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Stability of linear KdV equation in a network with bounded and unbounded lengths

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***Abstract*—In this work, we study the exponential stability of a system of linear Korteweg-de Vries (KdV) equations interconnected through the boundary conditions on a star-shaped network structure. On each branch of the network we define a linear KdV equation defined on a bounded domain $(0, \ell_j)$ or the half-line $(0, \infty)$. We start by proving well-posedness using semigroup theory and then some hidden regularity results. Then, we state the exponential stability of the linear KdV equation by acting with a damping term on not all the branches. This is proved by using compactness argument deriving a suitable observability inequality.**

I. Introduction

In [1], the Korteweg-de Vries (KdV) equation was first proposed to model the behavior of long water waves in a channel. This famous nonlinear third-order dispersive equation arises in various physical systems, including water waves, tsunamis, transmission of electrical signals in nerve fibers, plasma, cosmology, etc (see for instance [2]–[4]). It is a prototypical example of a soliton equation, which admits solutions in the form of solitary waves that preserve their shape and speed during propagation. If we study the KdV equation in a bounded domain, the following model was suggested in [5]

$$\partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0.$$

The KdV equation has been the subject of extensive research in recent years, with particular focus on its controllability and stabilization properties, which are detailed for instance in [6] and [7]. When it is defined on a network, the KdV equation was proposed to model the pressure of the arterial tree [8]. We also mention [9] where controllability properties were studied and [10], [11] where the exponential stability was achieved by acting with damping terms with time-delay and saturation, respectively (see [12] for more problems related to KdV in networks). The main difference of this work with the previously cited is that, we consider a star-shaped network mixing bounded and unbounded lengths as for example [13], [14] in the case of wave equation.

With respect to the KdV equation defined on the half-line, we can mention, for instance, [5], [15] which focus on the well-posedness properties. In [16], the exact controllability of the linear KdV equation defined on the half-line was obtained by using Carleman estimates. A first result of exponential stability of the KdV equation in the half-line considering a localized damping was derived in [17] under the assumption that the damping term $a(x) \geq c > 0$ in $(0, \delta) \cup (b, \infty)$ with $b > \delta$ (see [18] for a similar problem in the context of KdV-Burger equation in the whole-line and half-line). Then, in [19] exponential decay of the energy in weighted spaces was derived, and it was noticed that the interval $(0, \delta)$ can be dropped. We can mention also [20] where similar ideas of [19] were applied in the case of a Gear-Grimshaw system modeling long waves. To ease the reading of the paper, some technical notations are recapitulated in the Appendix.

In [8] the stabilization problem for the KdV equation on a star-shaped network with bounded lengths was addressed. See also [17] where the asymptotic behaviour of the KdV equation in the half-line was investigated. Inspired by theses works, we study the exponential stabilization problem of the linear KdV equation defined on a star-shaped network, where the branches mix finite intervals and half-lines. We consider a network of $N = N_F + N_\infty$ damped linear KdV equations each one of them defined on I_j for $j \in [\![1, N]\!]$, i.e

$$(\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t, x) = 0,\ \forall x \in I_j,\ t > 0,$$

where the intervals $I_j = (0, \ell_j)$, with $\ell_j > 0$ for $j \in [\![1, N_F]\!]$ and $I_j = (0, \infty)$ for $j \in]\!]N_F, N]\!]$. These equations are connected by the transmission conditions at 0 as follows

$$\begin{cases} u_j(t,0) = u_{j'}(t,0), \quad j, j' \in [\![1, N]\!], & \begin{pmatrix}\text{continuity}\\ \text{condition}\end{pmatrix}, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0),\ t > 0, & \begin{pmatrix}\text{null flux}\\ \text{condition}\end{pmatrix}, \end{cases}$$

where $\alpha > \frac{N}{2}$. The central node conditions are inspired by [8], [10], [11]. In the case $j = [\![1, N_F]\!]$, we

