2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

A Lyapunov-based small-gain theorem for finite time input-to-state stablity of discrete time infinite networks

Yuanqiu Mo, Wenwu Yu, Soura Dasgupta

Abstract— **This paper considers the finite time input-to-state stability (FTISS) with respect to a closed set for discrete time infinite networks composed of a potentially infinite finite-dimensional subsystems. Towards this end, FTISS Lyapunov functions are first provided for infinite networks, via leveraging the existing tools for discrete time finite networks. Further, a small gain condition is postulated so that FTISS Lyapunov functions for the overall system can be constructed from the FTISS Lyapunov-like functions for each subsystem. The established small gain result is scale-free as it can be applied to any truncation of the original infinite network while maintaining quantitative stability properties.**

I. Introduction

This paper considers the Finite Time Input to State Stability (FTISS) of networks of potentially infinite number of discrete time systems. We will define FTISS in the sequel. For the moment we note that the notion of input-to-state stability (ISS), introduced by [1], has been a significant tool in the stability analysis of nonlinear systems. Among various tools used to analyze the ISS of nonlinear systems, Lyapunov-based small gain approaches [2], [3] have attracted broad attention due to their wide applications in networked systems. The first such result involved continuous time nonlinear feedback configuration where both constituents were assumed to have separate Lyapunov-like functions admitting small gain condition that permitted a Lyapunov function for the closed loop. Similar results were obtained for discrete time nonlinear feedback systems [4].

With the rapid development of large-scale systems in the last decade, the ISS notion has been generalized to networked systems. ISS results based on Lyapunov approaches for continuous time and discrete time networks were established in [5], [6] and [7], [8], respectively. As in [3], each subsystem has a Lyapunov-like function and combines with its neighbor's *gain functions* to obey a cyclic small gain condition. Generally, an indispensable step for the above Lyapunov-based small gain methods is the formulation of an ISS Lyapunov function for the overall networked system. To further facilitate the construction of ISS Lyapunov functions, finite step ISS Lyapunov functions were proposed [9], in such a function, decaying at each time step required by the classical Lyapunov function is relaxed by a strict decrease after a finite number of steps rather than at every step. In [9], [10], four types of finite step ISS Lyapunov functions, as well as their equivalence have been introduced. ISS results using finite step ISS Lyapunov functions are in [8], [10].

Recently, this line of research has been extended to infinite networks, which feature in various applications, e.g., representing spatially invariant systems [11]. Infinite networks consist of an infinite number of subsystems, each communicating with a finite number of neighbors. [12] has been proven that continuous time infinite networks are ISS if the gain function in each subsystem's Lyapunov-like function is less than identity. This result was extended to the discrete time scenario in [13], where finite step ISS Lyapunov functions were used instead of ISS Lyapunov function. [14] further reduced the design conservatism in [12] by proving that, continuous time infinite networks are exponentially ISS, provided the gain operator generated from the internal Lyapunov gains has a spectral radius less than one. Though fruitful ISS results have been well studied for infinite networks, FTISS results for discrete time infinite networks are scarce. Existing literature either focuses on infinite networks in continuous time domain, such as finite time or fixed time ISS of continuous time infinite networks [15], [16], or are finite networks oriented [17]. Roughly speaking, FTISS indicates the system state falls below an input-dependent upper bound after a finite time, rather than asymptotically or exponentially as in classical ISS. While finite time ISS implies ISS, the converse need not hold.

Motivated by our previous results on FTISS Lyapunov functions for discete time systems [18], in this paper Lyapunov-based small gain theorems addressing FTISS of discrete time infinite networks are proposed. We first prove that FTISS Lyapunov functions in [18] remain useful for establishing FTISS with respect to a closed set of discrete time infinite networks. Further, a small gain condition is designed and imposed on the FTISS Lyapunov-like functions of each subsystem, such that FTISS Lyapunov function of the overall system can be established. The proposed Lyapunov-based small gain method is scale-free in the sense that quantitative stability indices are preserved for any truncation of the infinite network.

The rest of this paper is organized as follows: Section II introduces notations and definitions. Section III proposes FTISS Lyapunov functions for FTISS with respect to a closed set for discrete time infinite networks. Section IV provides a small gain theorem, Section V provides a numerical example and Section VI concludes.

II. Notations and definitions

Define $\mathbb{R}, \mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{N}_0$ as the set of real numbers, the set of nonnegative real numbers, the set of positive integers,

Mo and Yu are with the School of Mathematics, Southeast University, Nanjing 211189, China (email: yuanqiumo@seu.edu.cn, wwyu@seu.edu.cn). Yu is also with the Purple Mountain Laboratories, Nanjing 211102, China. Dasgupta is with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, Iowa 52242, USA (email: soura-dasgupta@uiowa.edu). This work was supported by the National Natural Science Foundation of China under Grant No. 62303112, and Natural Science Foundation of Jiangsu Province under Grant No. BK20230826.

and the set of nonnegative integers, respectively. We use $|\cdot|$ to denote the vector norms on (in)finite-dimensional vector spaces. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is in class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty}\alpha(s) = \infty$. It is in generalized $\mathcal{K}$ ($\mathcal{GK}$) if it is continuous, $\alpha(0) = 0$ and satisfies

$$\begin{cases} \alpha(s_1) > \alpha(s_2) & \text{if } \alpha(s_1) > 0 \text{ and } s_1 > s_2 \\ \alpha(s_1) = \alpha(s_2) & \text{if } \alpha(s_1) = 0 \text{ and } s_1 > s_2 \end{cases} \quad (1)$$

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is in class $\mathcal{KL}$ if for a fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$, and for a fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to\infty}\beta(\cdot, s) = 0$. It is in generalized class $\mathcal{KL}$ ($\mathcal{GKL}$) if for a fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{GK}$, and for a fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to T}\beta(\cdot, s) = 0$ for some $T \le \infty$; $\mathrm{id} : \mathbb{R}_+ \to \mathbb{R}_+$ obeys $\mathrm{id}(s) = s, \forall s \in \mathbb{R}_+$.

We consider an interconnection of countable (potentially infinite) set of systems, each of which is described by a finite-dimensional difference equation. Let $\mathcal{N}_i$ denote the set of neighboring subsystems that influence dynamics of the i-th subsystem. Then the dynamic of subsystem i is in the following form:

$$x_i(k+1) = f_i(x_i(k), \bar{x}_i(k), u_i(k)),\ i \in \mathbb{N}, k \in \mathbb{N}_0 \quad (2)$$

where $f_i : \mathbb{R}^{n_i} \times X(\mathcal{N}_i) \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$, $x_i(k) \in \mathbb{R}^{n_i}$ is the state of subsystem i at time k, $\bar{x}_i(k) = (x_j(k))_{j\in\mathcal{N}_i} \in X(\mathcal{N}_i)$, with $X(\mathcal{N}_i) = \Pi_{j\in\mathcal{N}_i}\mathbb{R}^{n_j}$ the state space of all i's neighbors, and $u_i(k) \in \mathbb{R}^{m_i}$ is the input. We have the following assumption throughout the paper.

