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# Nash-equilibrium Seeking Algorithm for Power Allocation Games on Networks of International Relations

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Abstract— In the field of international security, understanding the strategic interactions between countries within a networked context is crucial. Our previous research has introduced a “games-on-signed graphs [10]” framework to analyze these interactions. While the framework is intended to be basic and general, there is much left to be explored, particularly in capturing the complexity of strategic scenarios in international relations. Our paper aims to fill this gap in two key ways. First, we modify the existing preference axioms to allow for a more nuanced understanding of how countries pursue self-survival, defense of allies, and offense toward adversaries. Second, we introduce a novel algorithm that proves the existence of a pure strategy Nash equilibrium for these revised games. To validate our model, we employ historical data from the year 1940 as the game input and predict countries’ survivability. Our contributions thus extend the real-world applicability of the original framework, offering a more comprehensive view of strategic interactions in a networked security environment.

Index Terms— power allocation, resource allocation, security, network games, international relations, Nash equilibrium

## I. Introduction

In recent years, the study of “international systems” within the realm of systems and control has gained traction, although research in this area is still nascent. An international system is a complex network of interactions and relationships between various countries, organizations, and other actors on the global stage. Such systems are dynamic, with inputs like national power and political ideologies leading to outputs like shifts in cooperation, power dynamics, and conflict resolution. Notably, these outputs are neither fixed nor deterministic but are subject to change over time.

Pioneering work in the development of international systems theory has been contributed by scholars like Karl Deutsch, whose seminal writings [1]–[3] advocate for formal modeling in international relations [4]. Deutsch posits that human and social communication, despite being more complex than machine communication, are subject to the same fundamental principles. For example, Karl Deutsch and David Singer assessed the stability of multipolar international systems, drawing significant conclusions on the relationship between great powers [5]. Similarly, Richard Rosecrance discussed how states’ characteristics and the global environment have changed inter-state relations [6], thereby introducing new technical challenges to systems science.

To advance research in international systems, we have developed a “games-on-signed graphs” framework [7]–[15]. The power allocation games in this framework aim to elucidate how countries strategically allocate resources – primarily national power – to maximize favorable outcomes in an increasingly complex international environment. Nowadays, given the increasing demand for resources and the growing complexity of international relations, understanding the strategic resource allocation has become essential for policymakers, academics, and practitioners alike. The instruments of national power must be applied in an environment characterized by mutual vulnerabilities, demanding careful calculation, prudence, and strategic restraint. This is in line with the network games literature (e.g., [16]–[29]), where network resource allocation games have been extensively studied. These studies cover diverse fields from transportation to wireless communication (e.g., [30]–[33]).

In the power allocation games, countries strategically deploy their national powers to support their allies and oppose their adversaries in order to survive and succeed in the international environment. These games can be regarded as a system, whose input or “game parameters” – including countries’ national power, is transformed into the output, the achievement of their objectives in the international environment, such as survival and success. The transformation process would be shaped by the structure of the system itself, as well as the actions and strategies of other states.

To better capture the complexity of strategic scenarios in international relations, this paper further investigates countries’ preference structures in the power allocation games. It seeks to refine the framework by making two modifications to the preference axioms. First, the power allocation games in [10] assume countries strictly prefer states in which they survive over states where they do not, regardless of their allies’ or adversaries’ situations. Consequently, they always prioritize self-survival, and the utility functions satisfying this preference axiom have a “jump discontinuity [12].” The first modification

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relaxes this assumption – countries still prioritize self-survival but only weakly prefer surviving states to non-surviving ones. Thus, the corresponding utility functions are not necessarily discontinuous. The second modification is motivated from influence maximization. In the new games, beyond self-survival, countries optimize the number of survived allies and the number of vulnerable adversaries. This modification renders the countries' behavior more strategic. For instance, in the previous model [10], each country will not protect the interest of one ally by jeopardizing another ally's interest. On the contrary, in our new model, such strategic choice is feasible if it increases the number of survived allies, which is more consistent with many real-world scenarios.

We propose a novel algorithm to prove the existence of pure strategy Nash equilibrium for this new game and simulate predictions by using the data of relevant countries in the year of 1940 as game input. By taking these steps, we aim to expand the real-world applicability of the power allocation games in the original framework and provide a more comprehensive picture of power deployment between countries in international security.

