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Robust quantum coding for thermal noise via dissipative dynamics

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***Abstract*— This paper proposes an improved quantum coding method based on Markovian dissipative dynamics that is robust against thermal noise. A method for correcting information errors is indispensable for the practical use of quantum information devices, and stabilizer codes for quantum systems have been devised. One of the quantum coding methods, namely, quantum coding based on Markovian dissipative dynamics, has a favorable feature that it does not require strict time control, however it is known that these methods are susceptible to thermal noise. In this paper, we propose a quantum coding method that incorporates a new mechanism to correct the disturbance of the quantum state caused by thermal noise into the quantum coding method using dissipative dynamics. We also analyze the stability of the proposed quantum dynamics in the quantum state corresponding to the target code word. Numerical experiments confirm the effectiveness of the proposed method.**

***Index Terms*— Quantum coding, dissipative dynamics and thermal noise**

I. Introduction

In recent years, with the development of nanotechnology, new technologies based on quantum information theory [3], [9], [12], such as quantum cryptography, quantum communication, and quantum computers, have become feasible and have attracted considerable attention. However, it remains difficult to realize these technologies in the same manner as classical technologies. One of the main reasons for this is the vulnerability of quantum systems to external noise. It is essential to address the problem of noise to realize quantum information technology, and various studies have been conducted.

One strategy for preventing the effect of noise is to use error-correcting codes. Error-correcting codes are schemes that can correct original information by adding redundant bits, even if the error is caused by noise in the communication channel. However, compared with the case of classical systems, careful consideration is required to constitute coding scheme for quantum information. This is because quantum systems have the property that their states change depending on the observation, and the original information may be corrupted by the observation of incorrect information. A solution to this problem is the stabilizer code [5], [7]. Stabilizer codes confine information in a space called the stabilizer state and allow the measurement of noise without destroying the information, and much research has been conducted on coding theory and its correction methods.

Many of these studies were based on the argument that a unitary transformation of quantum systems is possible. However, the unitary transformation of physical qubits requires strict time control and accurate manipulation, which is challenging to realize. Ticozzi et al. [10] proposed another method of encoding called continuous-time dissipative encoding (CDE), which does not require strict time control or manipulation by constituting dissipative systems and evolving the quantum system over time such that it asymptotically converges to objective code words. A discrete-time version of this concept was reported in [2]. This method was further extended to decoding by dissipative systems, called continuous-time dissipative decoding (CDD) [8], [11]. However, [8] pointed out that CDE and CDD are sensitive to thermal noise, and under its effect, the original information gradually disappears during the time evolution, and the objective code words are not derived.

With this background, this paper proposes a method to suppress the effect of thermal noise by extending the CDE. Specifically, when the original information to be encoded is known, by adding a new term to the original Lindblad equation of CDE that has the effect of correcting the disturbed quantum state to the original encoding dynamics, the quantum system can evolve in time to a neighborhood of the desired encoded state while suppressing the effects of thermal noise.

The remainder of this paper is organized as follows: Section II summarizes the fundamentals of quantum information theory. Section III summarizes previous studies on the coding of stabilizer codes using dissipative dynamics, whereas Section IV analyzes the methods of previous studies and proposes a new coding method that can suppress the effect of thermal noise. Section V presents the proposed method's numerical simulations, and Section VI concludes the paper.

In this paper, we omit the proofs of all the theorems from the page limitations.

Notations

$\mathbb{Z}_2$: the set of integers modulo 2, i: the imaginary unit, $\mathbb{C}$: the set of complex numbers, $\mathrm{tr}(X)$: the trace of matrix X, X^*: the Hermitian transpose of matrix X, $A \otimes B$: the tensor product of matrices A and B, $[A, B] = AB - BA$: the commutator of matrices A and B, $\{A, B\} = AB + BA$: the anticommutator of matrices A and B, $a \,\mathrm{xor}\, b$: the exclusive OR of $a, b \in \mathbb{Z}_2$.

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II. Quantum information

A. Fundamentals of quantum information

In this section, we revisit the basics of quantum information in [7].

Elements of the complex Hilbert space $\mathcal{H} = \mathbb{C}^n$ are denoted by $|\phi\rangle \in \mathcal{H}$ and the elements of the dual space are denoted as $\langle\psi| \in \mathcal{H}^*$, where the former is called a ket and the latter is a bra. The ket and bra satisfy $\langle\phi| = |\phi\rangle^*$, where $*$ denotes complex conjugate transpose. Notation $\langle\psi|\phi\rangle$ denotes the inner product and $\||\phi\rangle\| := \sqrt{\langle\phi|\phi\rangle}$ denotes the norm. In addition, $|\psi\rangle\langle\phi|$ represents a linear operator from $|x\rangle \in \mathcal{H}$ to $|\psi\rangle\langle\phi|x\rangle \in \mathcal{H}$.

The quantum state is denoted by the unit vector $|\phi\rangle$ in the appropriate Hilbert space $\mathcal{H}$. A quantum state multiplied by a phase factor $e^{i\theta}$, $e^{i\theta}|\phi\rangle$, is identified with $|\phi\rangle$.

The information in a quantum state represented by an element in the Hilbert space $\mathcal{H} = \mathbb{C}^2$ is called 1-qubit. Its standard basis is $\{|0\rangle, |1\rangle\}$, where

$$|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, \ |1\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top. \tag{1}$$

Similarly, the information represented by an n-qubit quantum system is described by an element of $\mathcal{H}^{\otimes n}$. The 2^n-standard basis is described by $|i_1 i_2 \cdots i_n\rangle (:= |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle)$, $\forall i_k \in \{0, 1\}$.

