We abused the notation and took $\bar{x} \mapsto \bar{x}_k$ and $\bar{u} \mapsto \bar{u}_k$.

[2]Note that (4f) has been, for further use, put in a more compact form than the one used in (4).

Proposition 1. *[AH-polytope inclusion, [1, Thm. 1]] Let* $\mathbb{U} = \{\bar{u}\} \oplus U\mathbb{P}_u \subset \mathbb{R}^n$, $\mathbb{V} = \{\bar{v}\} \oplus V\mathbb{P}_v \subset \mathbb{R}^n$, *defined as in* (7), *where* $\mathbb{P}_u = \{u \in \mathbb{R}^{n_u} : H_u u \leq h_u\}$, $\mathbb{P}_v = \{v \in \mathbb{R}^{n_v} : H_v v \leq h_v\}$ *and* $(H_u, h_u) \in \mathbb{R}^{q_u \times n} \times \mathbb{R}^{q_u}$, $(H_v, h_v) \in \mathbb{R}^{q_v \times n} \times \mathbb{R}^{q_v}$, $(U, \bar{u}) \in \mathbb{R}^{n \times n_u} \times \mathbb{R}^n$, $(V, \bar{v}) \in \mathbb{R}^{n \times n_v} \times \mathbb{R}^{n_v}$.

Then set inclusion $\mathbb{U} \subseteq \mathbb{V}$ *holds if*

$$\exists\, \Gamma \in \mathbb{R}^{n_v \times n_u},\ \beta \in \mathbb{R}^{n_v}, \Lambda \in \mathbb{R}_+^{q_v \times q_u} \tag{8}$$

such that

$$U = V\Gamma, \qquad \bar{v} - \bar{u} = V\beta, \tag{9a}$$

$$\Lambda H_u = H_v \Gamma \qquad \Lambda h_u \leq h_v + H_v \beta. \tag{9b}$$

Zonotopes (in one interpretation, the affine mappings of the unit ball of the infinity norm [11]) are a particular case of an AH-polytope:

$$\langle \bar{t}, T \rangle := \{\bar{t}\} \oplus T\mathbb{B}_\infty^{n_t}, \tag{10}$$

with $\bar{t} \in \mathbb{R}^n$ the zonotope's center and $T \in \mathbb{R}^{n \times n_t}$ its generator matrix. The particularities of representation (10) lend to a simplified reformulation of Prop. 1.

Proposition 2. *[Zonotope inclusion, [1, Cor. 4]] Let* $\mathbb{U} = \langle \bar{t}, T \rangle \subset \mathbb{R}^n$, $\mathbb{V} = \langle \bar{v}, V \rangle \subset \mathbb{R}^n$, *defined as in* (10), *where* $(U, \bar{u}) \in \mathbb{R}^{n \times n_u} \times \mathbb{R}^n$, $(V, \bar{v}) \in \mathbb{R}^{n \times n_v} \times \mathbb{R}^{n_v}$.

Then set inclusion $\mathbb{U} \subseteq \mathbb{V}$ *holds if*

$$\exists\, \Gamma \in \mathbb{R}^{n_v \times n_u},\ \beta \in \mathbb{R}^{n_v}, \tag{11}$$

such that

$$U = V\Gamma,\ \bar{v} - \bar{u} = V\beta,\ \left\| \begin{bmatrix} \Gamma & \beta \end{bmatrix} \right\|_\infty \leq 1. \tag{12}$$

A. Containment reformulations

We aim to provide linear and compact sufficient conditions for the set inclusions checks from (5). We start by assuming that $\mathcal{T} \subseteq \mathcal{X}$, the MPI set for closed-loop dynamics $x_{k+1} = (A - BK)x_k$, has already been computed and is given in half-space representation: $\mathcal{T} = \{x : H_\mathcal{T} x \leq h_\mathcal{T}\}$, with $H_\mathcal{T} \in \mathbb{R}^{q_t \times n}$, $h_\mathcal{T} \in \mathbb{R}^{q_t}$. Furthermore, recall that both $\mathcal{U}$ and $\mathcal{X}$ are given in half-space form around (6). It is then straightforward to adapt Proposition 1 to set inclusions (5) in the next corollary.