Assumption 1. *Each subsystem has a finite number of neighbors, i.e., $|\mathcal{N}_i|$ is finite.*

Define $x(k) = (x_i(k))_{i\in\mathbb{N}}$ and $u(k) = (u_i(k))_{i\in\mathbb{N}}$ as the state vector and the overall input at time k, respectively. The composite system comprises a countable set of subsystems and is described by

$$x(k+1) = f(x(k), u(k)). \quad (3)$$

Define the state space $X := \ell^\infty(\mathbb{N}, (n_i))$ for the composite system (3) as

$$\ell^\infty(\mathbb{N}, (n_i)) := \{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i}, \sup_{i\in\mathbb{N}} |x_i| < \infty\}, \quad (4)$$

and similarly define $U := \ell^\infty(\mathbb{N}, (m_i)) = \{u = (u_i)_{i\in\mathbb{N}} : u_i \in \mathbb{R}^{m_i}, \sup_{i\in\mathbb{N}} |u_i| < \infty\}$ as the input space for (3). Note that f may not be well-defined with respect to above state and input spaces, i.e., $f : X \times U \to X$ may not hold (see Example 3.2 in [13]). In particular, if f is a map from $X \times U$ to X, we say f is *well-posed*. Define $X_E := \Pi_{i\in\mathbb{N}}\mathbb{R}^{n_i}$ as the extended state space, as each subsystem's state is determined by a finite number of subsystems, f is well-defined with respect to X_E and U, i.e., $f : X_E \times U \to X_E$. We use $x_i(\cdot, \xi, u)$ and $x(\cdot, \xi, u)$ to denote the corresponding solutions to (2) and (3), respectively, with $\xi \in X$ the initial state and $u \in U$ the input. For simplicity, denote $x_i(k) := x_i(k, \xi, u)$ and $x(k) = x(k, \xi, u)$.

We introduce the following norms on state and input spaces of (3), as well as the space $X(\mathcal{N}_i)$ in (2).

Definition 1. *For $\bar{x}_i = (x_j)_{j\in\mathcal{N}_i} \in X(\mathcal{N}_i)$, define $|\bar{x}_i| = |\bar{x}_i|_{X(\mathcal{N}_i)} := \sup_{j\in\mathcal{N}_i} |x_j|$. For $x = (x_i)_{i\in\mathbb{N}} \in X$ with X defined in (4), define $|x|_\infty := \sup_{i\in\mathbb{N}} |x_i|$, and the norm on U is also denoted by $|\cdot|_\infty$. Further, for sequences $u : \mathbb{N}_0 \to U$ define $||u||_\infty := \sup_{k\ge 0} |u(k)|_\infty$.*

From Definition 1, it can be readily verified that $|\bar{x}_i| \le |x|_\infty$ with $\bar{x}_i = (x_j)_{j\in\mathcal{N}_i} \in X(\mathcal{N}_i)$ and $x \in X$. Note that in this paper we only consider admissible inputs $u(k)$ in (3) that obey $||u||_\infty < \infty$ with $||\cdot||_\infty$ as in Definition 1.

A. Distances in sequence spaces

Before we introduce the notion of finite time ISS with respect to a closed set, we first define the distance from bounded sequences in X defined in (4) to a nonempty closed set. For a closed set $\emptyset \ne \mathcal{A}_i \subset \mathbb{R}^{n_i}$, for $x_i \in \mathbb{R}^{n_i}$, define the distance between x_i and $\mathcal{A}_i$ as

$$|x_i|_{\mathcal{A}_i} := \inf_{y_i\in\mathcal{A}_i} |x_i - y_i|. \quad (5)$$

Define $\mathcal{A}(\mathcal{N}_i) = \Pi_{j\in\mathcal{N}_i}\mathcal{A}_j$. For any $\bar{x}_i \in X(\mathcal{N}_i)$, there holds

$$|\bar{x}_i|_{\mathcal{A}(\mathcal{N}_i)} := \inf_{z\in\mathcal{A}(\mathcal{N}_i)} |\bar{x}_i - z| = \inf_{(z_j)_{j\in\mathcal{N}_i}\in\mathcal{A}(\mathcal{N}_i)} \max_{j\in\mathcal{N}_i} |x_j - z_j| = \max_{j\in\mathcal{N}_i} |x_j|_{\mathcal{A}_j} \quad (6)$$

where (6) uses that the choice of $z_j \in \mathcal{A}_j$ with $j \in \mathcal{N}_i$ is independent of each other. Further define the set

$$\mathcal{A} := \{x \in X : x_i \in \mathcal{A}_i, i \in \mathbb{N}\} = X \cap \Pi_{i\in\mathbb{N}}\mathcal{A}_i. \quad (7)$$

If $\mathcal{A} \ne \emptyset$, the distance from any $x \in X$ to $\mathcal{A}$ is defined as

$$|x|_\mathcal{A} := \inf_{y\in\mathcal{A}} |x - y|_\infty = \inf_{y\in\mathcal{A}} \sup_{i\in\mathbb{N}} |x_i - y_i| \quad (8)$$

We next give a crucial result from [13].

Lemma 1. *With $\mathcal{A}$ and X defined in (7) and (4), respectively, if $\mathcal{A}$ is nonempty, then for any $x \in X$, there is a $y^* \in \mathcal{A}$ such that*

$$|x|_\mathcal{A} = \sup_{i\in\mathbb{N}} |x_i|_{\mathcal{A}_i} = |x - y^*|_\infty. \quad (9)$$

It follows from (9) that $|x|_\mathcal{A}$ reduces to $|x|_\infty$ if $\mathcal{A} = \{0\}$.

B. finite time input-to-state stability

Now we give formal definitions of $\mathcal{K}$-boundedness and finite time input-to-state stability (FTISS) with respect to a closed set for (3). Note that in this subsection we have assumed that (3) is well-posed.

$\mathcal{K}$-boundedness, a commonly used notion in ISS for discrete time systems, together with $\mathcal{K}$-boundedness with respect to a closed set is defined as follows.

Definition 2. *Given a map $f : X_1 \times X_2 \to X_1$ with X_1, X_2 subsets of normed vector spaces, f is called $\mathcal{K}$-bounded if there exist $\omega_1, \omega_2 \in \mathcal{K}$ such that for all $\xi \in X_1, \mu \in X_2$ there holds*

$$|f(\xi, \mu)| \le \omega_1(|\xi|) + \omega_2(|\mu|). \quad (10)$$

Further, f is called $\mathcal{K}$-bounded with respect to a closed set $\mathcal{A} \ne \emptyset$ if f obeys

$$|f(\xi, \mu)|_\mathcal{A} \le \omega_1(|\xi|_\mathcal{A}) + \omega_2(|\mu|). \quad (11)$$

In particular, consider the well-posed $f : X \times U \to X$, f is $\mathcal{K}$-bounded with respect to a closed set $\emptyset \ne \mathcal{A} \subset X$ if

$$|f(\xi, \mu)|_\mathcal{A} \le \omega_1(|\xi|_\mathcal{A}) + \omega_2(|\mu|_\infty), \xi \in X, \mu \in U \quad (12)$$

with $\omega_1, \omega_2 \in \mathcal{K}$.