To provide a comprehensive examination of these issues, the remainder of this paper is organized as follows: Section II presents the definitions and the setup of the game, introducing the specific modifications we propose and elaborating on the new preference axioms. Section III presents a novel algorithm for finding a pure strategy Nash equilibrium in the game and Section IV validates it through simulations using historical data from the year of 1940. Finally, Section V concludes the paper, summarizing our contributions and suggesting avenues for future research.

## II. Definitions and Model Setup

Denote by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. Let $\mathbb{1}_{\{\cdot\}}$ be the indicator function, i.e., $\mathbb{1}_{\{C\}} = 1$ if the condition $C$ holds, and $\mathbb{1}_{\{C\}} = 0$ otherwise. In this section, we formalize our model of international relations as the following networked power-allocation game.

International relations as a signed graph: Consider a world of $n$ countries, indexed as $i \in \{1, \dots, n\}$, and assume that their mutual relations are fixed. For any country $i$, denote the set of its allies by $\mathcal{A}_i$ and the set of its enemies by $\mathcal{E}_i$. We assume that all the international relations are symmetric. Namely, $j \in \mathcal{A}_i$ iff $i \in F_j$, and $j \in \mathcal{E}_i$ iff $i \in \mathcal{E}_j$. For convenience, let $i \in \mathcal{A}_j$ for any $i \in \{1, \dots, n\}$. These $n$ countries and the sets $\mathcal{E}_1, \dots, \mathcal{E}_n$, $\mathcal{A}_1, \dots, \mathcal{A}_n$ induce an undirected, unweighted, and signed graph, where the nodes represent the countries and the positive (negative resp.) edges represent the friendly (antagonistic resp.) relations. Note that we do not assume a complete graph. That is, $\mathcal{E}_i \cup \mathcal{A}_i = \{1, \dots, n\}$ does not necessarily hold for any $i$.Noting that relations between countries are generally two-way and consistent, it is reasonable to assume that relations between countries are the same.

Power allocation strategies: For each country $i$, we define its power as a positive value $p_i$. Country $i$ can arbitrarily divide its power and spend them on either attacking its enemies or supporting its allies. For any country $j \in \mathcal{E}_i$ (or $j \in \mathcal{A}_i$ respectively), denote by $x_{ij}$ the amount of country $i$'s power spent on attacking $j$ (supporting $j$ respectively). The vector $(x_{i1}, \dots, x_{in})$ is considered as the strategy of country $i$. Denote by $X = (x_{ij})_{n \times n}$ the strategies of all countries and call it a strategy matrix. By definition,

$$X \in \Omega = \Big\{ X \in \mathbb{R}^{n \times n}_{\geq 0} \,\Big|\, \text{For any } i,\ x_{ij} = 0 \text{ if } j \notin \mathcal{E}_i \cup \mathcal{A}_i, \text{ and } \sum_{j \in \mathcal{E}_i \cup \mathcal{A}_i} x_{ij} = p_i \Big\}.$$

States of countries: Given any strategy matrix $X \in \Omega$, the state of each country $i$, denoted by $s_i(X)$, is either safe, or precarious, or unsafe, following the rules below:

$$s_i(X) = \begin{cases} \text{safe}, & \text{if } \sum_{j \in \mathcal{E}_i} x_{ij} + \sum_{j \in \mathcal{A}_i} x_{ji} > \sum_{j \in \mathcal{E}_i} x_{ji}, \\ \text{precarious}, & \text{if } \sum_{j \in \mathcal{E}_i} x_{ij} + \sum_{j \in \mathcal{A}_i} x_{ji} = \sum_{j \in \mathcal{E}_i} x_{ji}, \\ \text{unsafe}, & \text{if } \sum_{j \in \mathcal{E}_i} x_{ij} + \sum_{j \in \mathcal{A}_i} x_{ji} < \sum_{j \in \mathcal{E}_i} x_{ji}. \end{cases}$$

That is, the state of a country depends on the power it spends on attacking its enemies, plus the support from its allies, and minus the attack from its enemies.

Preference axioms: To make our model more general and inclusive, we do not specify the countries' utility functions, but instead assume they obey the following preference axiom.