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complement the system with the classical null boundary conditions at the right end,

$$u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0,\ t>0,$$

and initial condition $u_j(0,x) = u_j^0(x),\ x \in I_j$. We denote the network structure by $\mathcal{T}$ (see Figure 1). The system studied in this work reads as:

(LKdV)

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t,x) = 0,\ \forall x \in I_j, \\ \qquad\qquad\qquad\qquad t>0,\ j \in [\![1,N]\!], \\ u_j(t,0) = u_{j'}(t,0), \qquad j,j' \in [\![1,N]\!], \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), \quad t>0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, \quad t>0,\ j\in[\![1,N_F]\!], \\ u_j(0,x) = u_j^0(x), \qquad x\in I_j, j\in[\![1,N]\!], \end{cases}$$

where $\alpha > \frac{N}{2}$ and the damping terms $(a_j)_{j\in[\![1,N]\!]} \in \mathbb{L}^\infty(\mathcal{T})$, act locally on the branches.

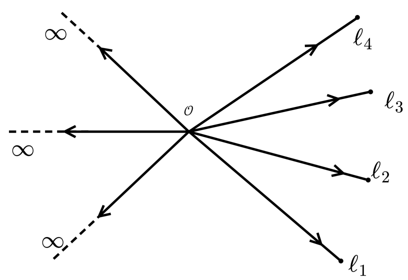

Fig. 1. Network structure for $N_F = 4$ and $N_\infty = 3$.

Our purpose is to achieve the exponential stability by acting with the damping terms not necessarily in all the branches. Let $\mathbb{I}_{act} \subset [\![1,N]\!]$ the set of action index, formally the damping terms are taken in the following way.

- **Not action index:** For $j \in [\![1,N]\!] \setminus \mathbb{I}_{act}$, $a_j \equiv 0$.
- **Locally action:** For $j \in \mathbb{I}_{act}$, $a_j(x) \geq c_j > 0$ in a nonempty open set ω_j of $\mathbb{I}_{act}$.
- **Structure of action set in the half-line case:** For the index $j \in]\!]N_F, N]\!] \cap \mathbb{I}_{act}$, we take a specific structure of the set $\omega_j = (\beta_j, \infty)$, for $\beta_j > 0$ given.

This properties are summarized in

$$(1)\qquad \begin{cases} a_j \equiv 0 \ \text{ for } \ j \in [\![1,N]\!] \setminus \mathbb{I}_{act}, \\ a_j(x) \geq c_j > 0 \ \text{ in } \omega_j \subset I_j, \text{ for } \ j \in \mathbb{I}_{act}, \\ \omega_j = (\beta_j, \infty), \text{ for } j \in]\!]N_F, N]\!] \ \cap \mathbb{I}_{act}. \end{cases}$$

For the system (LKdV) we define the natural $\mathbb{L}^2(\mathcal{T})$ energy of a solution by

$$(2)\qquad E(t) = \frac{1}{2}\sum_{j=1}^{N}\int_{I_j} |u_j(t,x)|^2 dx.$$

Formally, we can check after several integrations by parts (see [10], [11] for similar computations) that for every sufficiently smooth solution of (LKdV) the energy satisfies[1]

$$(3)\qquad \begin{aligned} \dot{E}(t) =& -\left(\alpha - \frac{N}{2}\right)|u_1(t,0)|^2 - \frac{1}{2}\sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 \\ & - \sum_{j=1}^{N}\int_{I_j} a_j(x)|u_j(t,x)|^2 dx. \end{aligned}$$

Observe that, as $a_j \geq 0$, the term $a_j u_j$ provides dissipation to the energy, then $\dot{E}(t) \leq 0$. The main contribution of this work is to prove that indeed the terms $a_j u_j$ provide exponential stability of (LKdV).

The article is organized as follows. In Section II, the well-posedness of (LKdV) is proven using semi-group theory. In Section III a hidden regularity result is obtained. In Section IV, the stabilization problem is studied, and an observability inequality is used to prove exponential stability. Some concluding remarks and perspectives are collected in Section V.

