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Penalized Least-Squares Method for LQR Problem of Singular Systems

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Abstract— **The linear quadratic regulator (LQR) algorithms devised using the Riccati equation possess two key attributes—they are recursive and have easily met conditions of existence. Nevertheless, these features only apply for the transformed structure of the regulated dynamics in singular systems, otherwise their optimal performance will be compromised under violation of constraints in non-singular versions. This technical note presents the LQR problem for a time-varying discrete linear singular system in a direct manner avoiding any transformations. This approach eliminates the requirement of making regularity assumptions for the system. To achieve this, first, we formulate a quadratic cost function for LQR derivation based on a penalized weighted least-squares method. Then, by using Bellman's principle of optimality and performing variable substitutions, we connect the formulation to a constrained and recursive minimization problem. We then proceed with investigating the existence conditions and using dynamic programming in a backward strategy at the finite horizon to derive a recursive regulator algorithm for the original system in a matrix array framework, without degrading its optimal performance. The achieved algorithm has more general features compared to the classical LQR problem for standard systems. This study concludes with numerical evaluation of the algorithm and confirmation of the results.**

I. INTRODUCTION

Studies on singular systems—also known as *descriptor* and *semi-state* systems—have mainly been conducted in the realm of mathematics [1], yet recently singular systems have received significant recognition in various fields of engineering and industry such as power systems, robotics, biology, and economics [2]. The complexity of these systems leads to a number of challenges when attempting to analyze and synthesize them for control purposes [3]. Nevertheless, scientists relying on the latest advancements in mathematical tools have been able to handle the challenges and extend some of investigations [4], such as existence and solvability [5], stability [6], observer design [7] and filtering [8] problems, which have enabled the transition from classical control to those achieved with singular techniques. To describe more complex dynamics along with long interconnection properties and constraint elements [9], nonlinear [10] and fractional [11] dynamics have been considered in the studies together with singular systems.

The linear quadratic regulator (LQR) problem with a singular system is an extension of the classic LQR problem, which is a highly significant category of optimal control problems in theory and practice [12]. This problem involves finding a control algorithm that minimizes a quadratic objective function in linear form, while considering either continuous or discrete systems [13]–[16]. A significant amount of research has been conducted in the field of solving the LQR problem subject to singular systems. In [17], by performing singular value decomposition (SVD) on the singular matrix, the optimization problem was solved by applying the Hamiltonian strategy for a standard system. Using a similar strategy coupled with transforming the original system to a coordinate system using SVD, the LQR problem and its associated Riccatti differential equation were investigated in [18] for a continuous singular system. Relying on the equivalence transformation technique, studies were conducted where some developments and analyses related to the existence [19], uniqueness [20], and stability [21] of solutions, and extension to rectangular [22] and fractional order [23] cases with cross product of performance and actuator effort were considered.

However, most of the results take into account the regularity requirement of the singular system and the requirement for transformation into a restricted system equivalent using the SVD technique. This can cause the optimal performance of the developed LQR algorithms to be compromised under violation of constraints in non-singular versions. For example, possible faults can cause sudden shifts in structure or layout of a system (like a power line outage in power grids), the algebraic conditions become different, thus the transformed systems associated with the constraints are no longer able to sync with each other [24]. This posed a major challenge in convincing researchers to keep the singular structure intact and utilize it for different control problems. In the light of this gap, to prevent the efficiency of LQR algorithm from degrading and make it more practical, we propose here a recursive algorithm for LQR problem of linear singular systems by using penalized weighted least-squares approach [25] and performing the following steps:

- Formulate a quadratic cost function for LQR derivation based on a penalized weighted least-squares method.
- Connect the formulation to a recursive constrained minimization problem based on Bellman's principle;
- Investigate the existence conditions for the constructed minimization problem;
- Derive a recursive regulator algorithm using backward dynamic programming at the finite horizon.

Compared with the existing LQR algorithms, our computationally simple algorithm is designed to minimize the

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regularized residual norm in its penalized worst-case scenario in which no singular problem needs to be solved in the derivation process. Also, we demonstrate that no regularity assumption of the system is required in this process, and the algorithm is directly derived subject to the original system instead of a restricted equivalent form of the system. Furthermore, the control law, feedback gain, and Riccati equation are derived in an array of matrices which can motivate us to develop an algorithm for the recursive optimal solution of robust LQRs in future research.