If the system takes state $|\phi_i\rangle$ with probability p_i, then the density operator is described as follows:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| \tag{2}$$

The density operator has the following properties.

$$\mathrm{tr}(\rho) = 1, \quad \rho = \rho^* \geq O \tag{3}$$

When a quantum system has a density operator ρ, The expected value $\langle A\rangle$ of the observed physical quantity A is given by

$$\langle A\rangle = \mathrm{tr}(A\rho). \tag{4}$$

When a quantum system is in a quantum state $|\phi\rangle$ with a probability 1, it is referred to as a pure state, whereas the other cases are referred to as mixed states. This can be verified by calculating purity $\mathrm{tr}(\rho^2)$ as follows:

- $\mathrm{tr}(\rho^2) = 1 \Leftrightarrow$ pure state
- $\mathrm{tr}(\rho^2) < 1 \Leftrightarrow$ mixed state

B. Stabilizer code

The following three operators are called Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ Y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

These are Hermitian matrices on 1-qubit quantum system with eigenvalues $+1$ and -1, and satisfy the following:

$$\begin{aligned} &X^2 = Y^2 = Z^2 = I, \ XY = -YX = \mathrm{i}Z \\ &YZ = -ZY = \mathrm{i}X, \ ZX = -XZ = \mathrm{i}Y \end{aligned} \tag{6}$$

Thus, the following set G_1,

$$G_1 := \{\pm 1, \pm \mathrm{i}\} \times \{I, X, Y, Z\}, \tag{7}$$

constitutes a group with product operations, which is called the 1-qubit Pauli group. Similarly

$$G_n := \{\pm 1, \pm \mathrm{i}\} \times \{I, X, Y, Z\}^{\otimes n} \tag{8}$$

is known as the n-qubit Pauli group. The operator of n-qubit Pauli group that operates X on jth qubit is denoted by X_j; that is,

$$X_j := I \otimes \cdots \otimes I \otimes \underbrace{X}_{j\text{th qubit}} \otimes I \otimes \cdots \otimes I. \tag{9}$$

The operators Y_j and Z_j are defined similarly.

A commutative subgroup S of the Pauli group G_n such as

$$\forall s_i, s_j \in S \subset G_n, \quad [s_i, s_j] = 0 \tag{10}$$

is called the stabilizer group.

The elements of the maximum set of independent elements of the stabilizer group S are called the generators of S. When the generators of S are given by g_i ($i = 1, 2, \ldots, r$), S is denoted as $S = \langle g_1, g_2, \ldots, g_r\rangle$. The simultaneous eigenspace V_S of the eigenvalues $+1$ of any generator g_i of S; that is,

$$V_S := \{|\phi\rangle \mid g_i |\phi\rangle = |\phi\rangle, \forall g_i \in S\} \tag{11}$$

is called the stabilizer space of S and its elements $|\phi\rangle \in V_S$ are called stabilizer states. The following property of stabilizer spaces is known [7].

Proposition 2.1 ([7]): Let $S = \langle g_1, g_2, \ldots, g_r\rangle$ be a stabilizer group without $-I$. The dimension of the stabilizer space V_S of S is 2^{n-r}.

The projection of an n-qubit Pauli matrix $A \in G_n$ onto the stabilizer space is denoted by $\frac{1}{2}(I+A)$, and the projection onto the eigenspace with eigenvalue -1 is denoted by $\frac{1}{2}(I - A)$. In particular, the projection onto the stabilizer space of $S = \langle g_1, g_2, \ldots, g_r\rangle$ is given by $\prod_{i=1}^{l} \frac{1}{2}(I + g_i)$.

The stabilizer code consists of the stabilizer group $S = \langle g_1, g_2, \ldots, g_r\rangle \subset G_n$ and logical operators $\{\bar{X}_i, \bar{Z}_i\}_{i=1}^{n-r}$ ($\forall \bar{X}_i, \bar{Z}_i \in G_n$) which satisfy the following equations:

$$\begin{aligned} &\left[\bar{X}_i, g_j\right] = \left[\bar{Z}_i, g_j\right] = O, \forall i, j \\ &\{\bar{X}_i, \bar{Z}_i\} = O, \forall i, \quad \left[\bar{X}_i, \bar{Z}_j\right] = O, \forall i \neq j \\ &\left[\bar{X}_i, \bar{X}_j\right] = \left[\bar{Z}_i, \bar{Z}_j\right] = O, \forall i, j \end{aligned} \tag{12}$$

Then, each code word $\boldsymbol{x} = \langle x_1 x_2 \cdots x_{n-r}\rangle$, which constitutes the standard basis of the logic word represented by $(n-r)$-qubit, is defined as the stabilizer state of $S_{\boldsymbol{x}} := \left\langle g_1, \ldots, g_r, (-1)^{x_1}\bar{Z}_1, \ldots, (-1)^{x_{n-r}}\bar{Z}_{n-r}\right\rangle$ and denote it as $|x_1 x_2 \cdots x_{n-r}\rangle_L$. As $S_{\boldsymbol{x}}$ is known to be a stabilizer group from (12), the dimension of its stabilizer space $V_{S_{\boldsymbol{x}}}$ is 1 from Proposition 2.1. Subsequently, the following equations hold for the logical operator $\{\bar{X}_i, \bar{Z}_i\}_{i=1}^{n-r}$:

$$\bar{X}_i |x_1 \cdots x_{n-r}\rangle_L = |x_1 \cdots (x_i \operatorname{xor} 1) \cdots x_{n-r}\rangle_L \tag{13}$$

$$\bar{Z}_i |x_1 x_2 \cdots x_{n-r}\rangle_L = (-1)^{x_i} |x_1 x_2 \cdots x_{n-r}\rangle_L \tag{14}$$

Then, $\bar{X}_i$ is regarded as a NOT operator and $\bar{Z}_i$ as the operator used to identify ith logical bit.