II. Well-posedness of LKdV

In what follows, we use the well-known definitions of *classical* and *mild* solutions, we refer to [21, Chapter 4]. Note that (LKdV) can be written as

$$(4)\qquad \begin{cases} \underline{u}_t(t) = \mathcal{A}\underline{u}(t), \quad t>0 \\ \underline{u}(0) = \underline{u}^0, \end{cases}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\underline{u} = -(\partial_x + \partial_x^3 + \underline{a})\underline{u},$$

$$\mathcal{D}(\mathcal{A}) = \left\{\underline{u} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{j=1}^{N}\frac{d^2u_j}{dx^2}(0) = -\alpha u_1(0)\right\}.$$

Let $\underline{u} \in \mathcal{D}(\mathcal{A})$, then, after some integrations by parts

$$\begin{aligned} (\underline{u}, \mathcal{A}\underline{u})_{\mathbb{L}^2(\mathcal{T})} =& -\left(\alpha - \frac{N}{2}\right)|u_1(0)|^2 - \frac{1}{2}\sum_{j=1}^{N}|\partial_x u_j(0)|^2 \\ & - \sum_{j=1}^{N}\int_{I_j} a_j|u_j|^2 dx \\ \leq & 0. \end{aligned}$$

Easy calculations show that $\mathcal{A}^*$ is defined by

$$\mathcal{A}^*\underline{v} = (\partial_x + \partial_x^3 - \underline{a})\underline{v},$$

$$\mathcal{D}(\mathcal{A}^*) = \left\{\underline{v} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T}), \frac{dv_j}{dx}(0) = 0 \right. \\ \left. \forall j \in [\![1,N]\!], \sum_{j=1}^{N}\frac{d^2v_j}{dx^2}(0) = (\alpha - N)v_1(0)\right\}.$$

Similarly, we get that for all $\underline{v} \in \mathcal{D}(\mathcal{A}^*)$

$$\begin{aligned} (\underline{v}, \mathcal{A}^*\underline{v})_{\mathbb{L}^2(\mathcal{T})} =& -\left(\alpha - \frac{N}{2}\right)|v_1(0)|^2 - \frac{1}{2}\sum_{j=1}^{N_F}|\partial_x v_j(\ell_j)|^2 \\ & - \sum_{j=1}^{N}\int_{I_j} a_j|v_j|^2 dx \\ \leq & 0. \end{aligned}$$

[1]We consider solutions such that for $j \in]\!]N_F, N]\!]$ satisfies $u_j(\cdot,x) \to 0$ when $x \to \infty$.

Finally, $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, and $\mathcal{A}$ is a densely defined closed operator, thus by [21, Corollary 4.4, Chapter 1] $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on $\mathbb{L}^2(\mathcal{T})$. Systems (LKdV) and (4) are equivalent, thus we deduce the following result.

Theorem 1. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, then, there exists a unique mild solution $\underline{u} \in C([0,\infty);\mathbb{L}^2(\mathcal{T}))$ of* (LKdV). *Moreover, if $\underline{u}^0 \in \mathcal{D}(\mathcal{A})$, then $\underline{u}$ is a classical solution and $\underline{u} \in C([0,\infty);\mathcal{D}(\mathcal{A})) \cap C^1([0,\infty);\mathbb{L}^2(\mathcal{T}))$.*

III. Hidden regularity

The main idea of this section is to prove the following regularity result:

Proposition 1. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$. Consider $\underline{u}$ the associate mild solution of* (LKdV), *then, $\underline{u} \in \mathbb{X}^1$. Moreover, the following estimates hold*

- *There exists $C > 0$ such that, for all $j \in [\![1, N_F]\!]$;*

(5) $$\int_0^T \int_{I_j} |\partial_x u_j|^2 dx dt \leq C \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})},$$

- *For any $x_0 > 0$, there exists $C_{x_0} > 0$ such that, for all $j \in]\!]N_F, N]\!]$;*

(6) $$\int_0^T \int_{x_0}^{x_0+1} |\partial_x u_j|^2 dx dt \leq C_{x_0} \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}.$$