Corollary 1. *Inclusions* (5) *are verified by the sufficient formulations:*

i) state inclusion (5a)

$$\Lambda_x H_\mathcal{T} = H_\mathcal{X} \lambda_k,\ \Lambda_x h_\mathcal{T} \leq h_\mathcal{X} - H_\mathcal{X} \bar{x}_k, \tag{13}$$

ii) input inclusion (5b)

$$\Lambda_u H_\mathcal{T} = -H_\mathcal{U} K \lambda_k,\ \Lambda_u h_\mathcal{T} \leq h_\mathcal{U} - H_\mathcal{U} \bar{u}_k, \tag{14}$$

with $\Lambda_x \in \mathbb{R}_+^{q_x \times q_t}$ *and* $\Lambda_u \in \mathbb{R}_+^{q_u \times q_t}$.

Proof: We adapt the sets from Prop. 1 to those used in (5) and observe the simplifications that appear.

i) taking $\{\bar{u}, U, \mathbb{P}_u, \bar{v}, V, \mathbb{P}_v\} \leftarrow\!\!\shortmid \{\bar{x}_k, \lambda_k \cdot I, \mathcal{T}, 0, I, \mathcal{X}\}$ we get that $\Gamma = \lambda_k \cdot I, \beta = -\bar{x}_k$, which, put in (9), directly give relations (13);

ii) taking $\{\bar{u}, U, \mathbb{P}_u, \bar{v}, V, \mathbb{P}_v\} \leftarrow\!\!\shortmid \{\bar{u}_k, \lambda_k \cdot K, \mathcal{T}, 0, I, \mathcal{U}\}$ we get that $\Gamma = \lambda_k \cdot K, \beta = -\bar{u}_k$, which, put in (9), directly give relations (14). ■

Remark 3. *Note that, while* λ_k *is a variable changing at each MPC run, it is still a scalar. It is then possible to pre-compute* $\Lambda_x^0 \geq 0$ *verifying* $\Lambda_x^0 H_\mathcal{T} = H_\mathcal{X}$ *and replace the inequality from* (13) *with*

$$\Lambda_x^0 h_\mathcal{T} \cdot \lambda_k \leq h_\mathcal{X} - H_\mathcal{X} \bar{x}_k \tag{15}$$

via notation $\Lambda_x \leftarrow\!\!\shortmid \Lambda_x^0 \lambda_k$. *The same procedure may be applied for* $\Lambda_u^0 \geq 0$ *verifying* $\Lambda_u^0 H_\mathcal{T} = -H_\mathcal{T} K$ *to subsequently replace* (14) *with*

$$\Lambda_u^0 h_\mathcal{T} \cdot \lambda_k \leq h_\mathcal{U} - H_\mathcal{U} \bar{u}_k. \tag{16}$$

The advantage of these reformulations is that we no longer introduce Λ_x, Λ_u *as variables into the optimization problem. They are obtained in a pre-processing step and, within the MPC optimization problem* (13) *and* (14) *are replaced by* (15) *and* (16). *This shows that we do not need to introduce any new variables in the MPC problem to describe the sufficient conditions from Cor. 1.* ♦

Typically, constraint sets employed in MPC are symmetric to the origin (perhaps after a change of coordinates). Thus, we propose the next corollary of Prop. 1.