Notations: $\mathbb{R}$ denotes the real numbers set, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ hold out n-dimensional vector and $n\times m$ matrix, respectively, whose elements are in $\mathbb{R}$, $A^\top$ is the transpose of the matrix A, $A = A^\top > 0$ denotes symmetric positive definite (PD) matrix, $\mathbf{I}_n$ is the identity matrix of order n, $(x \oplus y)$ and $\text{col}\{x, y\}$ denote a block diagonal matrix and a column vector, respectively, with entries x and y.

II. PROBLEM STATEMENT

Consider the state-space representation of the time varying discrete linear singular model as

$$\mathbf{E}_{k+1}\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k \tag{1}$$

for $0 \le k \le N$, where $\mathbf{x}_k \in \mathbb{R}^n$ is a state vector, $\mathbf{u}_k \in \mathbb{R}^m$ is an input control vector, $\mathbf{A}_k \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_k \in \mathbb{R}^{n\times m}$ are assumed nominal parameter known matrices, $\mathbf{x}_0$ is the initial state vector, and $\{\mathbf{u}_k\}_{k=0}^N$ is an unrestricted sequence of control inputs. Also, $\mathbf{E}_k \in \mathbb{R}^{n\times n}$ is a singular matrix, i.e., $\operatorname{rank}\mathbf{E}_k < n$, for $0 \le k \le N$. To formulate the LQR problem subject to (1), define the auxiliary expression

$$\mathbf{L}_i(\mathbf{x}_i, \mathbf{u}_i) = \mathbf{x}_i^\top\mathbf{Q}_i\mathbf{x}_i + \mathbf{u}_i^\top\mathbf{R}_i\mathbf{u}_i \tag{2}$$

for $0 \le i \le N$, and the well-known quadratic functional as

$$\mathbf{J} = \mathbf{x}_{N+1}^\top\mathbf{P}_{N+1}\mathbf{x}_{N+1} + \sum_{i=0}^{N}\mathbf{L}_i(\mathbf{x}_i, \mathbf{u}_i), \tag{3}$$

where the weights $\mathbf{Q}_i$, $\mathbf{R}_i$, and $\mathbf{P}_{N+1}$ are symmetric PD matrices. Then, consider the optimization problem

$$\min_{\mathbf{x}_{k+1},\mathbf{u}_k}\{\mathbf{x}_{N+1}^\top\mathbf{P}_{N+1}\mathbf{x}_{N+1} + \sum_{i=0}^{N}\mathbf{L}_i(\mathbf{x}_i, \mathbf{u}_i)\} \tag{4}$$

subject to (1). The objective is to determine an optimal sequences $\{\mathbf{x}_{k+1}^*\}_{k=0}^N$ and $\{\mathbf{u}_k^*\}_{k=0}^N$ in order to minimize (3). The proposed problem of optimal control differs from typical formulations as the minimization is not just based on the $\mathbf{u}_k$ parameter but also on the $\mathbf{x}_{k+1}$.

III. TECHNICAL RESULTS

Here, some supporting lemmas, which provide useful links in the solution of the optimization problem and that will lead us to obtaining the recursive optimal solution of the LQR problem, will be presented. Consider the following minimization problem

$$\min_{\mathbf{x}}\{(\mathbf{F}\mathbf{x} - \mathbf{g})^\top\mathbf{V}(\mathbf{F}\mathbf{x} - \mathbf{g})\}, \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^m$ is the unknown vector, and $\mathbf{F} \in \mathbb{R}^{n\times m}$, $\mathbf{g} \in \mathbb{R}^n$, and $\mathbf{V}$ are respectively known information matrix, measurement vector, and a symmetric PD weighting matrix. We denote the optimal solution of (5) as $\hat{\mathbf{x}}$.