The stabilizer group $S = \langle g_1, g_2, \ldots, g_r\rangle$ defines an $r \times 2n$-dimensional check matrix R. The ith row of R corresponds to generator g_i, and when g_i is an n-fold tensor product $g_i = A_i^1 \otimes \cdots \otimes A_i^n (A_i^j \in G_1)$, the (i,j) and $(i,j+n)$ components of R are determined as in Table I.

Table I: Component rule in check matrix

A_i^j	$R(i,j)$	$R(i,j+n)$
I	0	0
X	1	0
Y	1	1
Z	0	1

Let Λ be a matrix, such that

$$\Lambda = \begin{bmatrix} O & I_n \\ I_n & O \end{bmatrix}, \tag{15}$$

then, a condition

$$R \Lambda R^\top = O \tag{16}$$

is necessary and sufficient for that all generators of S are independent. Similarly, the independence between the generators of two stabilizer groups S_1 and S_2 is equivalent to

$$R_1 \Lambda R_2^\top = O, \tag{17}$$

where R_1 and R_2 are the corresponding check matrices. Furthermore, the check matrices can be transformed into the following standard form using elementary transformations:

$$S = \left[\begin{array}{ccc|ccc} A_1 & A_2 & I_k & B & O & C \\ O & O & O & D & I_{r-k} & E \end{array}\right], \tag{18}$$

where $A_1 \in \mathbb{Z}_2^{k\times(n-r)}$, $A_2 \in \mathbb{Z}_2^{k\times(r-k)}$, $B \in \mathbb{Z}_2^{k\times(n-r)}$, $C \in \mathbb{Z}_2^{k\times k}$, $D \in \mathbb{Z}_2^{(r-k)\times(n-r)}$, and $E \in \mathbb{Z}_2^{(r-k)\times k}$.

III. Previous research

In this section, we summarize the coding of the stabilizer codes using the Markovian dissipative dynamics proposed by Ticozzi et al. [10].

A. Markovian dissipative dynamics

The target code word space C_E can be decomposed into $C_E \simeq \mathcal{H}_L \otimes \mathcal{H}_F \oplus \mathcal{H}_R$, where $\mathcal{H}_L$ denotes the Hilbert space corresponding to the encoded logical information, $\mathcal{H}_F$ denotes the Hilbert space corresponding to the redundancy term of the coded word, and $\mathcal{H}_R$ denotes the Hilbert space orthogonal to them. Similarly, the Hilbert space $\mathcal{H}_P$ corresponding to the physical system can be decomposed as $\mathcal{H}_P \simeq \mathcal{H}_{L'} \otimes \mathcal{H}_{F'} \oplus \mathcal{H}_{R'}$, where $\mathcal{H}_{L'}$ and $\mathcal{H}_{F'}$ need not be isomorphic to $\mathcal{H}_L$ and $\mathcal{H}_F$, respectively.

The continuous-time Markov dissipative dynamics of a quantum system is represented by the Lindblad equation [1]:

$$\dot{\rho} = -\mathrm{i}\,[H,\rho] + \sum_k \left(L_k \rho L_k^* - \frac{1}{2}\left\{L_k^* L_k, \rho\right\}\right) := \mathcal{L}(\rho), \tag{19}$$

where H is the Hamiltonian of the quantum system and L_k is the appropriate matrix of dissipative terms with the environment.

B. Continuous time dissipative encoding

Given a stabilizer group S, the check matrix is transformed into the standard form (18) and is denoted as R. Next, using R, we determine the generator $g_i\,(i = 1,2,\dots,r)$ of S and the following two check matrices, G_X and G_Z:

$$G_X = \left[\begin{array}{ccc|ccc} I_{n-r} & D^\top & O & O & O & B^\top \end{array}\right] \tag{20}$$

$$G_Z = \left[\begin{array}{ccc|ccc} O & O & O & I_{n-r} & O & A_1^\top \end{array}\right] \tag{21}$$

As G_X and G_Z satisfy $R\Lambda G_X^\top = R\Lambda G_Z^\top = O$ and $G_X \Lambda G_Z^\top = I$, the logical operators $\{\bar{X}_i\}_{i=1}^{n-r}$ and $\{\bar{Z}_i\}_{i=1}^{n-r}$ corresponding to the check matrices G_X and G_Z are derived, respectively. Thus, we can construct the stabilizer code from S.