Corollary 2 (Symmetric AH Polytope inclusion)**.** *Let* $\mathbb{U}, \mathbb{V}$ *be defined as in Prop. 1 but change* $P_u = \{u \in \mathbb{R}^{n_u} : |H_u u| \leq h_u\}$, *and* $P_v = \{v \in \mathbb{R}^{n_v} : |H_v v| \leq h_v\}$ *to symmetric (w.r.t. the origin) polytopes, with* $H_u \in \mathbb{R}^{\frac{q_u}{2} \times n_u}$, $H_v \in \mathbb{R}^{\frac{q_v}{2} \times n_v}$, $h_u \in \mathbb{R}_+^{\frac{q_u}{2}}$ *and* $h_v \in \mathbb{R}_+^{\frac{q_v}{2}}$. *Then we have* $\mathbb{U} \subseteq \mathbb{V}$ *if:*

$$\exists\, \Gamma \in \mathbb{R}^{n_v \times n_u},\ \beta \in \mathbb{R}^{n_v}, \bar{\Lambda} \in \mathbb{R}^{\frac{q_v}{2} \times \frac{q_u}{2}} \tag{17}$$

such that the following hold:

$$U = V\Gamma, \qquad \bar{v} - \bar{u} = V\beta, \tag{18a}$$

$$\bar{\Lambda} H_u = H_v \Gamma, \qquad |\bar{\Lambda}| h_u \leq h_v - |H_v \beta|. \tag{18b}$$

Proof: Noting that P_u may be rewritten as $P_u = \{u \in \mathbb{R}^{n_u} : \begin{bmatrix} H_u^\top & -H_u^\top \end{bmatrix} u \leq \begin{bmatrix} h_u^\top & h_u^\top \end{bmatrix}\}$ (similarly for P_v) we have that $\Lambda \in \mathbb{R}_+^{q_v \times q_u}$, as defined in Prop. 1, may be decomposed into $\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_1 \end{bmatrix}$ with $\Lambda_1, \Lambda_2 \in \mathbb{R}_+^{\frac{q_v}{2} \times \frac{q_u}{2}}$. This allows to reformulate (9) into

$$U = V\Gamma, \qquad \bar{v} - \bar{u} = V\beta,$$

$$\pm(\Lambda_1 - \Lambda_2) H_u = \pm H_v \Gamma, \quad (\Lambda_1 + \Lambda_2) h_u \leq h_v \pm H_v \beta.$$

Taking $\bar{\Lambda} = \Lambda_1 - \Lambda_2$ and observing that $|\bar{\Lambda}| h_u \leq (|\Lambda_1| + |\Lambda_2|) h_u = (\Lambda_1 + \Lambda_2) h_u$ we arrive at (18). ■

Cor. 2 may be adapted for set inclusions (5), as was done with Cor. 1 for Prop. 1. Without repeating the same reasoning, the end result are relations

$$\bar{\Lambda}_x H_\mathcal{T} = \quad H_\mathcal{X} \lambda_k, \qquad |\bar{\Lambda}_x| h_\mathcal{T} \leq h_\mathcal{X} - |H_\mathcal{X} \bar{x}_k|, \tag{19a}$$

$$\bar{\Lambda}_u H_\mathcal{T} = -H_\mathcal{U} K \lambda_k, \qquad |\bar{\Lambda}_u| h_\mathcal{T} \leq h_\mathcal{U} - |H_\mathcal{U} \bar{u}_k|, \tag{19b}$$

with $\bar{\Lambda}_x \in \mathbb{R}^{\frac{q_x}{2} \times \frac{q_t}{2}}$ and $\bar{\Lambda}_u \in \mathbb{R}^{\frac{q_u}{2} \times \frac{q_t}{2}}$. ♦