From Definition 2, $\mathcal{K}$-boundedness of f implies $f(0, 0) = 0$ and continuity of f at $(0, 0)$, but it does not require f to be continuous elsewhere.

Definition 3. *(FTISS) Given a nonempty and closed set $\mathcal{A} \subset X$ with X defined in (4). (3) is called finite time input-to-state stable with respect to $\mathcal{A}$ if there exist $\beta \in \mathcal{GKL}$ and $\lambda \in \mathcal{K}$ such that for all initial states $\xi \in X$ and all inputs $u \in U$ there holds:*

$$|x(k, \xi, u)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, k) + \lambda(||u||_\infty), \ \forall k \in \mathbb{N}_0. \quad (13)$$

Further, there exists a function $T : \mathbb{R}_+ \to \mathbb{N}_0$ such that: (i) for all $r \in \mathbb{R}_+ \setminus \{0\}$, $\beta(r, k) \equiv 0$ whenever $k \geq T(r) \in \mathbb{N}_0$ and (ii) $T(0) = 0$.

Remark 1. *An equivalent form of (13) is*

$$|x(k)|_{\mathcal{A}} \leq \max\{\bar{\beta}(|\xi|_{\mathcal{A}}, k), \bar{\lambda}(||u||_\infty)\}, \ \forall k \in \mathbb{N}_0, \quad (14)$$

where $\bar{\beta}(r, s) = \beta(2r, s)$ is a $\mathcal{GKL}$ function and $2\mathrm{id} \circ \lambda = \bar{\lambda} \in \mathcal{K}$, with β and λ defined in (13).

Clearly, FTISS implies ISS, i.e., when β in (13) or $\bar{\beta}$ in (14) is in class $\mathcal{KL}$, (3) is input-to-state stable (ISS) with respect to $\mathcal{A}$, while the converse may not hold.

III. Finite time input-to-state Lyapunov functions

We aim to prove FTISS with respect to a closed set for the discrete time infinite networks described by (3), via a Lyapunov-based small gain approach. The main idea is to impose a small gain condition on each subsystem's trajectory involving FTISS Lyapunov-like function for the subsystem, that permits FTISS Lyapunov functions to be established. To this end, in this section we first introduce four types of FTISS Lyapunov functions from our previous results [18] and assume that (3) is well-posed.

We call a function $V : X \to \mathbb{R}_+$ proper with respect to a nonempty closed set $\mathcal{A} \subset X$ if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in X$

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}). \quad (15)$$

Then FTISS Lyapunov functions are given as follows.

Definition 4. *Let $V : X \to \mathbb{R}_+$ be proper with respect to $\mathcal{A} \subset X$. For a given input $u \in U$, V is said to be*

- *an implication-form FTISS Lyapunov function I for (3) if for all $\xi \in X$ and all $k \in \mathbb{N}_0$, $V(x(k)) \geq \phi_{\mathrm{imp1}}(|u(k)|_\infty)$ implies, with $c > 0$, $0 < a < 1$ and $\phi_{\mathrm{imp1}} \in \mathcal{K}$,*
$$V(x(k+1)) \leq \max\{V(x(k)) - cV(x(k))^a, 0\} \quad (16)$$
- *a max-form FTISS Lyapunov function I for (3) if for all $\xi \in X$ and all $k \in \mathbb{N}_0$ we have*
$$V(x(k+1)) \leq \max\{V(x(k)) - cV(x(k))^a, \lambda_{\max1}(||u||_\infty)\} \quad (17)$$
with $c > 0$, $0 < a < 1$ and $\lambda_{\max1} \in \mathcal{K}$.
- *an implication-form FTISS Lyapunov function II for (3) if for all $\xi \in X$ and all $k \in \mathbb{N}_0$ we have*
$$V(x(k)) \geq \phi_{\mathrm{imp2}}(|u(k)|_\infty) \implies V(x(k+1)) \leq \max\{V(x(k)) - b, 0\} \quad (18)$$
with $b > 0$ and $\phi_{\mathrm{imp2}} \in \mathcal{K}$.
- *a max-form FTISS Lyapunov function II for (3) if for all $\xi \in X$ and all $k \in \mathbb{N}_0$ we have*
$$V(x(k+1)) \leq \max\{V(x(k)) - b, \lambda_{\max2}(||u||_\infty)\} \quad (19)$$
with $b > 0$ and $\lambda_{\max2} \in \mathcal{K}$.

Remark 2. *The above FTISS Lyapunov functions are extensions of those introduced in [18]. The main difference is that here we require V to be proper with respect to $\mathcal{A}$, i.e., (15) holds, while in [18] only properness is required, i.e., $|\cdot|_{\mathcal{A}}$ in (15) is replaced by $|\cdot|$. Moreover, V introduced in Definition 4 need not to be continuous.*

Then we have the following theorem.

Theorem 1. *With X and $\mathcal{A}$ defined in (4) and (7), respectively, suppose $f : X \times U \to X$ is $\mathcal{K}$-bounded with respect to $\mathcal{A}$. Then (3) is finite time input-to-state stable if it admits one of the FTISS Lyapunov functions defined in Definition 4.*

Proof. As f is $\mathcal{K}$-bounded w.r.t. $\mathcal{A}$, it follows from a trivial modification of Lemma 1 in [18] that an implication-form FTISS Lyapunov function I (resp. II) implies the existence of a max-form FTISS Lyapunov function I (resp. II).

For max-form FTISS Lyapunov function I, let $v = \lambda_{\max1}(||u||_\infty)$. We consider two cases: 1) $V(\xi) \leq v$; and 2) $V(\xi) > v$. In the former case, either $V(k) \leq V(k-1)$ or $V(k) \leq v$ for all $k \in \mathbb{N}$ by (17), there holds $|x(k)|_{\mathcal{A}} \leq \alpha_2^{-1} \circ \lambda_{\max1}(||u||_\infty)$ by (15), and thus the $\mathcal{GKL}$ functio β in (13) obeys $\beta(r, s) \equiv 0$ with $T(r) = 0$.

In the latter case, we consider three subcases: 1) $c^{\frac{1}{1-a}} \geq V(\xi) > v$; 2) $V(\xi) > c^{\frac{1}{1-a}} > v$; and 3) $V(\xi) > v \geq c^{\frac{1}{1-a}}$. For the first subcase, it follows from (17) that $V(x(1)) \leq v$, leading to $V(x(k)) \leq v$ for all $k \in \mathbb{N}$ by the above arguments. Then it follows from (15) that the $\mathcal{GKL}$ function in (13) can be set as $\beta(r, s) = \alpha_1^{-1}(\alpha_2(r) \max\{1-s, 0\})$, with $T(r) = 1$ for $r \neq 0$ and $T(0) = 0$.