Next, the correction matrices $\{C_i\}_{i=1}^r$ are constructed as follows: First, we construct the matrix $\bar{R}$ as follows:

$$\bar{R} = \begin{bmatrix} R \\ \hline G_Z \\ \hline G_X \end{bmatrix} \in \mathbb{Z}_2^{(2n-r)\times 2n} \tag{22}$$

Because each row of $\bar{R}$ is linearly independent, there exists a unit row vector $c_i \in \mathbb{Z}_2^{2n}$, $i = 1,2,\dots,r$ that satisfies the following equation [7]:

$$\bar{R}\Lambda c_i^\top = e_i \tag{23}$$

$$e_i := [\; 0 \;\cdots\; 0 \; \underbrace{1}_{i\text{th element}} \; 0 \;\cdots\; 0 \;]^\top \tag{24}$$

Define the correction matrix $C_i \in G_n$ such that c_i is a row vector corresponding to C_i in a check matrix. Subsequently, $\{C_i\}_{i=1}^r$ satisfies the equation

$$\left[C_i, \bar{X}_j\right] = \left[C_i, \bar{Z}_j\right] = O, \ \forall i,j \tag{25}$$

$$\{C_i, g_i\} = O, \ \forall i \tag{26}$$

$$\left[C_i, g_j\right] = O, \ \forall i \neq j. \tag{27}$$

Using $\{C_i\}_{i=1}^r$ and the generators of the stabilizer group $\{g_i\}_{i=1}^r$, we construct a continuous-time dynamics as follows: First, $L_{\boldsymbol{b}}$ is constructed as

$$L_{\boldsymbol{b}} = \prod_{i=1}^r C_i^{(1+(-1)^{1-b_i})/2} \cdot \prod_{i=1}^r \frac{1}{2}(I + (-1)^{b_i} g_i), \tag{28}$$

where, $\boldsymbol{b} = (b_1, b_2, \dots, b_r) \in \mathbb{Z}_2^r$. Consider the Lindblad equation for $H = O$ with $L_{\boldsymbol{b}}$ as the dissipative term:

$$\begin{aligned} \dot{\rho}(t) &= \sum_{\boldsymbol{b}} \left(L_{\boldsymbol{b}} \rho L_{\boldsymbol{b}}^* - \frac{1}{2}\{L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}, \rho\}\right) \\ &= \sum_{\boldsymbol{b}} L_{\boldsymbol{b}} \rho L_{\boldsymbol{b}}^* - \rho =: \Phi(\rho) - \rho \end{aligned} \tag{29}$$

The derivations are presented in Appendix I. This dynamics is called continuous-time dissipative encoder (CDE) and Ticozzi et al. [10] show that ρ in the CDE converges asymptotically to the code word space C_E and that with an appropriate redundancy term in the initial state, the original information can be encoded appropriately by the time evolution of (29).

C. Effect of noise for encoding

The described encoding process CDE is assumed to be noise-free. Shimomura [8] demonstrated that this process is susceptible to thermal noise. The time evolution of a quantum system under the influence of thermal noise is described by a quantum stochastic differential equation, and for CDE, the expected time evolution by using Jaynes–Cummings model [6] is expressed as follows [8], [4],

$$\begin{aligned}\dot{\rho} = &\sum_{\boldsymbol{b}}\Big(L_{\boldsymbol{b}}\rho L_{\boldsymbol{b}}^* - 1/2\big\{L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}, \rho\big\}\Big) \\ &+ \gamma(\bar{N}+1)\sum_{i=1}^{n}\Big(D_i\rho D_i^* - 1/2\big\{D_i^* D_i, \rho\big\}\Big) \\ &+ \gamma\bar{N}\sum_{i=1}^{n}\Big(D_i^*\rho D_i - 1/2\big\{D_i D_i^*, \rho\big\}\Big) \end{aligned} \tag{30}$$

$$\bar{N} := \frac{1}{e^{\kappa}-1},\ D_i := I\otimes\cdots\otimes I\otimes\underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{i\text{th component}}\otimes I\otimes\cdots\otimes I,$$

where the additional terms in the second and third lines in (30) represent the effect of thermal noise. The coefficient γ represents the magnitude of the effect of thermal noise and κ is proportional to the inverse of the absolute temperature. The numerical simulation of (30) shows that the original information is lost over time owing to thermal noise, and the quantum state is moved away from the target state [8].

IV. Reducing noise in dissipative encoding

In this section, we analyze previous studies and then propose a noise suppression method for encoding by dissipative systems.

A. Analysis of previous researches

First, we analyze the space affected by the dissipation term $L_{\boldsymbol{b}}$. For $\boldsymbol{a} = (a_1, a_2, \ldots, a_r)$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_{n-r})$, a group

$$\begin{aligned} S_{\boldsymbol{a},\boldsymbol{x}} := \langle &(-1)^{a_1} g_1, \ldots, (-1)^{a_r} g_r, \\ &(-1)^{x_1}\bar{Z}_1, \ldots, (-1)^{x_{n-r}}\bar{Z}_{n-r}\rangle \end{aligned} \tag{31}$$

represents a commutative subgroup of G_n, therefore, is a stabilizer group. Then, we can derive the following theorem for the corresponding stabilizer space $V_{S_{\boldsymbol{a},\boldsymbol{x}}}$:

Theorem 4.1: The dimension of $V_{S_{\boldsymbol{a},\boldsymbol{x}}}$ is 1. Let $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ denote its standard basis. If $(\boldsymbol{a},\boldsymbol{x}) \neq (\boldsymbol{a}',\boldsymbol{x}')$, then: $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ and $|\boldsymbol{a}',\boldsymbol{x}'\rangle_L$ are orthogonal. For each pair $(\boldsymbol{a},\boldsymbol{x})$, the corresponding $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ is the standard basis for the 2^n-dimensional vector space of n-qubit.