Remark 4. *Sometimes, as is the case for the first example of Sec. IV, sets $\mathcal{U},\mathcal{X},\mathcal{T}$ are symmetric w.r.t. centers u_0, x_0, x_{t0}. This may be handled by mappings $\bar{u} \hookleftarrow \bar{u}+Uu_0$ and $\bar{v} \hookleftarrow \bar{v}+Vv_0$ in Cor. 2 which, ultimately, lead to the following reformulation of inequalities (19):*

$$|\bar{\Lambda}_x|h_{\mathcal{T}} + |H_{\mathcal{X}}\left(\bar{x}_k + \lambda_k x_{t0} - x_0\right)| \leq h_{\mathcal{X}}, \tag{20a}$$
$$|\bar{\Lambda}_u|h_{\mathcal{T}} + |H_{\mathcal{U}}\left(\bar{u}_k - \lambda_k K x_{t0} - u_0\right)| \leq h_{\mathcal{U}}. \tag{20b}$$

♦

B. Maximal positive invariant set approximation

Let us briefly recall the notion of set invariance [14]. A set $\Omega \subset \mathbb{R}^n$ is called positive invariant (PI) w.r.t. dynamics (3) iff the implication

$$x_k \in \Omega \implies x_{k'} \in \Omega, \forall k' \geq k, \tag{21}$$

holds for some finite index k. Equivalently stated, the set-inclusion relation[3] $(A-BK)\Omega \subseteq \Omega$ has to hold. The maximal positive invariant (MPI) set is simply the largest PI set respecting the state constraints ($\Omega \subseteq \mathcal{X}$). By construction, the MPI set is given (in possibly redundant half-space form) as [15]

$$\mathcal{T} = \bigcap_{\ell=0}^{L}(A-BK)^{\ell}\mathcal{X}, \tag{22}$$

with L being a finite (under standard assumptions) scalar which verifies $(A-BK)^{L+1}\mathcal{X} \subseteq \mathcal{X}$.

The goal of (22) is to find the largest positive invariant set which fits the feasible domain $\mathcal{X}$. This makes sense in the standard MPC construction where the terminal set should 'fit' as well as possible within $\mathcal{X}$. Yet, the variable terminal set employed in (4) is often scaled to a smaller value (via λ_k), which makes it no longer a 'maximal' PI set. We may ask then whether using (22) as terminal set is still justified. Hence, we propose to:

i) compute a PI set for the closed dynamics (3) using a scalable zonotopic formulation;
ii) employ the zonotopic linear encodings from Prop.2 to further reduce the computational effort.

Recalling the definition of a zonotope from (10), we take

$$\mathcal{T}_z(\delta) = \langle 0, G\text{diag}(\delta)\rangle, \tag{23}$$

with the generator matrix G a priori fixed and $\delta \in \mathbb{R}^D$, positive scaling factors which are to be determined. For further use we make the notation $\Delta := \text{diag}(\delta)$. We have to check two inclusions:

i) the invariance condition

$$(A-BK)\mathcal{T}_z(\delta) \subseteq \mathcal{T}_z(\delta), \tag{24}$$

ii) and the state admissibility condition

$$\mathcal{T}_z(\delta) \subseteq \mathcal{X}. \tag{25}$$

[3]We consider the closed-loop dynamics described by $A-BK$. No unstable system can have a bounded PI set.

For further use, we abuse the notation and we assume that both $\mathcal{U},\mathcal{X}$ may be written in zonotopic form (10):

$$\mathcal{U} = \langle 0, U\rangle, \quad \mathcal{X} = \langle 0, X\rangle. \tag{26}$$

Proposition 3. *The largest zonotopic set $\mathcal{T}_z(\bar{\delta})$, given as in (23) and respecting (24)–(25) is defined by*

$$\bar{\delta} = \max_{\delta} \sum_{1\leq i_1<\ldots i_n \leq D}\left[\left|\det(G^{i_1\ldots i_n})\right| \cdot \prod_{j=1}^{n}\delta_{i_j}\right] \tag{27a}$$
$$\text{s.t. } (A-BK)G\Delta = G\Delta\Gamma_1,\ \|\Gamma_1\|_\infty \leq 1, \tag{27b}$$
$$G\Delta = X\Gamma_2,\ \|\Gamma_2\|_\infty \leq 1, \tag{27c}$$

with $G^{i_1\ldots i_n}$ denoting the sub-matrix obtained from G by extracting columns of indices $\{i_1,\ldots,i_n\}$.