Definition 1 (Preference): Given two strategy matrices $X$ and $Y$, country $i \in \{1, \dots, n\}$ prefers $X$ to $Y$, denoted by $X \geq_i Y$, if either of the following two conditions hold:

(i) $s_i(X) \in \{\text{safe}, \text{precarious}\}$ and $s_i(Y) = \text{unsafe}$;
(ii) Either $s_i(X), s_i(Y) \in \{\text{safe}, \text{precarious}\}$ or $s_i(X) = s_i(Y) = \text{unsafe}$. Moreover,

$$\sum_{j \in \mathcal{E}_i} \mathbb{1}_{\{s_j(X) \neq \text{safe}\}} + \sum_{l \in \mathcal{A}_i} \mathbb{1}_{\{s_l(X) \neq \text{unsafe}\}} \geq \sum_{j \in \mathcal{E}_i} \mathbb{1}_{\{s_j(Y) \neq \text{safe}\}} + \sum_{l \in \mathcal{A}_i} \mathbb{1}_{\{s_l(Y) \neq \text{unsafe}\}}$$

There exist various utility functions obeying the above preference axioms, e.g., the utility $u_i(X)$ of $i$ defined as:

$$\begin{aligned} u_i(X) = & n \mathbb{1}_{\{s_i(X) = \text{safe or precarious}\}} \\ & + \sum_{j \in \mathcal{A}_i} \mathbb{1}_{\{s_j(X) = \text{safe or precarious}\}} \\ & + \sum_{j \in \mathcal{E}_i} \mathbb{1}_{\{s_j(X) = \text{unsafe or precarious}\}}. \end{aligned}$$

Nash equilibrium: With Definition 1, a strategy $X^*$ is called a pure strategy Nash equilibrium if, for any $i$, $X^* \geq_i X$ for any $X \in \Omega$ satisfying that the $j$-th row of $X$ is equal to the $j$-th row of $X^*$ for any $j \neq i$.

## III. Existence and Computation of Nash Equilibrium

In this section, we establish the main result of this paper. That is, any game satisfying the preference axioms given by Definition 1 admits at least one Nash equilibrium. We establish this result via the following steps:

(i) Step 1: we first construct a special subset of strategy matrices, referred to as no-support-no-unsafe (NSNU) strategy matrices.
(ii) Step 2: we show that the set of NSNU strategy matrices is invariant under a certain type of strategy adjustments, which will be specified later.
(iii) Step 3: by manipulating the countries' strategy adjustments, we show that any strategy matrix in $\Omega$ can be changed to an NSNU strategy matrix and then reach a Nash equilibrium via an iterative process.

As Step 1, we define the no-support-no-unsafe (NSNU) strategy matrices as follows.

Definition 2 (NSNU strategy matrix): Given the $n$ countries' powers $p_1, \dots, p_n$, a strategy matrix $X \in \Omega$ is called a no-support-no-unsafe (NSNU) strategy matrix, if it satisfies that i) $X = X^\top$; 2) $x_{ij} = 0$ for any $i$, $j$ such that $j \notin \mathcal{E}_i \cup \{i\}$.

The following facts are straightforward consequences of the definition of countries' states and the preference axioms.

Fact 1: Under any NSNU strategy matrix $X$, there is no support between allies, i.e., $x_{ij} = 0$ for any $i$ and any $j \in \mathcal{A}_i \setminus \{i\}$. Moreover, no country is at the unsafe state. A country $i$ is safe if and only if $x_{ii} > 0$, and is precarious if and only if $x_{ii} = 0$.

Fact 2: For any NSNU strategy matrices $X$ and $Y$, for any $i \in \{1, \dots, n\}$, $X \geq_i Y$ if and only if

$$\begin{aligned}\sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{x_{jj}=0\}} &= \sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{s_j(X)=\text{precarious}\}} \\ &\geq \sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{s_j(Y)=\text{precarious}\}} = \sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{y_{jj}=0\}}.\end{aligned}$$

As Step 2, we characterize the best response of a country $i$ under an NSNU strategy matrix.

Definition 3 (Preferable adjustment): Given any NSNU strategy matrix $X \in \Omega$ and any country $i$, define $Y$ as the preferable adjustment of $X$ for $i$, if $Y$ is constructed from $X$ as follows.