Proof. Let $j \in [\![1, N]\!]$, multiplying (LKdV) by u_j and integrating on $[0, s] \times I_j$, after some integrations by parts and summing from $j \in [\![1, N]\!]$ we get

$$\sum_{j=1}^N \int_{I_j} |u_j(s,x)|^2 dx + \sum_{j=1}^N \int_0^s |\partial_x u_j(t,0)|^2 dt$$
$$+ (2\alpha - N) \int_0^s |u_1(t,0)|^2 dt + 2 \sum_{j=1}^N \int_0^s \int_{I_j} a_j |u_j|^2 dx dt$$
$$= \sum_{j=1}^N \int_{I_j} |u_j(0,x)|^2 dx,$$

which gives us that

(7) $$\begin{aligned} &\|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} + (2\alpha - N)\|u_1(\cdot,0)\|^2_{L^2(0,T)} \\ &+ \|\partial_x \underline{u}(\cdot,0)\|^2_{L^2(0,T)} + 2\sum_{j=1}^N \int_0^T \int_{I_j} a_j |u_j|^2 dx dt \\ &\leq \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}. \end{aligned}$$

Similarly, multiplying the first line of (LKdV) by $x u_j$, integrating on $[0, T] \times I_j$ after some integrations by parts we get

(8) $$\begin{aligned} &\frac{1}{2}\int_{I_j} x|u_j(T,x)|^2 dx + \frac{3}{2}\int_0^T \int_{I_j} |\partial_x u_j|^2 dx dt \\ &+ \int_0^T \int_{I_j} a_j x |u_j|^2 dx dt = \frac{1}{2}\int_{I_j} x|u_j^0|^2 dx \\ &+ \frac{1}{2}\int_0^T \int_{I_j} |u_j|^2 dx dt - \int_0^T u_1(t,0)\partial_x u_j(t,0) dt. \end{aligned}$$

We get from (8) for all $j \in [\![1, N_F]\!]$,

$$\begin{aligned} \int_0^T \int_{I_j} |\partial_x u_j|^2 dx dt \leq & C \Big(\|u_j^0\|^2_{L^2(I_j)} + \|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} \\ &+ \|u_1(\cdot,0)\|^2_{L^2(0,T)} \\ &+ \|\partial_x u_j(\cdot,0)\|^2_{L^2(0,T)} \Big). \end{aligned}$$

In this case, we derive (5) by using (7). We focus now on the case, $j \in]\!]N_F, N[\![$. Inspired by [22, Theorem 2.1], let $x_0 > 0$ and K_{1,x_0}, $K_{1,x_0} > 0$ depending only x_0. Consider $\psi \in C^\infty(\mathbb{R})$ an increasing function such that (see Figure 2)

(9) $$\begin{cases} \psi(x) = 0, & \text{for } x \leq \frac{x_0}{2}, \\ \psi(x) = 1, & \text{for } x \geq \frac{3x_0}{2} + 1, \\ \psi'(x) \geq K_{1,x_0}, & \text{for } x \in [x_0, x_0+1], \\ \psi'(x) \geq 0, \ |\psi^{(k)}(x)| \leq K_{2,x_0}, & \text{for } x \in \mathbb{R}, k \in [\![0, 3]\!]. \end{cases}$$

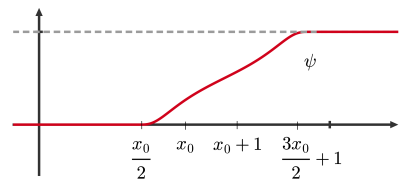

Fig. 2. Graph of the function $\psi(x)$.