Proof: The cost (27a) describes (23)'s volume as defined in [16]. Terms (27b)–(27c) are sufficient reformulations of conditions (24)–(25) via Prop. 2. ■

Having a new, zonotopic, terminal set ($\mathcal{T}_z(\bar{\delta})$ instead of $\mathcal{T}$) allows us to adapt Prop. 2 into the following corollary.

Corollary 3. *Consider $\mathcal{T}_z(\bar{\delta})$ defined as in (23) with $\bar{\delta}$ obtained from Prop. 3. With the notation from (26), inclusion conditions (5) are guaranteed to hold if*

$$\lambda_k G\bar{\Delta} = X\Gamma_x,\ -\bar{x}_k = X\beta_x,\ \|[\Gamma_x, \beta_x]\|_\infty \leq 1, \tag{28a}$$
$$\lambda_k KG\bar{\Delta} = U\Gamma_u,\ -\bar{u}_k = U\beta_u,\ \|[\Gamma_u, \beta_u]\|_\infty \leq 1, \tag{28b}$$

hold, with notation $\bar{\Delta} := \text{diag}(\bar{\delta})$.

Proof: Set inclusions (5) become[4], with the redefinition from (26) and replacement of $\mathcal{T}$ with $\mathcal{T}_z(\bar{\delta})$:

$$\left\langle \bar{x}_k, \lambda_k \cdot G\bar{\Delta}\right\rangle \subseteq \langle 0, X\rangle,$$
$$\left\langle \bar{u}_k, \lambda_k \cdot KG\bar{\Delta}\right\rangle \subseteq \langle 0, U\rangle.$$

Applying Prop. 2 to these relations directly leads to (28), thus concluding the proof. ■

IV. Illustrative example

We test the improvements in computation time for various implementations of set inclusions (5):

S1) the exact, vertex / half-space constraints from (6);
S2) the polyhedral formulations from (13)–(14);
S3) the reduced polyhedral forms from (15)–(16);
S4) the polyhedral forms exploiting symmetry in (19);

For illustration purposes we use both the simplified aircraft example from [2] and the 'CSE1' example from the $COMPl_eib$ [17]. The first is relevant for comparison purposes (w.r.t. [2], [10]) and the later better emphasis the computation time gains (due its higher dimension).

We used the MPT3 [18] and CasADi [19] tools to solve the MPC problem in Matlab.

[4]The minus sign may be discarded in the first equality of (28b) since zonotopes are symmetric to the origin.

A. *Simplified aircraft example from [10]*

The system's state has two components: the attack angle (between the airplane's orientation and the velocity vector), α, and the pitch rate (the rate of change of the airplane's angle over the x-axis), q. The system's input is the elevator angle of the airplane, δ.

The discretized (with a sampling time of 60ms) LTI system is given by

$$\begin{bmatrix}\alpha_{k+1}\\ q_{k+1}\end{bmatrix} = \underbrace{\begin{bmatrix}0.9719 & 0.0155\\ 0.2097 & 0.9705\end{bmatrix}}_{\text{A}} \begin{bmatrix}\alpha_k\\ q_k\end{bmatrix} + \underbrace{\begin{bmatrix}0.0071\\ 0.3263\end{bmatrix}}_{\text{B}} \delta_k$$