For subcase 2), by Example 3.2 in [19] if $V(x(k)) > c^{\frac{1}{1-a}}$,

$$\begin{aligned}\alpha_1(|x(k)|_{\mathcal{A}}) &\leq V(x(k)) < V(\xi)(1 - cV(\xi)^{a-1})^k \\ &\leq \alpha_2(|\xi|_{\mathcal{A}})(1 - c\alpha_2(|\xi|_{\mathcal{A}})^{a-1})^k \end{aligned} \quad (20)$$

implying that for $k \geq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{V(\xi)} \right\rceil$, $V(x(k)) \leq c^{\frac{1}{1-a}}$. Using the arguments for the first subcase, there holds $V(x(k)) \leq v$ for $k \geq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{V(\xi)} \right\rceil + 1$. From (20), $\mathcal{GKL}$ function in (13) can be set as $\beta(r, s) = \alpha_1^{-1}\big(\alpha_2(r)(1 - c\alpha_2(r)^{a-1})^s\big)$ for $s = s^* := \left\lceil \log_{[1-c\alpha_2(r)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{\alpha_2(r)} \right\rceil$ and $\beta(r, s) = \alpha_1^{-1}\big(\alpha_2(r)(1 - c\alpha_2(r)^{a-1})^{s^*} \max\{1 + s^* - s, 0\}\big)$ for $s > s^*$ when $r > \alpha_2^{-1}(c^{\frac{1}{1-a}})$, $\beta(r, s) = \alpha_1^{-1}(\alpha_2(r))$ when $0 \leq r \leq \alpha_2^{-1}(c^{\frac{1}{1-a}})$ and $s = 0$, and $\beta(r, s) = 0$ otherwise. Further, $T(r) = \left\lceil \log_{[1-c\alpha_2(r)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{\alpha_2(r)} \right\rceil + 1$ for $r > \alpha_2^{-1}(c^{\frac{1}{1-a}})$, $T(r) = 1$ when $0 < r \leq \alpha_2^{-1}(c^{\frac{1}{1-a}})$, and $T(0) = 0$.

For subcase 3), as (20) still holds when $V(x(k)) > v$, i.e.,

$$\begin{aligned}&\alpha_1(|x(k)|_{\mathcal{A}}) \leq V(x(k)) < V(\xi)(1 - cV(\xi)^{a-1})^k - v + v \\ &\leq \alpha_2(|\xi|_{\mathcal{A}})(1 - c\alpha_2(|\xi|_{\mathcal{A}})^{a-1})^k - v + v \implies \\ &|x(k)|_{\mathcal{A}} \leq \alpha_1^{-1}\big(2\max\{\alpha_2(|\xi|_{\mathcal{A}})(1 - c\alpha_2(|\xi|_{\mathcal{A}})^{a-1})^k - v, 0\}\big) \\ &+ \alpha_1^{-1}(2v). \end{aligned} \quad (21)$$

As $V(x(k)) \leq v$ for all $k \geq k_0$ if $V(x(k_0)) \leq v$ for some $k_0 \in \mathbb{N}_0$, $V(x(k)) \leq v$ for $k \geq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{v}{V(\xi)} \right\rceil$. Let λ in (13) be $\lambda = \alpha_1^{-1} \circ 2\mathrm{id}$, from (21), the $\mathcal{GKL}$ function β can be set as $\beta(r, s) = \alpha_1^{-1}\big(2\max\{\alpha_2(r)(1 - c\alpha_2(r)^{a-1})^s - v, 0\}\big)$ when $r > \alpha_2^{-1}(v)$, and $\beta(r, s) = 0$ otherwise. Further,

$T(r) = \left\lceil \log_{[1-c\alpha_2(r)^{a-1}]} \frac{v}{\alpha_2(r)} \right\rceil$ for $r > \alpha_2^{-1}(v)$ and $T(r) = 0$ otherwise.

For the max-form FTISS Lyapunov function II, when $V(\xi) \leq v$ with $v = \lambda_{\max 2}(||u||_\infty)$, we can set $\beta(r,s) \equiv 0$ with $T(r) = 0$. When $V(x(k)) > v$, by (19)

$$\begin{aligned}&\alpha_1(|x(k)|_{\mathcal{A}}) \leq V(x(k)) \leq V(\xi) - k * b \\ &\leq \alpha_2(|\xi|_{\mathcal{A}}) - k * b - v + v \implies \quad (22)\\ &|x(k)|_{\mathcal{A}} \leq \alpha_1^{-1}(2\max\{\alpha_2(|\xi|_{\mathcal{A}}) - k*b - v), 0\}) + \alpha_1^{-1}(2v).\end{aligned}$$

Let λ in (13) be $\lambda = \alpha_1^{-1} \circ 2\mathrm{id}$, then β can be set as $\beta(r,s) = \alpha_1^{-1}(2\max\{\alpha_2(r) - s*b - v), 0\})$ when $r > \alpha_2^{-1}(v)$, and $\beta(r,s) = 0$ otherwise. Further, $T(r) = \left\lceil \frac{\alpha_2(r) - v}{b} \right\rceil$ for $r > \alpha_2^{-1}(v)$ and $T(r) = 0$ otherwise. ■

Note that, the $\mathcal{K}$-boundedness with respect to $\mathcal{A}$ of f is indeed necessary for (3) to be FTISS with respect to $\mathcal{A}$. Suppose (3) is FTISS with respect to $\mathcal{A}$, by (13)

$$|f(\xi,\mu)|_{\mathcal{A}} = |x(1,\xi,\mu)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, 1) + \lambda(|\mu|_\infty)$$

with $\beta(\cdot, 1) \in \mathcal{GK}$, Then by definition of the $\mathcal{GK}$ function, there always exists a $\mathcal{K}$ function ω_1 such that $\omega_1(s) > \beta(s,1)$ for all $s \in \mathbb{R}_+$, and thus f is $\mathcal{K}$-bounded with respect to $\mathcal{A}$.

IV. The Small Gain Theorem

With the FTISS Lyapunov functions for the composite system (3) established in the previous section, we now consider the subsystems like (2). Based on the assumption that each subsystem admits a FTISS Lyapunov-like function, a small gain condition provided that ensures FTISS. We do not assume the well-posedness of (3). We first assume that the trajectory of each subsystem is in the following form.