Lemma 1 ([26]): Let $\mathbf{V} \in \mathbb{R}^{n\times n}$ be a PD matrix and $\mathbf{F} \in \mathbb{R}^{n\times m}$ be a full column rank matrix. Then, if $\begin{bmatrix}\mathbf{V}^{-1} & \mathbf{F}\end{bmatrix}$ has full row rank, the matrix

$$\begin{bmatrix}\mathbf{V}^{-1} & \mathbf{F}\\ \mathbf{F}^\top & \mathbf{0}\end{bmatrix} \tag{6}$$

is invertible. Also, $\xi = \mathbf{F}^\top\mathbf{V}\mathbf{F}$ is invertible, and

$$\begin{bmatrix}\xi^{-1} & \xi^{-1}\mathbf{F}^\top\mathbf{V}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{I}_m\end{bmatrix}^\top \begin{bmatrix}\mathbf{V}^{-1} & \mathbf{F}\\ \mathbf{F}^\top & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{0} & \mathbf{I}_n\\ -\mathbf{I}_m & \mathbf{0}\end{bmatrix}. \tag{7}$$

We observe that the optimization quadratic problem (5) can admit a more suitable representation with respect to the optimal solution structure. This alternative representation is presented in the following lemma which will be a fundamental approach towards a recursive solution in this study.

Lemma 2 ([26]): Let the matrix (6) be invertible. Therefore, according to the maximum likelihood linear estimation approach and Lagrange multipliers, the optimal solution of (5) can be obtained by solving the equation

$$\begin{bmatrix}\mathbf{V}^{-1} & \mathbf{F}\\ \mathbf{F}^\top & \mathbf{0}\end{bmatrix}\begin{bmatrix}\lambda\\ \mathbf{x}\end{bmatrix} = \begin{bmatrix}\mathbf{I}_n\\ \mathbf{0}\end{bmatrix}\mathbf{g}, \tag{8}$$

which can be uniquely given by

$$\hat{\mathbf{x}} = \begin{bmatrix}\mathbf{0} & \mathbf{I}_m\end{bmatrix}\begin{bmatrix}\mathbf{V}^{-1} & \mathbf{F}\\ \mathbf{F}^\top & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{0}\end{bmatrix}\mathbf{g}. \tag{9}$$

In what follows and through the use of the penalty function technique, the optimal solution of the quadratic optimization problem (5) subject to a constraint will be investigated. In general, the penalty function technique is a procedure for approximating the constrained optimization problems. The approximation is performed by adding a term in objective function that imposes a high cost for violating the constraints of the problem. In this regard, a parameter μ plays a pivotal role in shaping the penalty's severity, thereby influencing how closely the unconstrained problem approximates the original problem. When $\mu \to \infty$, the accuracy of the approximation improves. Details of using this technique are generally found in the literature dealing with nonlinear theory with regard to constrained optimization methods [27].

Lemma 3 ([28], [29]): Consider the optimization problem (5) subject to the constraint $\psi\mathbf{x} = \theta$, where $\psi \in \mathbb{R}^{k\times m}$ and $\theta \in \mathbb{R}^{k\times 1}$. Using an auxiliary varying parameter μ, one can transform this minimization problem into an unconstrained optimization problem as

$$\hat{\mathbf{x}}_\mu = \arg\min_{\mathbf{x}}(\mathcal{F}\mathbf{x} - \mathcal{G})^\top\mathcal{V}_\mu(\mathcal{F}\mathbf{x} - \mathcal{G}), \tag{10}$$

where $\mu > 0$, and

$$\mathcal{F} = \begin{bmatrix}\mathbf{F}\\ \psi\end{bmatrix}, \mathcal{V}_\mu = \begin{bmatrix}\mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mu\mathbf{I}\end{bmatrix}, \mathcal{G} = \begin{bmatrix}\mathbf{g}\\ \theta\end{bmatrix}. \tag{11}$$