Multiplying the $j - th$ equation of (LKdV) by $u_j(t,x)\psi(x)$, and integrating over $(0,\infty)$ we get after some integrations by parts

(10) $$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^\infty |u_j|^2 \psi(x) dx + \frac{3}{2}\int_0^\infty |\partial_x u_j|^2 \psi'(x) dx \\ &+ \int_0^\infty a_j |u_j|^2 \psi(x) dx = \frac{1}{2}\int_0^\infty |u_j|^2 (\psi'(x) + \psi'''(x)) dx. \end{aligned}$$

Recalling the definition of ψ (9), we observe that

$$\begin{aligned} K_{1,x_0} \int_{x_0}^{x_0+1} |\partial_x u_j|^2 dx \leq & \int_{x_0}^{x_0+1} |\partial_x u_j|^2 \psi'(x) dx \\ \leq & \int_0^\infty |\partial_x u_j|^2 \psi'(x) dx. \end{aligned}$$

With this in mind and (9) we obtain from (10)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^\infty |u_j|^2 \psi(x) dx + \frac{3}{2}K_{1,x_0}\int_0^\infty |\partial_x u_j|^2 \psi'(x) dx \\ = \frac{3}{2}K_{2,x_0}\int_0^\infty |u_j|^2 dx. \end{aligned}$$

We conclude the proof of (6) integrating t between $[0, T]$ and using (7). □

Remark 1. *We can build a function ψ satisfying* (9) *in the following way: consider the bump function $\kappa \in C^\infty(\mathbb{R})$ defined by $\kappa(x) = e^{-\frac{1}{x}}$, for $x > 0$ and $\kappa(x) = 0$ for $x \leq 0$. Then, letting*

$$\psi(x) = \frac{\kappa(x - \frac{x_0}{2})}{\kappa(x - \frac{x_0}{2}) + \kappa(\frac{3x_0}{2} + 1 - x)},$$

it is not difficult to check that the above function satisfies all the hypotheses of (9).

IV. Exponential stability

In this section, we prove our results related with the exponential stability in $\mathbb{L}^2(\mathcal{T})$. First, note that to prove the exponential stability, it is enough to prove the following observability inequality, with E defined in (2),

$$E(0) \leq C_{obs} \int_0^T \left(|u_1(t,0)|^2 + \sum_{j=1}^N |\partial_x u_j(t,0)|^2 + \sum_{j=1}^N \int_{I_j} a_j |u_j|^2 dx \right) dt, \tag{11}$$

for a suitable $C_{obs} > 0$ that does not depend on $\underline{u}$. Indeed, using (11) and dissipation law (3) we can show the existence of $0 < \gamma < 1$ such that $E(t) \leq \gamma E(0)$, finally as (LKdV) is invariant by translation in time, we derive the exponential decay. This idea was used in several works as [10], [11], [17], [23]. We recall the set of critical lengths for the KdV equation $\mathcal{N}$ introduced by Rosier in [24] and defined by

$$\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N}^* \right\},$$

and we define $I_c = \{j \in [\![1, N_F]\!];\ \ell_j \in \mathcal{N}\}$ the set of critical lengths, $I_\infty =]\!]N_F, N]\!]$ and I_c^* (resp I_∞^*) be the subset of I_c (resp I_∞) where we remove one index. Now, we establish our main results.

Theorem 2. *Let $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$ or $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j \in [\![1,N]\!]}$ satisfy* (1). *Then, there exist C, $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, the energy of the unique mild solution of* (LKdV) *satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

Remark 2. ***About the choice of action index:*** *Theorem 2 states that to achieve the exponential stability we can act as any of the following cases:*

- *Take active damping in all except one branches with critical length and in all the half-lines. This is case $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$.*
- *Take active damping in all except one half-lines a and in all the branches with critical length. This is case $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$.*

This action set is inspired by [8] where in the case of N_F KdV equations on a star-shaped network, an exponential stability result was proved with set of action index I_c^*. Moreover as we will see in Proposition 2, this choice is optimal.