Assumption 2. *Consider a nonempty and closed set $\mathcal{A}^{n_i} \subset \mathbb{R}^{n_i}$. Then for each $i \in \mathbb{N}$, there exists a function $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that W_i satisfies the following conditions:*

- *there exist $\bar{\omega}_i, \underline{\omega}_i \in \mathcal{K}_\infty$ such that*
$$\underline{\omega}_i(|\xi_i|_{\mathcal{A}_i}) \leq W_i(\xi_i) \leq \bar{\omega}_i(|\xi_i|_{\mathcal{A}_i}), \ \forall \xi_i \in \mathbb{R}^{n_i}; \quad (23)$$
- *with $\lambda_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $\lambda_{iu} \in \mathcal{K}$, $c > 0, b > 0$ and $0 < a < 1$, for all $\xi \in X$ and all $k \in \mathbb{N}_0$, W_i satisfies one of the following four inequalities:*

$$\begin{aligned}&a) \exists j \in \mathcal{N}_i(k), W_j(x_j(k)) \geq \lambda_{ju}(|u(k)|_\infty) \Longrightarrow W_i(x_i(k+1)) \\ &\leq \sup_{j \in \mathcal{N}_i(k+1)} \lambda_{ij}\big(\max\{W_j(x_j(k)) - cW_j(x_j(k))^a, 0\}\big) \quad (24)\end{aligned}$$

$$\begin{aligned}&b) W_i(x_i(k+1)) \leq \max\{\sup_{j \in \mathcal{N}_i(k+1)} \lambda_{ij}\big(\max\{W_j(x_j(k)) - \\ &\quad cW_j(x_j(k))^a, 0\}\big), \lambda_{iu}(||u||_\infty)\} \quad (25)\end{aligned}$$

$$\begin{aligned}&c) \exists j \in \mathcal{N}_i(k), W_j(x_j(k)) \geq \lambda_{ju}(|u(k)|_\infty) \Longrightarrow W_i(x_i(k+1)) \\ &\leq \sup_{j \in \mathcal{N}_i(k+1)} \lambda_{ij}\big(\max\{W_j(x_j(k)) - b, 0\}\big) \quad (26)\end{aligned}$$

$$\begin{aligned}&d) W_i(x_i(k+1)) \leq \max\{\sup_{j \in \mathcal{N}_i(k+1)} \lambda_{ij}\big(\max\{W_j(x_j(k)) - \\ &\quad b, 0\}\big), \lambda_{iu}(||u||_\infty)\} \quad (27)\end{aligned}$$

where $\mathcal{N}_i(k+1) \subseteq \mathcal{N}_i \cup \{i\}$ in (24)-(27) is a finite set, denoting the set of subsystems that have influence on subsystem i's dynamics at time $k+1$.

Moreover, $\bar{\omega}_i, \underline{\omega}_i, \lambda_{ij}$ and λ_{iu} should also satisfy a uniformity condition as described below.

Assumption 3. *There exist $\bar{\omega}, \underline{\omega} \in \mathcal{K}_\infty, \bar{\lambda}_u \in \mathcal{K}$ such that for all $i, j \in \mathbb{N}$,*

$$\underline{\omega} \leq \underline{\omega}_i \leq \bar{\omega}_i \leq \bar{\omega}, \ \lambda_{ij} \leq \mathrm{id}, \text{ and } \lambda_{iu} \leq \bar{\lambda}_u. \quad (28)$$

with $\bar{\omega}_i, \underline{\omega}_i, \lambda_{ij}$ and λ_{iu} defined in Assumption 2. Further, f_i defined in (2) is uniformly $\mathcal{K}$-bounded w.r.t. $\mathcal{A}_i$, i.e., for all $i \in \mathbb{N}$ and for all $\xi_i \in \mathbb{R}^{n_i}$, $\bar{\xi}_i \in X(\mathcal{N}_i)$ and $\mu_i \in \mathbb{R}^{m_i}$,

$$|f_i(\xi_i, \bar{\xi}_i, \mu_i)|_{\mathcal{A}_i} \leq \kappa_1(|\xi_i|_{\mathcal{A}_i}) + \kappa_2(|\bar{\xi}_i|_{\mathcal{A}(\mathcal{N}_i)}) + \kappa_3(|\mu_i|) \quad (29)$$

with $\kappa_i \in \mathcal{K}$ for $i \in \{1,2,3\}$.

The following assumption is needed to guarantee the well-poseness of (3).

Assumption 4. *$\mathcal{A}_i$ defined in (7) is uniformly bounded, i.e., there exists a $C > 0$ such that $|x| \leq C$ for all $x \in \mathcal{A}_i$.*

Remark 3. *Note that when $\mathcal{A}_i$ is uniformly bounded, $\mathcal{A}$ defined in (7) reduces to $\mathcal{A} = \Pi_{i \in \mathbb{N}} \mathcal{A}_i$. Additionally, it follows from Lemma 1 in [18] that (24) and (26) imply (25) and (27), respectively, once Assumption 3 holds.*

The following lemma shows the well-poseness of f.

Lemma 2. *Suppose Assumptions 2-4 hold, then f in (3) is well-posed.*

Proof. We prove the case where W_i satisfies (25), arguments for other cases (i.e., W_i obeys (24), (26) or (27)) are similar.

It follows from (23), (25) and (28) that for all $i \in \mathbb{N}$, all $\xi_i \in \mathbb{R}^{n_i}$, all $\bar{\xi}_i \in X(\mathcal{N}_i)$ and all $u_i \in \mathbb{R}^{m_i}$

$$\begin{aligned}&|f_i(\xi_i, \bar{\xi}_i, u_i)|_{\mathcal{A}_i} = |x_i(1)|_{\mathcal{A}_i} \\ &\leq \underline{\omega}^{-1}\big(\max\{\sup_{j \in \mathcal{N}_i(1)} \bar{\omega}(|\xi_j|_{\mathcal{A}_j}), \bar{\lambda}_u(||u||_\infty)\}\big) \\ &\leq \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(|\xi_\ell|_{\mathcal{A}_\ell}) + \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\lambda}_u(||u||_\infty) \quad (30)\\ &\leq \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(|\xi_\ell| + \sup_{y_i \in \mathcal{A}_i} |y_i|) + \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\lambda}_u(||u||_\infty) \\ &\leq \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(2|\xi_\ell|) + \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(2C) + \\ &\quad \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\lambda}_u(||u||_\infty) \quad (31)\end{aligned}$$

where (30) assumes $\ell = \arg\max_{j \in \mathcal{N}_i(1)} |\xi_j|_{\mathcal{A}_j}$, (31) uses $\bar{\omega}(a+b) \leq \bar{\omega}(2a) + \bar{\omega}(2b)$ for $a, b \geq 0$ and Assumption 4.