(i) If $p_i \geq \sum_{j\in\mathcal{E}_i}(x_{jj} + x_{ji})$, let the entries of $Y$ be equal to the corresponding entries of $X$ except for the following changes:
  a) Let $y_{ii} = p_i - \sum_{j\in\mathcal{E}_i}(x_{jj} + x_{ji})$;
  b) For any $j \in \mathcal{E}_i$, let $y_{ij} = y_{ji} = x_{jj} + x_{ji}$ and $y_{jj} = 0$.
(ii) If $p_i < \min_{j\in\mathcal{E}_i}(x_{jj} + x_{ji})$, let the entries of $Y$ be equal to the corresponding entries of $X$ except for the following changes: Pick a $j_1$ from the set $\text{argmin}_{j\in\mathcal{E}_i}(x_{jj} + x_{ji})$ and
  a) Let $y_{ii} = 0$, $y_{ij_1} = y_{j_1 i} = p_i$, and $y_{j_1 j_1} = x_{j_1 j_1} + x_{j_1 i} - p_i$;
  b) For any $j \in \mathcal{E}_i \setminus \{j_1\}$, let $y_{ij} = y_{ji} = 0$ and $y_{jj} = x_{jj} + x_{ji}$.
(iii) If $\min_{j\in\mathcal{E}_i}(x_{jj} + x_{ji}) \leq p_i < \sum_{j\in\mathcal{E}_i}(x_{jj} + x_{ji})$, let $k = |\mathcal{E}_i|$, i.e., the number of country $i$'s enemies, and index the enemies of $i$ as $j_1, \dots, j_k$, with $x_{j_1 j_1} + x_{j_1 i} \leq x_{j_2 j_2} + x_{j_2 i} \leq \dots \leq x_{j_k j_k} + x_{j_k i}$. In this scenario, there exists $m \in \{1, \dots, k-1\}$ such that

$$\sum_{s=1}^{m}(x_{j_s j_s} + x_{j_s i}) \leq p_i < \sum_{s=1}^{m+1}(x_{j_s j_s} + x_{j_s i}).$$

Let the entries of $Y$ be equal to the corresponding entries of $X$ except for the following changes: Let $y_{ii} = 0$.
  a) For any $s \in \{1, \dots, m\}$, let $y_{ij_s} = y_{j_s i} = x_{j_s j_s} + x_{j_s i}$ and $y_{j_s j_s} = 0$;
  b) Let $y_{ij_{m+1}} = p_i - \sum_{s=1}^{m}(x_{j_s j_s} + x_{j_s i}) = y_{j_{m+1} i}$ and $y_{j_{m+1} j_{m+1}} = x_{j_{m+1} j_{m+1}} + x_{j_{m+1} i} - y_{j_{m+1} i}$;
  c) For any $s \in \{m+2, \dots, k\}$, let $y_{ij_s} = y_{j_s i} = 0$ and $y_{j_s j_s} = x_{j_s j_s} + x_{j_s i}$.

Despite its complicated form, the intuition behind preferable adjustment is quite clear: By re-allocating power, country $i$ aims to have as many precarious enemies as possible. In addition, each of $i$'s enemies also re-allocate its power spent on $i$ and itself, in order to maintain the symmetry of $Y$. As a result, $Y$ is still a NSNU strategy matrix and the allocation of $i$'s power in $Y$ constitutes $i$'s best response to $X$. The following lemma confirms this argument.

Lemma 1 (Properties of preferable adjustment): Given any NSNU strategy matrix $X \in \Omega$ and its preferable adjustment for $i$, denoted by $Y$,

(i) $Y$ is also an NSNU strategy matrix;
(ii) for any $\tilde{X} \in \Omega$ such that $(\tilde{x}_{j1}, \dots, \tilde{x}_{jn}) = (x_{j1}, \dots, x_{jn})$ for any $j \neq i$, country $i$ prefers $Y$ to $\tilde{X}$, i.e., $Y \geq_i \tilde{X}$.
(iii) $\sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{y_{jj}=0\}} \geq \sum_{j\in\mathcal{E}_i} \mathbb{1}_{\{x_{jj}=0\}}$.

*Proof:* We first prove statement (i). By carefully examining Definition 3, one could check that, in any scenario the preferable adjustment guarantees that $Y$ is symmetric and all the diagonals are non-negative. In addition, according to Definition 3, for any $l \neq i$ and any $j \notin \mathcal{E}_l \cup \{l\}$, $x_{lj}$ is not adjusted, i.e., $y_{lj} = x_{lj} = 0$. Moreover, $x_{ij}$ is not adjusted for any $j \notin \mathcal{E}_i \cup \{i\}$, i.e., $y_{ij} = x_{ij} = 0$. That is, for any $l$, $s$, $y_{ls} = 0$ as long as $s \notin \mathcal{E}_l \cup \{l\}$. Therefore, according to Definition 2, $Y$ is an NSNU strategy matrix.