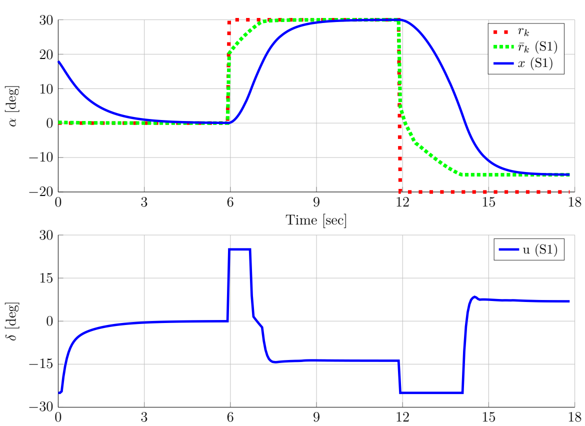
$$y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{\text{C}} \begin{bmatrix}\alpha_k\\ q_k\end{bmatrix},$$

with state and input constraints

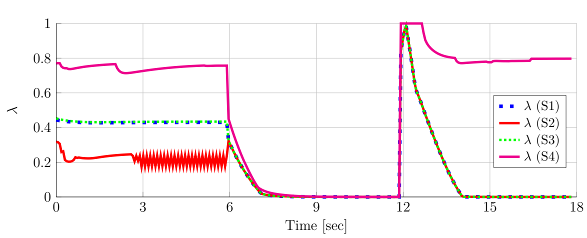
$$\mathcal{X} = \{(\alpha,\ q)^T \mid -15 \le \alpha \le 15, -100 \le q \le 100\},$$
$$\mathcal{U} = \{\delta \mid -25 \le \delta \le 25\}.$$

For the MPC problem (4), we take, similarly with [2], the stage penalty matrices $Q = \mathrm{diag}(10, 1)$, $R = 1$, $W = 10^4$, and terminal ingredients, the penalty matrix $P = [124.24,\ 5.17;\ 5.17,\ 3.47]$ and terminal control gain $K = \begin{bmatrix}1.96, & 0.84\end{bmatrix}$. The prediction horizon is chosen $N_{\text{pred}} = 20$, and $\epsilon = 10^{-5}$. On a simulation horizon of 18 sec we consider a piecewise constant, over each slot of 6 seconds, taking the values $\{0,\ 30,\ -20\}$, respectively. The initial state is taken to be $x_0 = [18;\ 0]$.

The results are depicted in Fig. 1. The top plot in Fig. 1a illustrates the ideal reference (loosely dashed, red), the feasible reference (dotted green) and actual output (solid blue). The bottom plot illustrates the input. The feasible reference, output and input are obtained with scenario S1). We do not depict the signals for the remaining scenarios to avoid cluttering the figure and because they have very similar values. Fig.'1b shows the scaling factor λ_k under each of the scenarios. We observe that the values range, as expected, between 0 and 1 and we observe that the values for the scenarios S1), S2), and S3) fully overlap over the segment $[6, 18]$ sec. Since we did not tune W which penalizes the difference between ideal and feasible reference and since, as noted in [1], the AH/zonotopic encodings describe only sufficient conditions, we consider these scaling values acceptable.



(a) reference tracking



(b) terminal set scaling

Fig. 1: Simulation results for the airplane model

The mean computation times are depicted in the following table. Due to the problem's small size ($x \in \mathbb{R}^2$) the values are close and quite small.

Scenario	S1)	S2)	S3)	S4)
Mean Computation Times [msec]	1.3	1.6	1.2	1.1

TABLE I: Computation times

B. *CSE1 example from [17]*

The 'CSE' dynamics describe a system combining coupled springs, dashpots and masses. The mass positions and velocities define the system state and its input are the two forces exerted at the ends of the coupled springs chain [17]. The continuous-time model is given by

$$\dot{x} = \underbrace{\begin{bmatrix}0 & I\\ -M_c^{-1}K_c & -M_c^{-1}L_c\end{bmatrix}}_{\text{A}} x + \underbrace{\begin{bmatrix}0\\ M_c^{-1}D_c\end{bmatrix}}_{\text{B}} u$$

$$y = \underbrace{\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\end{bmatrix}}_{\text{C}} x,$$