Further, as $|f_i(\xi_i, \bar{\xi}_i, u_i)| - \sup_{y_i \in \mathcal{A}_i} |y_i| \leq |f_i(\xi_i, \bar{\xi}_i, u_i)|_{\mathcal{A}_i}$, by (31) and Assumption 4,

$$\begin{aligned}&|f(\xi, u)|_\infty = \sup_{i \in \mathbb{N}} |f_i(\xi_i, \bar{\xi}_i, u_i)| \leq \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(2|\xi_\ell|) \\ &+ \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\omega}(2C) + C + \underline{\omega}^{-1} \circ 2\mathrm{id} \circ \bar{\lambda}_u(||u||_\infty), \quad (32)\end{aligned}$$

showing that $f(\xi, u) \in X$ and thus f is well-posed. ■

The statement in Lemma 2 does not hold without Assumption 4, here we use an example similar to Example 6.5 in [13] to illustrate. Consider

$$x_i(k+1) = \begin{cases} x_i(k) - |x_i(k)|_{\mathcal{A}_i} \mathrm{sgn}(x_i(k)), x_i(k) \notin \mathcal{A}_i \\ i x_i(k), x_i(k) \in [-\frac{1}{2}, \frac{1}{2}] \\ x_i(k), \text{otherwise} \end{cases} \quad (33)$$

In this zero-input example, by setting $\mathcal{A}_i = [-i, i]$ for $i \in \mathbb{N}$ and $W_i(\cdot) = |\cdot|_{\mathcal{A}_i}$, it can be readily verified that W_i satisfies Assumptions 2 and 3 with $\lambda_{ij} = 0, \bar{\omega} = \underline{\omega} = \mathrm{id}$ and $\bar{\lambda}_u$ any $\mathcal{K}$ function. However, if we choose the initial state as $\xi = \frac{1}{3}[1, \cdots]^\top \in \mathcal{A}$, we have $f(\xi, 0) = i[\frac{1}{3}, \cdots]^\top \in \Pi_{i \in \mathbb{N}} \mathcal{A}_i \setminus X$, and thus f in (33) is not well-posed.

Based on Lemma 2, we give the small gain theorem.

Theorem 2. *With $\mathcal{A}$ defined in (7), suppose Assumptions 2-4 hold. Then (3) is FTISS with respect to $\mathcal{A}$.*

Proof. Here we prove the case where W_i satisfies (25), by showing that in this case (3) admits a max-form FTISS Lyapunov function II. From Remark 3 and using the same arguments, it can be proved that (3) admits a max-form FTISS Lyapunov function II if W_i satisfies (24), (26) or (27).

Define $V : X \to \mathbb{R}_+$ as $V(\xi) = \sup_{i\in\mathbb{N}} W_i(\xi_i)$. Then it follows from (23) and (28) that

$$V(\xi) \le \sup_{i\in\mathbb{N}} \bar{\omega}_i(|\xi_i|_{\mathcal{A}_i}) \le \bar{\omega}(\sup_{i\in\mathbb{N}} |\xi_i|_{\mathcal{A}_i}) = \bar{\omega}(|\xi|_{\mathcal{A}}) \quad (34)$$

where the last inequality in (34) uses the monotonicity of $\bar{\omega}$, and the equality in (34) uses (9). Similarly, $V(\xi) \ge \underline{\omega}(|\xi|_{\mathcal{A}})$, and thus (15) holds for V. Further, it follows from (25) that

$$\begin{aligned} &V(x(k+1)) = \sup_{i\in\mathbb{N}} W_i(x_i(k+1)) \le \\ &\sup_{i\in\mathbb{N}}\max\{\sup_{j\in\mathcal{N}_i(k+1)} W_j(x_j(k)) - cW_j(x_j(k))^a, \lambda_{iu}(||u||_\infty)\} \\ &\le \max\{\sup_{j\in\mathbb{N}} W_j(x_j(k)) - cW_j(x_j(k))^a, \sup_{i\in\mathbb{N}} \lambda_{iu}(||u||_\infty)\} \\ &\le \max\{\sup_{j\in\mathbb{N}} W_j(x_j(k)) - cW_j(x_j(k))^a, \bar{\lambda}_u(||u||_\infty)\} \quad (35)\end{aligned}$$

where (35) uses (28). For $W_j(x_j(k))$ in (35), we consider two cases: 1) $W_j(x_j(k)) \le c^{\frac{1}{1-a}}$; and 2) $W_j(x_j(k)) > c^{\frac{1}{1-a}}$. In the former case, there holds: $W_j(x_j(k)) - cW_j(x_j(k))^a \le 0$; In the latter case, we have

$$W_j(x_j(k)) - cW_j(x_j(k))^a \le W_j(x_j(k)) - c^{\frac{1}{1-a}}. \quad (36)$$

Combining (36) and (35), we have

$$\begin{aligned} V(x(k+1)) &\le \max\{\sup_{j\in\mathbb{N}} W_j(x_j(k)) - b, \bar{\lambda}_u(||u||_\infty)\} \\ &= \max\{V(x(k)) - b, \bar{\lambda}_u(||u||_\infty)\} \quad (37)\end{aligned}$$

with $b = c^{\frac{1}{1-a}}$. Therefore, V is a max-form FTISS Lyapunov function II, together with Lemma 2, we can conclude that (3) is FTISS with respect to $\mathcal{A}$. ■

Remark 4. *In (26) and (27), we implicitly assume b is the uniform lower bound that each W_i decays. Without such uniformity, (3) may not be FTISS. Consider $\forall k \in \mathbb{N}_0$*

$$x_i(k+1) = \begin{cases} \max\{x_i(k) - (\frac{1}{k+i})^2, 0\}, i = 1 \\ \max\{x_{i-1}(k) - (\frac{1}{k+i})^2, 0\}, i \ne 1 \end{cases} \quad (38)$$

In this zero-input example, let $\mathcal{A} = \{0\}$ and $W_i(\xi_i) = |\xi_i|_{\mathcal{A}_i} = |\xi_i|$, then $W_1(x_1(k+1)) \le \max\{W_1(x_1(k)) - (\frac{1}{k+1})^2, 0\}$ and $W_i(x_i(k+1)) \le \max\{W_{i-1}(x_{i-1}(k)) - \frac{1}{(k+i)^2}, 0\}$ for $i \in \mathbb{N}\setminus\{1\}$. However, the network is not stable when the initial state of the first subsystem is large than $\frac{\pi^2}{6}$.

Remark 5. *Theorem 2 may not hold without the uniformity of λ_{iu} in Assumption 4. For instance, consider $x_i(k+1) = iu(k)$, let $\mathcal{A} = \{0\}$. Then $W_i(\xi_i) = |\xi_i|_{\mathcal{A}_i} = |\xi_i|$ satisfies Assumptions 2 and 4 with $\bar{\omega} = \underline{\omega} = \mathrm{id}, \lambda_{ij} = 0$, and $\lambda_{iu} = i\mathrm{id}$. However, in this case (3) is not FTISS w.r.t. $\mathcal{A}$.*

A. The small gain theorem for truncated networks

We next show that the quantitative stability results derived in Theorem 2 are preserved for any truncation of infinite networks, proving that our stability tools are scale-free.