We now prove statement (ii). Since $Y$ is an NSNU strategy matrix, country $i$ and all its allies are not unsafe. Therefore, according to Definition 1, $i$ does not prefer $Y$ to $\tilde{X}$ if and only if $i$ and all its allies are not unsafe under $\tilde{X}$ and $i$ has strictly more precarious or unsafe enemies under $\tilde{X}$ than under $Y$. We show that this is impossible by discussing the three scenarios in Definition 3 one by one.

Suppose $p_i \geq \sum_{j \in E_i}(x_{jj} + x_{ji})$. According to Definition 3, in this scenario, $y_{jj} = 0$ for any $j \in \mathcal{E}_i$, which together with Fact 1 implies that all the enemies of $i$ are precarious. Therefore, the number of $i$'s unsafe or precarious enemies under $Y$ could not be smaller than that under $\tilde{X}$. That is, $Y \geq_i \tilde{X}$.

Suppose $p_i < \min_{j \in \mathcal{E}_i}(x_{jj} + x_{ji})$. Then $p_i < x_{jj} + x_{ji}$ for any $j \in \mathcal{E}_i$. That is, given $X$, no matter how $i$ reallocates its power on attacking its enemies, none of its enemies will be precarious or unsafe. Therefore, in this scenario, the numbers of precarious or unsafe enemies of $i$ under both $Y$ and $\tilde{X}$ are 0, which leads to $Y \geq_i \tilde{X}$.

Suppose $\min_{j \in \mathcal{E}_i}(x_{jj} + x_{ji}) \leq p_i < \sum_{j \in \mathcal{E}_i}(x_{jj} + x_{ji})$. Rank the values $x_{jj} + x_{ji}$ with $j \in \mathcal{E}_i$ in ascending order, i.e, $x_{j_1 j_1} + x_{j_1 i} \leq \cdots \leq x_{j_k j_k} + x_{j_k i}$. Then there exists $m \in \{1, \ldots, k-1\}$ such that

$$\sum_{s=1}^{m}(x_{j_s j_s} + x_{j_s i}) \leq p_i < \sum_{s=1}^{m+1}(x_{j_s j_s} + x_{j_s i}).$$

According to Definition 3, for any $1 \leq s \leq m$,

$$y_{ij_s} = x_{j_s j_s} + x_{j_s i} \text{ and } y_{j_s j_s} = x_{j_s j_s} + x_{j_s i} - y_{ij_s} = 0,$$

and for any $m+1 \leq s \leq k$,

$$0 \leq y_{ij_s} < x_{j_s j_s} + x_{j_s i} \text{ and } y_{j_s j_s} = x_{j_s j_s} + x_{j_s i} - y_{ij_s} > 0.$$

Therefore, according to Fact 1, under $Y$, the number of $i$'s precarious enemies is $m$. Suppose that, under $\tilde{X}$, $i$ has $q$ precarious enemies, denoted by $d_1, \ldots, d_q$. Then we have

$$\sum_{s=1}^{q} \tilde{x}_{id_s} \geq \sum_{s=1}^{q}(\tilde{x}_{d_s d_s} + \tilde{x}_{d_s i}) = \sum_{s=1}^{q}(x_{d_s d_s} + x_{d_s i}).$$

Moreover, since $x_{j_1 j_1} + x_{j_1 i}, \ldots, x_{j_k j_k} + x_{j_k i}$ are ranked in ascending order, we have

$$\begin{aligned} \sum_{s=1}^{q}(x_{j_s j_s} + x_{j_s i}) &\leq \sum_{s=1}^{q}(x_{d_s d_s} + x_{d_s i}) \\ &\leq \sum_{s=1}^{q} \tilde{x}_{id_s} \leq p_i < \sum_{s=1}^{m+1}(\tilde{x}_{j_s j_s} + \tilde{x}_{j_s i}). \end{aligned}$$

Therefore, $q \leq m$. That is, $i$ has more precarious enemies under $Y$ than under $\tilde{X}$, which in turn leads to $Y \geq_i \tilde{X}$.