By Lemma 2, the optimum solution can be obtained as

$$\hat{\mathbf{x}}_\mu = \begin{bmatrix}\mathbf{0}\\ \mathbf{I}_m\end{bmatrix}^\top\begin{bmatrix}\mathcal{V}_\mu^{-1} & \mathcal{F}\\ \mathcal{F}^\top & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I}_{n+k}\\ \mathbf{0}\end{bmatrix}\mathcal{G},$$

where $\lim_{\mu\to\infty} \|\mathcal{F}\hat{\mathbf{x}}_\mu - \mathcal{G}\|^2_{\mathcal{V}} = \|\hat{\mathbf{x}}^*\|^2_{\mathbf{V}}$ with $\lim_{\mu\to\infty} \hat{\mathbf{x}}_\mu = \hat{\mathbf{x}}^*$. Here, $\hat{\mathbf{x}}^*$ is a solution of the optimization problem (5) subject to $\psi\mathbf{x} = \theta$ given as

$$\hat{\mathbf{x}}^* = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{I}_m\end{bmatrix}^\mathsf{T} \begin{bmatrix}\mathbf{V}^{-1} & \mathbf{0} & \mathbf{F}\\ \mathbf{0} & \mathbf{0} & \psi\\ \mathbf{F}^\mathsf{T} & \psi^\mathsf{T} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{g}\\ \theta\\ \mathbf{0}\end{bmatrix}. \tag{12}$$

Lemma 3 is inspired by a result presented in [30] and its proof is a direct application of the convergence theorem of the combined penalty function with Lemma 2. The following key result presents the structure of the optimal solution for a case of optimization problem subject to a constraint that fits the form of the problem addressed in the Lemma 3.

Lemma 4: Let $\mathbf{P} \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, and $\mathbf{R} \in \mathbb{R}^{m\times m}$ are known symmetric PD matrices, and $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$ are given matrices. Assume $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ are unknown vectors and $z \in \mathbb{R}^n$ is a given vector. Consider the minimization problem of the form

$$\mathcal{U}^* := \min_{\mathbf{x},\mathbf{u}} \{\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x} + \mathbf{z}^\mathsf{T}\mathbf{Q}\mathbf{z} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u}\}, \tag{13}$$

where $\mathcal{U}^* = \mathrm{col}\{\mathbf{x}^*, \mathbf{u}^*\}$, subject to $\mathbf{E}\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}$. Then, the optimal solution $\mathcal{U}^*$ is given by

$$\mathcal{U}^* = \mathcal{K}\mathbf{z}, \tag{14}$$

where $\mathcal{K} = \mathrm{col}\{\mathbf{L}_\mathbf{x}, \mathbf{K}_\mathbf{u}\}$, or equivalently,

$$\mathcal{K} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathcal{Q} & \mathcal{E}\\ \mathcal{E}^\mathsf{T} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathcal{A}\\ \mathbf{0}\end{bmatrix}, \tag{15}$$

where $\mathcal{Q} = (\mathcal{P} \oplus \mu)^{-1}$ with $\mu \to \infty$, $\mathcal{P} = \mathbf{P} \oplus \mathbf{R} \oplus \mathbf{Q}$, and

$$\mathcal{A} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ -\mathbf{I}\\ \mathbf{A}\end{bmatrix}, \qquad \mathcal{E} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{E} & -\mathbf{B}\end{bmatrix}.$$

Furthermore,

$$\begin{aligned}&\min_{\mathbf{x},\mathbf{u}} \{\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x} + \mathbf{z}^\mathsf{T}\mathbf{Q}\mathbf{z} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u}\}\\ &\quad \text{s.t.} \quad \mathbf{E}\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}\end{aligned} \tag{16}$$

is equal to $\mathbf{z}^\mathsf{T}\mathbf{S}\mathbf{z}$, where $\mathbf{S}$ is a PD matrix defined by $\mathbf{S} = \mathbf{L}_\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{L}_\mathbf{x} + \mathbf{K}_\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{K}_\mathbf{u} + \mathbf{Q}$.

Proof: It follows from Lemma 2 and rewriting (13) as

$$\begin{aligned}\begin{bmatrix}\mathbf{x}^*\\ \mathbf{u}^*\end{bmatrix} &:= \min_{\mathbf{x},\mathbf{u}} \Bigg\{ \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}^\mathsf{T} \begin{bmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}\\ &+ \left(\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{E} & -\mathbf{B}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} - \begin{bmatrix}-\mathbf{I}\\ \mathbf{A}\end{bmatrix}\mathbf{z} \right)^\mathsf{T} \begin{bmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &\times \left(\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{E} & -\mathbf{B}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} - \begin{bmatrix}-\mathbf{I}\\ \mathbf{A}\end{bmatrix}\mathbf{z} \right) \Bigg\}\\ &= \min_{\mathbf{x},\mathbf{u}} \left\{ \left(\mathcal{E}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} - \mathcal{A}\mathbf{z} \right)^\mathsf{T} \begin{bmatrix}\mathcal{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \left(\mathcal{E}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} - \mathcal{A}\mathbf{z} \right) \right\}.\end{aligned} \tag{17}$$