When $\boldsymbol{a} = O$, $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ is the code word corresponding to the information $|\boldsymbol{x}\rangle = |x_1x_2\cdots x_{n-r}\rangle$ of the stabilizer code defined by the stabilizer group $S = \langle g_1, g_2, \ldots, g_r\rangle$ and logical operators $\{\bar{X}_i, \bar{Z}_i\}_{i=1}^{n-r}$.

Next we can derive the following theorem for $L_{\boldsymbol{b}}$ and the standard basis $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ of $V_{\boldsymbol{a},\boldsymbol{x}}$:

Theorem 4.2: The following holds for the operator $L_{\boldsymbol{b}}$ and $|\boldsymbol{a},\boldsymbol{x}\rangle_L \in \mathcal{H} = \mathbb{C}^{2^n}$:

$$L_{\boldsymbol{b}}\,|\boldsymbol{a},\boldsymbol{x}\rangle_L = \begin{cases} |(0,0,\ldots,0),\boldsymbol{x}\rangle_L, & \boldsymbol{b}=\boldsymbol{a} \\ O, & \boldsymbol{b}\neq\boldsymbol{a}\end{cases} \tag{32}$$

From Theorem 4.2, it is known that $L_{\boldsymbol{b}}$ is an operator that selects a vector $|\boldsymbol{b},\boldsymbol{x}\rangle_L$ and transforms it into the corresponding code word $|\boldsymbol{0},\boldsymbol{x}\rangle_L$. In addition, $L_{\boldsymbol{b}}$ does not change $\boldsymbol{x}$ in $|\boldsymbol{b},\boldsymbol{x}\rangle_L$.

Because CDE (29) is a linear differential equation, the time evolution of ρ can be represented as the linear sum of several components. In addition, because ρ is an operator of $\mathcal{H} = \mathbb{C}^{2^n}$, it can be expressed using its standard basis $|\boldsymbol{a},\boldsymbol{x}\rangle_L$ as follows:

$$\rho(t) = \sum_{\boldsymbol{a},\boldsymbol{x},\boldsymbol{a}',\boldsymbol{x}'} k_{\boldsymbol{a},\boldsymbol{x},\boldsymbol{a}',\boldsymbol{x}'}(t)\,|\boldsymbol{a},\boldsymbol{x}\rangle_L\,\langle\boldsymbol{a}',\boldsymbol{x}'|_L\,, \tag{33}$$

where $\boldsymbol{a},\boldsymbol{a}' \in \mathbb{Z}_2^r$, $\boldsymbol{x},\boldsymbol{x}' \in \mathbb{Z}_2^{n-r}$, and $k_{\boldsymbol{a},\boldsymbol{x},\boldsymbol{a}',\boldsymbol{x}'}(t) \in \mathbb{C}$ are the coefficients of each component $|\boldsymbol{a},\boldsymbol{x}\rangle_L\,\langle\boldsymbol{a}',\boldsymbol{x}'|_L$ of $\rho(t)$. Then, with the results of Theorem 4.2, the time evolution of each component in CDE (29) can be represented by

$$\dot{\rho}(t) = \sum_{\boldsymbol{a}=\boldsymbol{a}',\boldsymbol{x},\boldsymbol{x}'} k_{\boldsymbol{a},\boldsymbol{x},\boldsymbol{a}',\boldsymbol{x}'}(t)\,|\boldsymbol{0},\boldsymbol{x}\rangle_L\,\langle\boldsymbol{0},\boldsymbol{x}'|_L - \rho(t). \tag{34}$$

From this, it is known that the components $|\boldsymbol{a},\boldsymbol{x}\rangle_L\,\langle\boldsymbol{a}',\boldsymbol{x}'|_L$ at $\boldsymbol{a} = \boldsymbol{a}'$ are transformed to $|\boldsymbol{0},\boldsymbol{x}\rangle_L\,\langle\boldsymbol{0},\boldsymbol{x}'|_L$ in the time evolution, while the other components are reduced. On the other hand, the CDE (29) does not affect $\boldsymbol{x}$ or $\boldsymbol{x}'$ elements. Therefore, for a CDE with thermal noise (30) when $\boldsymbol{x}$ or $\boldsymbol{x}'$ changes to a different state owing to noise it cannot be corrected, and then the CDE is susceptible to thermal noise.

B. Robust CDE for thermal noise

In this subsection, we propose a new mechanism to render the CDE robust against thermal noise. We consider the case where the initial quantum state of the original information before coding is known and a new dissipative term $M_{\boldsymbol{y}}$ is added to the Lindblad equation of CDE such that the quantum state converges to the target state. Hereafter, we suppose that the initial quantum state of the original information is in a pure state, as follows:

$$\rho_0 = |\psi\rangle\langle\psi|,\ \psi \in \mathcal{H} = \mathbb{C}^{2^{(n-r)}},\ \|\psi\| = 1 \tag{35}$$