Now we proceed to prove statement (iii). According to statement (ii), $i$ under $Y$ has more precarious enemies than under $X$. In addition, according to Fact 1, under any NSNU strategy matrix, a country is precarious if and only if the power allocated to itself is 0. Therefore,

$$\sum_{j \in \mathcal{E}_i} \mathbb{1}_{\{y_{jj}=0\}} \geq \sum_{j \in \mathcal{E}_i} \mathbb{1}_{\{x_{jj}=0\}}.$$

This concludes the proof. ■

Given the definitions of NSNU strategy matrix and preferable adjustment, we are now ready to present and prove the main result in this paper.

Theorem 2 (Existence of Nash Equilibrium): Given $n$ countries with powers $p_1, \ldots, p_n$ respectively, and given their ally and enemy sets $\mathcal{A}_1, \mathcal{E}_1, \ldots, \mathcal{A}_n, \mathcal{E}_n$, for any power-allocation game as described in Section II, there exists at least one pure strategy Nash equilibrium, at which no country is unsafe and there is no support between any pair of allies.

Proof: The proof is omitted due to the space limit. ■

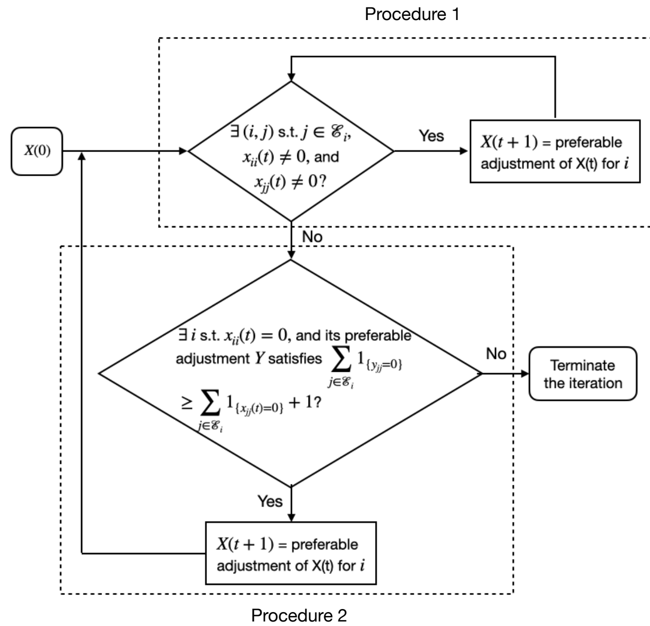


Fig. 1. The diagram of Procedure 1 and Procedure 2 defined in the proof of Theorem 2.

Theorem 2 indicates that, for any power-allocation game characterized in Section II, there exists a special Nash equilibrium, at which every country manages to keep themselves from being unsafe without any support from its allies.

## IV. Simulations

Though we have established pure strategy Nash equilibrium existence, there are many possible Nash equilibria for the power allocation game. The simulation algorithm as follows is used to generate the set of pure strategy Nash equilibria for the game. The inputs required for the algorithm are the parameters of the game, including the power of each country, the relationships between them, and their preference axioms. The algorithm assumes a simple utility function as below:

$$u_i(X) = \begin{cases} 0 & \text{if } s_i(X) = \text{unsafe} \\ \sum_{j \in \mathcal{E}_i \cup \mathcal{A}_i} u_{ij}(X) & \text{if } s_i(X) \in \{\text{safe}, \text{precarious}\} \end{cases}$$

where $u_{ij}(X) = 0$ if $j \in \mathcal{A}_i$ and $s_j(X) = \text{unsafe}$, or if $j \in \mathcal{E}_i$ and $s_j(X) = \text{safe}$. $u_{ij}(X) = 1$ if $j \in \mathcal{A}_i$ and $s_j(X) \in \{\text{safe}, \text{precarious}\}$, or if $j \in \mathcal{E}_i$ and $s_j(X) \in \{\text{unsafe}, \text{precarious}\}$. Note that this utility function satisfies the preference axioms in Definition 1.

The process involves each country sequentially updating its power allocation strategy in response to the strategies of the other countries, until a pure strategy Nash equilibrium is reached or a predetermined stopping criterion, such as 10000 rounds of iteration, is met. The output of the algorithm is an approximation of the set of pure strategy Nash equilibria.