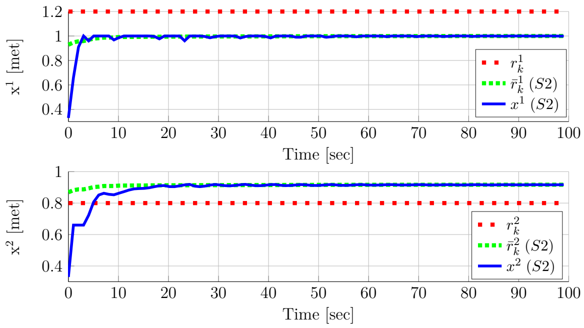
, where $M_c = \mu I$, $L_c = \delta I$,

$$K_c = k\begin{bmatrix}1 & -1 & \cdots & 0 & 0\\ -1 & -2 & \ddots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \ddots & -2 & -1\\ 0 & 0 & \cdots & -1 & 1\end{bmatrix}, \text{ and } D_c = \begin{bmatrix}1 & 0\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\\ 0 & -1\end{bmatrix}$$

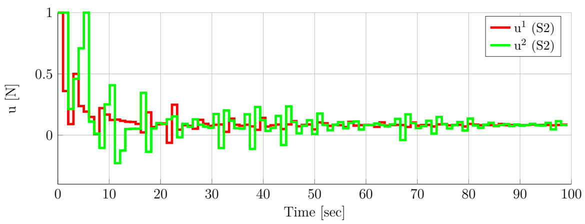
The 'CSE1' variant (for 10 springs), with parameters $n = 20$, $\mu = 4$, $\delta = 1$, $k = 1$, is discretized with the forward Euler method for a sampling time of 1 sec. The state and input constraints are:

$$\mathcal{X} = \{x \in \mathbb{R}^{20} : \|x\|_\infty \le 1\}, \quad \mathcal{U} = \{u \in \mathbb{R}^2 : \|u\|_\infty \le 1\}.$$

For the MPC problem (4), the penalty matrices are $Q = I_{20}$, $R = I_2$, $W = 10^4$, and the terminal ingredients P, K are computed accordingly. The prediction horizon is taken as $N_{\text{pred}} = 10$ and $\epsilon = 10^{-5}$. The initial state is taken as $x_0 = 0.33 \cdot [1,\ 1,\ \ldots,\ 1]^\top$. The reference signal is piecewise constant at $r_k = [1.2;\ 0.8]$.



(a) first and second output w.r.t. ideal and feasible reference



(b) first and second input forces

Fig. 2: Simulation results the coupled spring system

Figures 2a illustrates the ideal reference (loosely dashed, red), the feasible reference (dotted green) and actual output (solid blue). The first pseudo reference values stabilize around 1m, instead of 1.2m, because of the state constraints, and the pseudo reference for the second state goes to 0.9m, instead of 0.8m, because it is the closest feasible point. In both cases, the state of the system follows the pseudo reference closely. Fig 2b illustrates the two inputs, which respect the input constraints. The feasible reference, outputs and inputs are plotted for the scenario S2) since S1) fails to run and the remaining scenarios behave similarly. The computation times for the three scenarios are presented in II.

Scenario*	Computation Times [msec]		
	min	mean	max
S2)	153	210	316
S3)	17.6	30.3	46.8
S4)	26	36.4	60.6

TABLE II: Computation times for CSE1 system

*We stopped the application for the vertex / half-space scenario S1) after 24 hours because it was stuck in the pre-processing steps of the optimization problem. The other scenarios behave similarly (values around the same order of magnitude) and show promising behavior (fast computation time) for even larger problem sizes.

V. Conclusions

We have applied and adapted the linear encodings from [1] to the feasible-reference tracking MPC of [2] to reduce the computational cost of checking the various set inclusions employed in the scheme (by avoiding to explicitly use the vertex representation of the terminal set). We used both polytopic and zonotopic formulations and for the later case we proposed a zonotopic approximation of the MPI set. We observed clear computation time reductions. We plan to extend the current work to provide recursive feasibility guarantees.

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