First, for simplicity of the following derivation, a consider coordinate transformation of the logical operator $\{\bar{X}_i, \bar{Z}_i\}_{i=1}^{n-r}$ so that $|\boldsymbol{0},\boldsymbol{0}\rangle_L$, the standard basis of the stabilizer group $S_{\boldsymbol{a}=\boldsymbol{0},\boldsymbol{x}=\boldsymbol{0}}$, becomes the code word corresponding to $|\psi\rangle$. For that, let $T_0 \in \mathbb{C}^{2^{(n-r)}\times 2^{(n-r)}}$ be a coordinate transformation matrix that transforms $|0\,0\cdots 0\rangle$ into $|\psi\rangle$ as

$$|\psi\rangle = T_0\,|0\,0\cdots 0\rangle\,. \tag{36}$$

Subsequently, the coordinate transformation matrix $T \in \mathbb{C}^{2^n\times 2^n}$ is derived from the following $\mathcal{T}_0$

$$\mathcal{T}_0 := \text{block-diag}\underbrace{\big[T_0, T_0, \ldots, T_0\big]}_{2^r\text{ blocks}} \in \mathbb{C}^{2^n\times 2^n} \tag{37}$$

by swapping its rows and columns appropriately to satisfy $[T, g_i] = 0$, $\forall i = 1, 2, \ldots, r$. Then,

$$|\boldsymbol{a}, \boldsymbol{x}\rangle_L = T\, |\boldsymbol{a}, \boldsymbol{x}'\rangle_L\,, \tag{38}$$

where $\boldsymbol{x}'$ is the transformed $\boldsymbol{x}$ by T^{-1}. Then the transformed logical operator $\bar{X}_i'$ corresponding to $\bar{X}_i$ satisfies

$$\begin{aligned}\bar{X}_i'\, |\boldsymbol{a}, \boldsymbol{x}'\rangle_L &= \left|\boldsymbol{a}, (x_1', x_2', \ldots (x_i' \text{ xor } 1), \ldots, x_{n-r}')\right\rangle_L \\ &= \left|\boldsymbol{a}, T^{-1}(x_1, x_2, \ldots (x_i \text{ xor } 1), \ldots, x_{n-r})\right\rangle_L \\ &= T^{-1}\, |\boldsymbol{a}, (x_1, x_2, \ldots (x_i \text{ xor } 1), \ldots, x_{n-r})\rangle_L \\ &= T^{-1}\bar{X}_i\, |\boldsymbol{a}, \boldsymbol{x}\rangle_L = T^{-1}\bar{X}_i T\, |\boldsymbol{a}, \boldsymbol{x}'\rangle_L\,. \end{aligned} \tag{39}$$

This implies $\bar{X}_i' = T^{-1}\bar{X}_i T$. Similarly, $\bar{Z}_i' = T^{-1}\bar{Z}_i T$, and $\{\bar{X}_i', \bar{Z}_i'\}_{i=1}^{n-r}$ are the new logical operators by the new coordinates.

By using $\{\bar{X}_i', \bar{Z}_i'\}_{i=1}^{n-r}$, constitute $M_{\boldsymbol{y}}$ as follows:

$$M_{\boldsymbol{y}} := \prod_{i=1}^{n-r} \bar{X}_i'^{(1+(-1)^{1-y_i})/2} \cdot \prod_{i=1}^{n-r} \frac{1}{2}\left(I + (-1)^{y_i}\bar{Z}_i'\right) \tag{40}$$

Then, the following theorem holds for $M_{\boldsymbol{y}}$.

Theorem 4.3: Let $|\boldsymbol{a}, \boldsymbol{x}\rangle_L \in \mathcal{H} = \mathbb{C}^{2^n}$. $M_{\boldsymbol{y}}$ is an operator such that

$$M_{\boldsymbol{y}}\, |\boldsymbol{a}, \boldsymbol{x}\rangle_L = \begin{cases} |\boldsymbol{a}, (0, 0, \ldots, 0)\rangle_L\,, & \boldsymbol{y} = \boldsymbol{x}, \\ 0, & \boldsymbol{y} \neq \boldsymbol{x}. \end{cases} \tag{41}$$

By adding a new term using $M_{\boldsymbol{y}}$ to CDE (29), we construct a new CDE as follows:

$$\begin{aligned}\dot{\rho} &= \sum_{\boldsymbol{b}} \left(L_{\boldsymbol{b}}\rho L_{\boldsymbol{b}}^* - 1/2\{L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}, \rho\}\right) \\ &\quad + \sum_{\boldsymbol{y}} \left(M_{\boldsymbol{y}}\rho M_{\boldsymbol{y}}^* - 1/2\{M_{\boldsymbol{y}}^* M_{\boldsymbol{y}}, \rho\}\right) \\ &= \sum_{\boldsymbol{b}} L_{\boldsymbol{b}}\rho L_{\boldsymbol{b}}^* + \sum_{\boldsymbol{y}} M_{\boldsymbol{y}}\rho M_{\boldsymbol{y}}^* - 2\rho \end{aligned} \tag{42}$$

The derivation of the last line in (42) is shown in Appendix I.