Proof. As we said at the beginning of Section IV, it is enough to prove (11). To prove it, we follow a contradiction argument as in [24]. Suppose that (11) is false, then there exists $(\underline{u}^{0,n})_{n\in\mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,n}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and such that

$$\begin{aligned} &\|u_1^n(t,0)\|_{L^2(0,T)} \longrightarrow 0, \\ &\|\partial_x \underline{u}^n(t,0)\|_{L^2(0,T)} \longrightarrow 0, \\ &\sum_{j=1}^N \int_0^T \int_{I_j} a_j |u_j^n|^2 dxdt \longrightarrow 0, \text{ as } n \to \infty, \end{aligned} \tag{12}$$

where $\underline{u}^n$, is the unique mild solution of (LKdV) given by Theorem 1 with initial data $\underline{u}^{0,n}$. By Proposition 1, we get that $(\underline{u}^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^1$ and as $\partial_t u_j^n = -\partial_x u_j^n - \partial_x^3 u_j^n - a_j u_j^n$, we get that $(\partial_t u_j^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^{-2}$. Using [25, Corollary 4] we can extract a subsequence still denoted by $(\underline{u}^n)_{n\in\mathbb{N}}$ which converges in $\mathbb{X}^0$. The idea now is to see that $(\underline{u}^{0,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{T})$. Multiplying the first line of (LKdV) by $(T-t)u_j$ and integrating on $[0,T] \times I_j$, after some integrations by parts we can get

$$\begin{aligned} \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C \bigg(&\frac{1}{T} \|\underline{u}\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} + \|u_1(\cdot,0)\|^2_{L^2(0,T)} \\ &+ \|\partial_x \underline{u}(\cdot,0)\|^2_{L^2(0,T)} \\ &+ \sum_{j=1}^N \int_0^T \int_{I_j} a_j |u_j|^2 dxdt \bigg). \end{aligned} \tag{13}$$

Thus, as $(\underline{u}^n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{X}^0$, with (12) and (13), we get that $(\underline{u}^{0,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{T})$. Let $\underline{u}^0 = \lim_{n\to\infty} \underline{u}^{0,n}$ and $\underline{u}$ the unique mild solution of (LKdV) associated to $\underline{u}^0$. Then, $\underline{u}$ is the solution of

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in I_j, \\ & t \in (0,T),\ j \in [\![1,N]\!], \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j \in [\![1,N]\!], \\ \sum_{j=1}^N \partial_x^2 u_j(t,0) = 0, & t \in (0,T), \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j \in [\![1,N_F]\!], \\ u_j \equiv 0 \text{ in } (0,T) \times \omega_j, & j \in \mathbb{I}_{act}, \\ u_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!], \\ \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1. \end{cases} \tag{14}$$

Here we have two cases:

∘ $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$. In this case, for $j \in I_\infty$, $w = u_j$ solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\ w(t,0) = \partial_x w(t,0) = 0, & t \in (0,T), \\ w \equiv 0 \text{ in } (0,T) \times \omega_j. \end{cases}$$

Then, by Holmgren's Theorem [26] (see also [17, Theorem 1.1]), $w \equiv 0$ in $(0,\infty) \times (0,T)$. Then, we have the following remaining problem

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T), \\ & j \in [\![1,N_F]\!], \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j \in [\![1,N_F]\!], \\ \sum_{j=1}^{N_F} \partial_x^2 u_j(t,0) = 0, & t \in (0,T), \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j \in [\![1,N_F]\!], \\ u_j \equiv 0 \text{ in } (0,T) \times \omega_j, & j \in I_c^*, \\ \sum_{j=1}^{N_F} \|u_j^0\|^2_{L^2(0,\ell_j)} = 1. \end{cases}$$

The above system is exactly the same studied in [8]. Thus, by [8, Theorem 3.1] as we are acting in I_c^* we get $u_j \equiv 0$ for $j \in [\![1,N_F]\!]$ and finally, $\underline{u} \equiv 0$ which is a contradiction with the fact $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1$.