- At time 0, initialize the pure strategy Nash equilibrium set $\mathcal{X}$ to be empty and a strategy matrix $X(0)$.
- At time $t$, each country $i$ updates its power allocation strategy $x_i(t-1)$ to optimize its total utility $u_i(X)$ that satisfies the preference axioms, by assuming the strategies of all the others $x_j(t-1), j \neq i$ to be fixed and the total power constraint to be time constant,

$$p_i = \sum_{j \in \mathbf{n}} x_{ij}(t).$$

- Stop updating if reaching a pure strategy Nash equilibrium $X^*$ or the maximum number of rounds $t = T$.
- Update the equilibrium set $\mathcal{X} \cup \{X^*\}$ and go back to initialize a different strategy matrix.
- At the end of the rounds, compute the number of pure strategy Nash equilibria for each distinctive prediction as well as the predictions' relative percentages.
- Obtain countries' likelihoods of survival based on the step above. As in [8], a country is believed to survive if it is safe or precarious.

In this example, we use data from the Correlates of War project [35] to denote countries' conflictual and cooperative relationships in the year of 1940. We also calculate countries' national power with their CINC (composite index for national capability) index. The results regarding countries' likelihoods of survival are in Table 1. After approximating the survival likelihoods from the power-allocation game, we compared it against the real-world situation. We analyzed each country's historical state from 1940 to 1941. A country that was not at war in its territory was generally safe, while a country that was in war had a chance of being unsafe. We recorded the state of each country as a result of the power allocations. By comparing the power allocation game's calculated results with the analysis of the historical situation, we obtained an accuracy rate of 0.7407 for the model. While international situation is complex, this quantitative analysis method makes it possible to achieve better accuracy through model and algorithm improvement, such as by varying the utility functions and increasing the number of iteration rounds.

Fig. 2. 1940: red (adversaries), green (alliances)

TABLE I

Likelihood of Survival By Country in 1940

| Country | Survival Likelihood | Country | Survival Likelihood |
|---|---|---|---|
| Turkey | 1.0 | Norway | 0.0 |
| USA | 1.0 | Ecuador | 1.0 |
| Australia | 1.0 | Iraq | 1.0 |
| Denmark | 0.99490 | Peru | 1.0 |
| Egypt | 1.0 | Brazil | 1.0 |
| Canada | 1.0 | Paraguay | 1.0 |
| New Zealand | 0.0 | Chile | 1.0 |
| Saudi Arabia | 1.0 | Thailand | 0.0 |
| Argentina | 1.0 | Uruguay | 1.0 |
| Haiti | 1.0 | DR | 1.0 |
| South Africa | 0.0 | Hungary | 0.96837 |
| Afghanistan | 1.0 | Italy | 0.0 |
| China | 1.0 | Mexico | 1.0 |
| UK | 0.0 | Ireland | 0.00306 |
| Netherlands | 1.0 | Belgium | 0.00816 |
| Luxembourg | 0.0 | Guatemala | 1.0 |
| Honduras | 1.0 | France | 1.0 |
| El Salvador | 1.0 | Greece | 1.0 |
| Panama | 1.0 | Nicaragua | 1.0 |
| Switzerland | 0.0 | Costa Rica | 1.0 |
| Bulgaria | 0.0 | Japan | 0.0 |
| Colombia | 1.0 | Spain | 0.00510 |
| Romania | 0.05306 | Portugal | 1.0 |
| Russia | 1.0 | Estonia | 0.0 |
| Latvia | 0.0 | Lithuania | 0.0 |
| Iran | 1.0 | Finland | 0.0 |
| Sweden | 0.00204 | Germany | 1.0 |

## V. Conclusion and Further Discussion

Similar to the approach in [10], this paper establishes the existence of a pure strategy Nash equilibrium for a new type of power allocation games. These games are adapted from the original models presented in [10]. Instead of relying on fixed-point theorems, we employ an algorithm that directly constructs such an equilibrium. Additionally, we simulate predictions for countries' survivability using real-world data.

There are several promising avenues for future research. Beyond generalizing the results of this paper to different contexts and static game scenarios, it would be valuable to explore the equilibrium properties and predictions in dynamic power allocation games. Another meaningful direction would be to examine a dynamical system that models countries' power evolution as they optimize a utility sum over time, making necessary trade-offs between "guns" and "butter." These topics will be addressed in a more expanded version of this paper.

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