As in (34), (42) can be also represented in a componentwise form as follows:

$$\begin{aligned}\dot{\rho}(t) = &\sum_{\boldsymbol{a}=\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{x}'} k_{\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{a}', \boldsymbol{x}'}(t)\, |\boldsymbol{0}, \boldsymbol{x}\rangle_L \langle \boldsymbol{0}, \boldsymbol{x}'|_L \\ &+ \sum_{\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{x}=\boldsymbol{x}'} k_{\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{a}', \boldsymbol{x}'}(t)\, |\boldsymbol{a}, \boldsymbol{0}\rangle_L \langle \boldsymbol{a}', \boldsymbol{0}|_L - 2\rho \end{aligned} \tag{43}$$

From this, it is known that the components $|\boldsymbol{a}, \boldsymbol{x}\rangle_L \langle \boldsymbol{a}', \boldsymbol{x}'|_L$ at $\boldsymbol{a} = \boldsymbol{a}'$ are transformed to $|\boldsymbol{0}, \boldsymbol{x}\rangle_L \langle \boldsymbol{0}, \boldsymbol{x}'|_L$ in the time evolution, ones at $\boldsymbol{x} = \boldsymbol{x}'$ are transformed into $|\boldsymbol{a}, \boldsymbol{0}\rangle_L \langle \boldsymbol{a}', \boldsymbol{0}|_L$ whereas those of the other components decrease. Then, it is expected that $\rho(t)$ finally becomes the target state

$$\rho_{\mathrm{f}} := |\boldsymbol{0}, \boldsymbol{0}\rangle_L \langle \boldsymbol{0}, \boldsymbol{0}|_L\,. \tag{44}$$

Remark 4.1: In the above case, $|\boldsymbol{0}, \boldsymbol{0}\rangle_L$ was the code word corresponding to a given original information $|\psi\rangle$. Note that even in the general case where the code word for $|\psi\rangle$ is different from $|\boldsymbol{0}, \boldsymbol{0}\rangle_L$, the following analysis is the same as for $|\boldsymbol{0}, \boldsymbol{0}\rangle_L$ by transforming the coordinates of logical operators.

Next, we rigorously analyze the stability of (42).

Theorem 4.4: Let $\rho(t)$ be the solution to (42) for an arbitrary initial $\rho(0)$. Then,

$$\lim_{t\to\infty} \rho(t) = \rho_{\mathrm{f}} \tag{45}$$

where ρ_{f} is given in (44).

In the case with thermal noise, the time evolution of CDE (30) is represented by

$$\begin{aligned}&\dot{\rho}(t) \\ &= \sum_{\boldsymbol{b}} L_{\boldsymbol{b}}\rho L_{\boldsymbol{b}}^* + \sum_{\boldsymbol{y}} M_{\boldsymbol{y}}\rho M_{\boldsymbol{y}}^* - 2\rho \\ &\quad + \gamma(\bar{N}+1)\sum_{i=1}^{n}\left(D_i\rho D_i^* - 1/2\left\{D_i^* D_i, \rho\right\}\right) \\ &\quad + \gamma\bar{N}\sum_{i=1}^{n}\left(D_i^*\rho D_i - 1/2\left\{D_i D_i^*, \rho\right\}\right) \end{aligned} \tag{46}$$

and it is known that noise terms are added to (42). From Theorem 4.4, we obtain that (42) satisfies the asymptotic stability on ρ_{f} for arbitrary initial $\rho(0)$. Therefore, on (46) even if $\rho(t)$ is disturbed by thermal noise during the time evolution, $\rho(t)$ is expected to approach the neighborhood of ρ_{f}. This was confirmed by the numerical simulations described in the next section.

V. Numerical simulation

A. Settings

Numerical experiments were performed to encode the appropriate initial state with the proposed CDE under the influence of thermal noise during time evolution. Consider the simple case of encoding 1-qubit information into a 3-qubit quantum system. By using the method proposed by Ticozzi et al. [10], the logical operators are determined as $\bar{X}_1 = X_1X_2X_3, \bar{Z}_1 = Z_1$, correction matrices as $C_1 = X_2, C_2 = X_3$, and code words as $|0\rangle_L = |000\rangle$ and $|1\rangle_L = |111\rangle$. Using the above settings, numerical experiments were conducted under the following two initial conditions:

- Case1: $\rho_0 = |0\rangle\langle 0|$
- Case2: $\rho_0 = |\psi\rangle\langle\psi|\,,\ |\psi\rangle = \frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle$

The noise constants in (30) or (46) were $\kappa = 0.1$ and $\gamma = 5.0 \times 10^{-4}$ or 5.0×10^{-3}. For each initial state, 30,000 steps of CDE were computed with a time step $dt = 0.001$.

B. Results and discussion

Time responses of coding starting from each initial state are plotted in Figures 1 and 2, where the horizontal and vertical axes represent the time steps and the distance from the target state, respectively. The distance from the target state ρ_{f} is denoted by $\|\rho - \rho_{\mathrm{f}}\|_2$.

In all cases, when there is no additional term $M_{\boldsymbol{y}}$, $\rho(t)$ initially moves closer to the target state; however, the thermal noise gradually causes a loss of the original information, and the distance from the target state increases. On the other hand, in the cases of our proposed method with term $M_{\boldsymbol{y}}$, even with continuous thermal noise, it approaches the target and remains in the range determined by the magnitude of the noise. The size of the range becomes large when the magnitude of the noise is large. These results indicate the distance from the ideal target state is reduced compared to the previous method without the additional term $M_{\boldsymbol{y}}$.