$\circ$ $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$. Let $j \in I_\infty^*$, the same argument used in the previous case shows that $u_j \equiv 0$ in $(0,\infty)\times(0,T)$. Similarly, by Holmgren's Theorem [26], for all $j \in I_c$ $u_j \equiv 0$ in $(0,\ell_j)\times(0,T)$. Now, for $j \in [\![1,N_F]\!] \cap (I_c)^c$, u_j solves

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T), \\ u_j(t,0) = \partial_x u_j(t,0) = 0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0. \end{cases}$$

Then as $\ell_j \notin \mathcal{N}$ by [24, Lemma 3.5], $u_j \equiv 0$ in $(0,\ell_j)\times(0,T)$. Finally, let j the unique integer in $]\!]N_F,N]\!] \cap (I_\infty^*)^c$, then $w = u_j$ solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\ w(t,0) = \partial_x w(t,0) = 0, & t \in (0,T), \\ \partial_x^2 w(t,0) = 0, & t \in (0,T). \end{cases}$$

It is enough to see that, due to the three null boundary conditions in 0[2], the unique solution is $w \equiv 0$. This conclude the proof of Theorem 2. $\square$

Theorem 2 is optimal in number of action branches. For instance, if we take a smaller set of action index, we cannot derive the result as shows the next proposition.

Proposition 2. *Let $\mathbb{I}_{act} \subsetneq I_c^* \cup I_\infty$ or $\mathbb{I}_{act} \subsetneq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j\in[\![1,N]\!]}$ satisfy* (1). *Then, there exists a nontrivial solution of* (14). *In particular, there exists a solution to* (LKdV) *different to $\underline{0}$, that does not converge to $\underline{0}$ as t tends to infinity and thus* (LKdV) *is not asymptotically stable with this weak assumption.*

Proof. It is enough to consider the case where we remove one index more in the action sets defined in the assumption of Theorem 2. Let I_c^{**} (resp I_∞^{**}) be the subset of I_c (resp I_∞) where we remove two indexes. Let us show that, under these conditions, there exists a solution of (14).

$\circ$ $\underline{\mathbb{I}_{act} = I_c^{**} \cup I_\infty}$. We get for all $j \in I_\infty$ $u_j \equiv 0$. Thus, we get the optimality from [8, Lemma 3.2].

$\circ$ $\underline{\mathbb{I}_{act} = I_c^* \cup I_\infty^*}$. Following the computations of the proof of Theorem 2, we obtain the following system,

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in I_j, t \in (0,T), j = 1,2, \\ u_1(t,0) = \partial_x u_1(t,0) = 0, & t \in (0,T), \\ u_2(t,0) = \partial_x u_2(t,0) = 0, & t \in (0,T), \\ u_1(t,2\pi) = \partial_x u_1(t,2\pi) = 0, & t \in (0,T), \\ \partial_x^2 u_1(t,0) + \partial_x^2 u_1(t,0) = 0, & t \in (0,T), \\ u_j(0,x) = u_j^0(x), & x \in I_j, j = 1,2, \\ \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1. \end{cases}$$

Consider the stationary functions $u_1(x) = K_1(1-\cos(x))$ and $u_2(x) = K_2(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_2(x) = 0$ for $x \geq 2\pi$. We observe that u_1 and u_2 satisfy a linear KdV equation. Moreover, $u_1(0) = \partial_x u_1(0) = u_1(2\pi) = \partial_x u_1(2\pi) = 0$, $u_2(0) = \partial_x u_2(0) = 0$ and $\partial_x^2 u_1(0) + \partial_x^2 u_1(0) = K_1 + K_2$. Therefore, if $K_1 = -K_2$ we get a nontrivial solution of (14).

$\circ$ $\underline{\mathbb{I}_{act} = I_c \cup I_\infty^{**}}$. Consider the case $N_F = 0$. Then we obtain the system

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\infty), t \in (0,T), j = 1,2, \\ u_1(t,0) = \partial_x u_1(t,0) = 0, & t \in (0,T), \\ u_2(t,0) = \partial_x u_2(t,0) = 0, & t \in (0,T), \\ \partial_x^2 u_1(t,0) + \partial_x^2 u_1(t,0) = 0, & t \in (0,T), \\ u_j(0,x) = u_j^0(x), & x \in (0,\infty), \\ \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1. \end{cases}$$