Specifically, consider the first n subsystems of the infinite network. Let $x^{(n)} \in \mathbb{R}^N$ with $N = \sum_{i\in\{1,\cdots,n\}} n_i$ represent the composite state of the first n subsystems, and $u^{(n)} \in \mathbb{R}^M$ with $M = \sum_{i\in\{1,\cdots,n\}} m_i$ the corresponding input. Further, denote $\mathcal{N}^{(n)} = \bigcup_{i\in\{1,\cdots,n\}} \mathcal{N}_i \setminus \{1, \cdots, n\}$ be the set of neighbors of the first n subsystems, which in general will affect the dynamics of the first n subsystems. Then the first n subsystems of the infinite network can be described by

$$x^{(n)}(k+1) = \bar{f}(x^{(n)}(k), \tilde{u}(k)) \quad (39)$$

where $\tilde{u}(k) = (\tilde{x}(k), u^{(n)}(k)) \in \mathbb{R}^{L+M}$ with $\tilde{x}(k) = (x_j(k))_{j\in\mathcal{N}^{(n)}} \in \mathbb{R}^L$ and $L = \sum_{j\in\mathcal{N}^{(n)}} n_i$, and $\bar{f} : \mathbb{R}^N \times \mathbb{R}^{L+M} \to \mathbb{R}^N$. Note that in (39) we do not ignore effects of other states x_j with $j \in \mathcal{N}^{(n)}$ but interpret them as additional external inputs instead. Our aim is to provide the conditions under which (39) is FTISS in terms of $x^{(n)}$ with respect to $\mathcal{A}^{(n)}$, as well as to investigate whether the quantitative results (e.g., the input gain $\bar{\lambda}_u$ in (37) of Theorem 2) derived for the FTISS of the overall system will be preserved for the truncated system, under the assumption that the infinite network is FTISS.

Define $\mathcal{A}^{(n)} = \Pi_{i\in\{1,\cdots,n\}}\mathcal{A}_i$. It follows from (6) that for $x^{(n)} \in \mathbb{R}^N$

$$|x^{(n)}|_{\mathcal{A}^{(n)}} = \max_{i\in\{1,\cdots,n\}} |x_i|_{\mathcal{A}_i} \quad (40)$$

The truncated system (39) admits the FTISS Lyapunov function given in the following theorem and is thus FTISS.

Theorem 3. *Suppose Assumptions 3-4, (23) and (25) in Assumption 2 hold. Consider (39) with $\tilde{u}(k) \in \mathbb{R}^{L+M}$ the external inputs, define $V^{(n)} : \mathbb{R}^N \to \mathbb{R}_+$ as*

$$V^{(n)}(\xi^{(n)}) = \max_{1\le i\le n} W_i(\xi_i). \quad (41)$$

Then for all $\xi^{(n)} \in \mathbb{R}^N, \tilde{u} \in \mathbb{R}^{M+L}$, V obeys

$$\underline{\omega}(|\xi^{(n)}|_{\mathcal{A}^{(n)}}) \le V^{(n)}(\xi^{(n)}) \le \bar{\omega}(|\xi^{(n)}|_{\mathcal{A}^{(n)}}), \quad (42)$$

$$\begin{aligned} &V^{(n)}(\bar{f}(\xi^{(n)}, \tilde{u})) \le \max\{V^{(n)}(\xi^{(n)}) - c^{\frac{1}{1-a}}, \bar{\omega}(2||\tilde{x}||_\infty) \\ &+ \bar{\omega}(2C) - c^{\frac{1}{1-a}}, \bar{\lambda}_u(||u^{(n)}||_\infty)\}, \quad (43)\end{aligned}$$

where C in (43) is defined in Assumption 4.

Proof. It follows from (23) that

$$V^{(n)}(\xi^{(n)}) \le \max_{1\le i\le n} \bar{\omega}(|\xi_i|_{\mathcal{A}_i}) = \bar{\omega}(\max_{1\le i\le n} |\xi_i|_{\mathcal{A}_i}) = \bar{\omega}(|\xi^{(n)}|_{\mathcal{A}^{(n)}}).$$

Similarly, we can obtain $V^{(n)}(\xi^{(n)}) \ge \underline{\omega}(|\xi^{(n)}|_{\mathcal{A}^{(n)}})$. From (25), we have

$$\begin{aligned} &V^{(n)}(x^{(n)}(k+1)) \le \max\{\max_{1\le j\le n} W_j(x_j(k)) - cW_j(x_j(k))^a, \\ &\quad \sup_{j>n} W_j(x_j(k)) - cW_j(x_j(k))^a, \lambda_{iu}(||u^{(n)}||_\infty)\} \\ &\le \max\{\max_{1\le j\le n} W_j(x_j(k)) - c^{\frac{1}{1-a}}, \sup_{j>n} W_j(x_j(k)) - c^{\frac{1}{1-a}}, \\ &\quad \bar{\lambda}_u(||u^{(n)}||_\infty)\} \quad (44) \\ &\le \max\{V^{(n)}(x^{(n)}(k)) - c^{\frac{1}{1-a}}, \sup_{j>n} \bar{\omega}(|x_j(k)|_{\mathcal{A}_j}) - c^{\frac{1}{1-a}}, \\ &\quad \bar{\lambda}_u(||u^{(n)}||_\infty)\} \quad (45)\end{aligned}$$

where (44) uses (35) and (36) in Theorem 2 and Assumption 3, (45) uses (23) and Assumption 3. By the monotonicity of $\bar{\omega}$, (5) and the uniform boundedness of $\mathcal{A}_i$ as in Assumption 4, there holds $\sup_{j>n} \bar{\omega}(|x_j(k)|_{\mathcal{A}_j}) \le \bar{\omega}(\sup_{j>n} |x_j(k)|_{\mathcal{A}_j}) \le \bar{\omega}(|\tilde{x}(k)|_\infty + C) \le \bar{\omega}(2||\tilde{x}||_\infty) + \bar{\omega}(2C)$, where the last inequality uses the fact that $\bar{\omega}(a+b) \le \bar{\omega}(2a) + \bar{\omega}(2b)$ for all $a, b \ge 0$, and our claim follows. ■

Taking $\tilde{x}$ as the external input, (42) and (43) imply that $V^{(n)}$, as defined in (41), indeed serves as a max-form FTISS Lyapunov function II for (39). Consequently, (39) is FTISS. Furthermore, if the external input has not effect on the truncated network (39), the input gain $\bar{\lambda}_u$ and the time required for the $\mathcal{GKL}$ function of (39) to decay to 0 will be fully preserved. Applying the arguments in Theorem 3, we can also prove the FTISS of (39) if (25) in Theorem 3 is replaced by (24), (26) or (27).