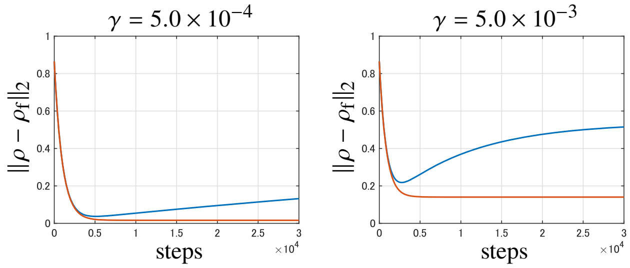

Fig. 1: The time responses of the distance from the target state for Case 1 ($\gamma = 5.0\times10^{-4}$ (left figure), $\gamma = 5.0\times10^{-3}$ (right figure), red line: dynamics with the term of $M_{\boldsymbol{y}}$, blue line: dynamics without the term of $M_{\boldsymbol{y}}$)

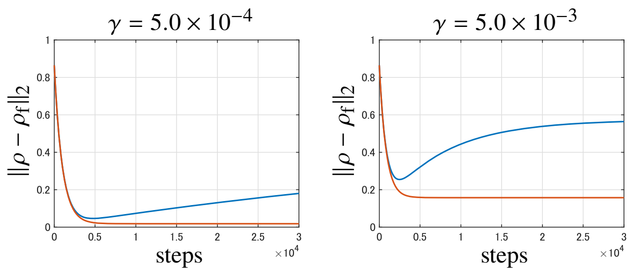

Fig. 2: The time responses of the distance from the target state for Case 2 ($\gamma = 5.0\times10^{-4}$ (left figure), $\gamma = 5.0\times10^{-3}$ (right figure), red line: dynamics with the term of $M_{\boldsymbol{y}}$, blue line: dynamics without the term of $M_{\boldsymbol{y}}$)

VI. Conclusion

In Section IV in this paper, we analyzed the existing coding methods and showed that they are fragile for unexpected state changes due to noise. Then, in the case where the initial state before encoding is known, we proposed a robust method for thermal noise by adding a new dissipative term to strengthen the convergence to the target states during encoding. In Section V, we presented numerical simulations based on the proposed method and confirmed that the new dissipation term reduces the effect of thermal noise. Note that the physical realization of the additional terms in our proposed method is left for the future work.

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Appendix I
Expression transformation in CDE

In the derivation of CDE, the following holds:

$$\sum_{\boldsymbol{b}} L_{\boldsymbol{b}}^* L_{\boldsymbol{b}} = I, \quad \sum_{\boldsymbol{y}} M_{\boldsymbol{y}}^* M_{\boldsymbol{y}} = I \tag{47}$$

This is confirmed by the following calculations. In the first equation, since $g_i, C_i \in G_n$, $g_i^* = g_i$, $C_i^* = C_i$, and $g_i^2 = C_i^2 = I$ follow. In addition, because the commutation relations $[g_i, g_j] = 0$ and $[C_i, C_j] = 0$ hold, the orders of the products of g_i and g_j and those of C_i and C_j are exchangeable. Subsequently, we obtain the following equation:

$$\begin{aligned}
L_{\boldsymbol{b}}^* L_{\boldsymbol{b}} &= \left(\prod_{i=1}^{r} C_i^{(1+(-1)^{1-b_i})/2} \cdot \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i)\right)^* \\
&\quad \cdot \left(\prod_{i=1}^{r} C_i^{(1+(-1)^{1-b_i})/2} \cdot \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i)\right) \\
&= \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i) \cdot \prod_{i=1}^{r} (C_i^2)^{(1+(-1)^{1-b_i})/2} \\
&\quad \cdot \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i) \\
&= \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i) \cdot \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i) \\
&= \prod_{i=1}^{r} \frac{1}{2}(I + (-1)^{b_i} g_i)
\end{aligned} \tag{48}$$

Thus, $L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}$ is the projection to the stabilizer space of stabilizer group $S_{\boldsymbol{b}} = \langle (-1)^{b_1} g_1, (-1)^{b_2} g_2, \ldots, (-1)^{b_r} g_r \rangle$. Summing over $\boldsymbol{b}$, $\sum_{\boldsymbol{b}} L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}$ is a projection onto the entire space $\mathcal{H} = \mathbb{C}^{2^n}$ and $\sum_{\boldsymbol{b}} L_{\boldsymbol{b}}^* L_{\boldsymbol{b}} = I$. For $M_{\boldsymbol{y}}$, the same transformation is possible by replacing $\{C_i\}_{i=1}^r$ with $\{\bar{X}_i\}_{i=1}^{n-r}$ and $\{g_i\}_{i=1}^r$ with $\{\bar{Z}_i\}_{i=1}^{n-r}$, and $\sum_{\boldsymbol{y}} M_{\boldsymbol{y}}^* M_{\boldsymbol{y}} = I$ is obtained.

From the above, in the derivation of CDE, we obtain

$$\frac{1}{2} \sum_{\boldsymbol{b}} \{L_{\boldsymbol{b}}^* L_{\boldsymbol{b}}, \rho\} = \frac{1}{2} \sum_{\boldsymbol{b}} \left(L_{\boldsymbol{b}}^* L_{\boldsymbol{b}} \rho + \rho L_{\boldsymbol{b}}^* L_{\boldsymbol{b}} \right) = \rho, \tag{49}$$

$$\frac{1}{2} \sum_{\boldsymbol{y}} \{M_{\boldsymbol{y}}^* M_{\boldsymbol{y}}, \rho\} = \frac{1}{2} \sum_{\boldsymbol{y}} \left(M_{\boldsymbol{y}}^* M_{\boldsymbol{y}} \rho + \rho M_{\boldsymbol{y}}^* M_{\boldsymbol{y}} \right) = \rho. \tag{50}$$