Similarly as in the previous case, $u_1(x) = K_1(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_1(x) = 0$ for $x \geq 2\pi$, $u_2(x) = K_2(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_2(x) = 0$ for $x \geq 2\pi$, such that $K_1 = -K_2$. Therefore in both cases we get a solution to (LKdV) different to $\underline{0}$ with does not converge to $\underline{0}$ as $t \to \infty$ as stated in Proposition 2. $\square$

V. Conclusion

This paper has studied the exponential stabilization of a linear KdV equation defined on a star-shaped network in the case where the branches model a bounded domain or the half-line. The well-posedness was shown by using linear semigroup theory and a hidden regularity result was also proved. The strategy to prove the exponential stability was to use this hidden regularity and compactness ideas in a contradiction argument. The optimality of the stabilization result in the sense of action edges was proved too.

Some future research lines include to extend this results in the case of the nonlinear KdV equation and implement a numerical scheme to illustrate our results. This both problems are challenging, in the case of the nonlinear KdV equation defined on a star-network [8] in addition to the internal nonlinearity, we have a nonlinear term in the transmission condition, that makes well-posedness and stabilization more complicated. In the case of a numerical scheme, even in the case of a single KdV equation defined on the half-line, it is not an easy problem. As was pointed in [27] this arises the issue of cutting the spatial domain and thus add two extra boundary conditions, by the propagation direction of the KdV equation the model only applies in a short time scale.

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APPENDIX

This appendix recaps some definitions used along the paper. Let $a < b$, two real numbers, we set

$$[\![a,b]\!] = \mathbb{N} \cap [a,b], \ \ ]\!]a,b]\!] = \mathbb{N} \cap (a,b].$$

Let $K = \{k_j : j \in [\![1,N]\!]\}$ be the set of the $N = N_F + N_\infty$ edges of a network $\mathcal{T}$ described as the intervals I_j for $j \in [\![1,N]\!]$, where

$$\begin{cases} I_j = (0,\ell_j) \text{ with } \ell_j > 0 & j \in [\![1,N_F]\!], \\ I_j = (0,\infty) & j \in]\!]N_F,N]\!]. \end{cases}$$

The network $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{j=1}^N k_j$. We introduce the product spaces: $\mathbb{H}^3(\mathcal{T}) = \prod_{j=1}^N H^3(I_j)$, $\mathbb{L}^\infty(\mathcal{T}) = \prod_{j=1}^N L^\infty(I_j)$ and $\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^N L^2(I_j)$, endowed with

$$(\underline{u},\underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^N \int_{I_j} u_j v_j dx, \quad \forall \underline{u},\underline{v} \in \mathbb{L}^2(\mathcal{T}).$$

Let $s = 1,2$ and for $j \in [\![1,N_F]\!]$ consider the space

$$H_r^s(I_j) = \left\{ v \in H^s(I_j), \ \left(\frac{d}{dx}\right)^{i-1} v(\ell_j) = 0, i \in [\![1,s]\!] \right\},$$

where the index r is related to the null right boundary conditions, and the space $\mathbb{H}_e^s(\mathcal{T})$ defined by

$$\mathbb{H}_e^s(\mathcal{T}) = \Bigg\{ \underline{u} = (u_1,\ldots,u_N)^\top \in \prod_{j=1}^{N_F} H_r^s(I_j) \times \prod_{j=N_F+1}^{N} H^s(I_j), \\ u_j(0) = u_{j'}(0), j, j' \in [\![1,N]\!] \Bigg\}, \text{with } s = 1,2,$$

$$\|\underline{u}\|^2_{\mathbb{H}_e^1(\mathcal{T})} = \sum_{j=1}^N \|u_j\|^2_{H^1(I_j)}, \text{ for } s = 1.$$

For $s = -2,-1,0,1$ we set

$$\mathbb{X}^s = \prod_{j=1}^{N_F} L^2(0,T;H^s(0,\ell_j)) \times (L^2(0,T;H^s_{loc}(0,\infty)))^{N_\infty}.$$