V. A Numerical Example

Consider the discrete time infinite network:

$$x_i(k+1)=\max\big\{x_{i+1}(k)-c\,\mathrm{sgn}(x_{i+1}(k))\min\big\{|x_{i+1}(k)|/c, |x_{i+1}(k)|^a\big\}, \sin(v)\big\}, \forall i\in\mathbb{N}, \forall k\in\mathbb{N}_0, \quad (46)$$

where $c>0, a\in(0,1)$ and $u(k)=\sin(v)$ with v randomly generated between 0 and k. Define $\mathcal{A}=\{0\}$ and let $W_i = |\cdot|_{\mathcal{A}_i} = |\cdot|$ for all $i\in\mathbb{N}$. Then there holds

$$W_i(x_i(k+1)) = |x_i(k+1)| \leq \max\Big\{|\sin(v)|, \underbrace{|x_{i+1}(k) - c\,\mathrm{sgn}(x_{i+1}(k))\min\big\{\frac{|x_{i+1}(k)|}{c}, |x_{i+1}(k)|^a\big\}|}_{(A)}\Big\}.$$

It follows from (18) in [19] that either $(A)=0$ when $|x_{i+1}(k)| \leq c^{\frac{1}{1-a}}$, or $(A) = |x_{i+1}(k)(1-c|x_{i+1}(k)|^{a-1})|$ when $|x_{i+1}(k)| > c^{\frac{1}{1-a}}$. In the latter case $(A) \leq |x_{i+1}(k)||1-c|x_{i+1}(k)|^{a-1}| = |x_{i+1}(k)|(1-c|x_{i+1}(k)|^{a-1})$. Thus, in both cases we have

$$W_i(x_i(k+1)) \leq \max\{W_{i+1}(x_{i+1}(k)) - cW_{i+1}(x_{i+1}(k))^a, |\sin(v)|\}. \quad (47)$$

Then it follows from Theorem 1 that (46) is FTISS.

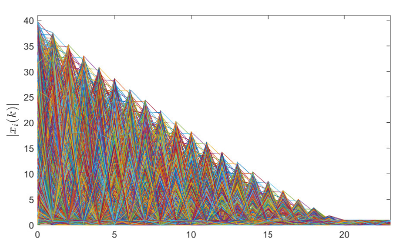

Fig. 1: Trajectories of states of the first 2000 subsystems of the infinite network. In this example we set $c=1.3, a=0.15$ and the initial state of each subsystem randomly generated from $[-40,40]$.

Figure 1 depicts states of the first 2000 subsystems of an infinite network defined by (46) with $c=1.3, a=0.15$ and the initial state randomly generated from $[-40,40]$. By Theorems 1 and 2, (46) has a max-form FTISS Lyapunov function II. Further by Theorem 2, α_1, α_2 in (15) and $\lambda_{\max2}, b$ in (19) obey $\alpha_1=\alpha_2=\lambda_{\max2}=\mathrm{id}$ and $b=c^{\frac{1}{1-a}}$, respectively. As the initial state is between $[-40,40]$ and $|\sin(v)|\leq 1$ for all $v\in\mathbb{R}$, the theoretical upper bound of time steps for each subsystem's state to drop below $\lambda_{\max2}(||u||_\infty)=1$ will not exceed $\left\lceil \frac{V(\xi)-\lambda_{\max2}(||u||_\infty)}{c^{\frac{1}{1-a}}} \right\rceil = 29$. Figure 1 shows that all states drop below the upper bound within 20 rounds.

VI. Conclusion

In this paper we have provided a Lyapunov-based small gain approach for FTISS with respect to a closed set for discrete time infinite networks. By assuming that each subsystem admits a finite time Lyapunov-like function, the small gain condition is given to ensure the existence of a FTISS Lyapunov function for the infinite network. Moreover, the proposed small gain theorem is scale-free in the sense that the upper bound defined by the input is preserved for any truncation of the original network, and under mild conditions the bound on time steps needed for the state trajectory to drop below the upper bound is also preserved.

References

[1] E. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.
[2] DJ Hill, "A generalisation of the small-gain theorem for nonlinear feedback systems," in *Nonlinear Control Systems Design 1989*, pp. 201–205. Elsevier, 1990.
[3] Zhong-Ping Jiang, Iven MY Mareels, and Yuan Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.
[4] Z. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.
[5] Sergey Dashkovskiy, Björn S Rüffer, and Fabian R Wirth, "An ISS small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.
[6] Sergey N Dashkovskiy, Björn S Rüffer, and Fabian R Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM Journal on Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.
[7] Tengfei Liu, David J Hill, and Zhong-Ping Jiang, "Lyapunov formulation of the large-scale, iss cyclic-small-gain theorem: The discrete-time case," *Systems & control letters*, vol. 61, no. 1, pp. 266–272, 2012.
[8] Roman Geiselhart and Fabian R Wirth, "Relaxed ISS small-gain theorems for discrete-time systems," *SIAM Journal on Control and Optimization*, vol. 54, no. 2, pp. 423–449, 2016.
[9] R. Geiselhart and N. Noroozi, "Equivalent types of ISS Lyapunov functions for discontinuous discrete-time systems," *Automatica*, vol. 84, pp. 227–231, 2017.
[10] N. Noroozi, R. Geiselhart, L. Grüne, B. Rüffer, and F. Wirth, "Nonconservative discrete-time ISS small-gain conditions for closed sets," *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1231–1242, 2017.
[11] Ruth Curtain, Orest V Iftime, and Hans Zwart, "System theoretic properties of a class of spatially invariant systems," *Automatica*, vol. 45, no. 7, pp. 1619–1627, 2009.
[12] Sergey Dashkovskiy and Svyatoslav Pavlichkov, "Stability conditions for infinite networks of nonlinear systems and their application for stabilization," *Automatica*, vol. 112, pp. 108643, 12, 2020.
[13] Navid Noroozi, Andrii Mironchenko, and Fabian R Wirth, "A relaxed small-gain theorem for discrete-time infinite networks," *Automatica*, vol. 142, pp. 110363, 8, 2022.
[14] Christoph Kawan, Andrii Mironchenko, Abdalla Swikir, Navid Noroozi, and Majid Zamani, "A lyapunov-based small-gain theorem for infinite networks," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 5830–5844, 2020.
[15] Svyatoslav Pavlichkov and Naim Bajcinca, "Decentralized fixed-time uniform iss stabilization of infinite networks of switched nonlinear systems with arbitrary switchings by small gain approach," *European Journal of Control*, vol. 74, pp. 100864, 10, 2023.
[16] Svyatoslav Pavlichkov, "A small gain theorem for finite-time input-to-state stability of infinite networks and its applications," *Visnyk of VN Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics*, vol. 94, pp. 40–59, 2021.
[17] Huanhuan Li, Tengfei Liu, and Zhong-Ping Jiang, "A Lyapunov-based small-gain theorem for a network of finite-time input-to-state stable systems," *IEEE Transactions on Automatic Control*, 2023.
[18] Yuanqiu Mo, Ran Xing, Huazhou Hou, and Soura Dasgupta, "Finite time input-to-state stability of discrete time autonomous systems," *IEEE Control Systems Letters*, 2023.
[19] Wassim M Haddad and Junsoo Lee, "Finite-time stability of discrete autonomous systems," *Automatica*, vol. 122, pp. 109282, 8, 2020.