Also, by substituting the optimal solution (14) in (17), the minimum value can be calculated as

$$\begin{aligned}\mathbf{J}^* &= \left(\mathcal{E}\begin{bmatrix}\mathbf{L}_\mathbf{x}\\ \mathbf{K}_\mathbf{u}\end{bmatrix}\mathbf{z} - \mathcal{A}\mathbf{z} \right)^\mathsf{T} \begin{bmatrix}\mathcal{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \left(\mathcal{E}\begin{bmatrix}\mathbf{L}_\mathbf{x}\\ \mathbf{K}_\mathbf{u}\end{bmatrix}\mathbf{z} - \mathcal{A}\mathbf{z} \right)\\ &= \left(\begin{bmatrix}\mathbf{L}_\mathbf{x}\\ \mathbf{K}_\mathbf{u}\\ \mathbf{I}\\ \mathcal{C}\end{bmatrix}\mathbf{z} \right)^\mathsf{T} \begin{bmatrix}\mathcal{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \left(\begin{bmatrix}\mathbf{L}_\mathbf{x}\\ \mathbf{K}_\mathbf{u}\\ \mathbf{I}\\ \mathcal{C}\end{bmatrix}\mathbf{z} \right)\\ &= \mathbf{z}^\mathsf{T}(\mathbf{L}_\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{L}_\mathbf{x} + \mathbf{K}_\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{K}_\mathbf{u} + \mathbf{Q})\mathbf{z} = \mathbf{z}^\mathsf{T}\mathbf{S}\mathbf{z},\end{aligned} \tag{18}$$

where $\mathcal{C} = \mathbf{E}\mathbf{L}_\mathbf{x} - \mathbf{B}\mathbf{K}_\mathbf{u} - \mathbf{A}$. ■

IV. LQR DERIVATION

The approach to solve the constrained problem (3) subject to (1) and derive LQR algorithm becomes quite simplified if one assumes that every optimal solution must satisfy the Bellman's principle of optimality [31]. By this principle, the optimization problem can be dealt recursively by minimizing the form enunciated in the following lemma.

Lemma 5: Optimization problem (4) subject to the linear singular model (1) can be solved recursively thorough the minimization of the form

$$\begin{aligned}\min_{\mathbf{x}_1,\mathbf{u}_0} \{\mathbf{L}_0(\mathbf{x}_0, \mathbf{u}_0) &+ \min_{\mathbf{x}_2,\mathbf{u}_1} \{\mathbf{L}_1(\mathbf{x}_1, \mathbf{u}_1) + \cdots\\ &+ \min_{\mathbf{x}_j,\mathbf{u}_{j-1}} \{\mathbf{L}_{j-1}(\mathbf{x}_{j-1}, \mathbf{u}_{j-1}) + \cdots\\ &+ \min_{\mathbf{x}_{N+1},\mathbf{u}_N} \{\mathbf{L}_N(\mathbf{x}_N, \mathbf{u}_N)\\ &+ \mathbf{x}_{N+1}^\mathsf{T}\mathbf{P}_{N+1}\mathbf{x}_{N+1}\}\} \cdots \}\}\end{aligned} \tag{19}$$

subject to (1).

Proof: The proof follows directly from the principle of optimality. ■

According to Lemmas 4 and 5 and with the help of dynamic programming, we obtain the optimal recursive solution of LQR problem as stated in the following theorem.

Theorem 1: The optimal recursive solution (LQR algorithm) for the constrained optimization problem (4) subject to the linear singular model (1) is given by

$$\mathcal{U}^*_{k+1} = \mathcal{K}_k\mathbf{x}_k, \quad k = 0, \cdots, N, \tag{20}$$

where $\mathcal{U}^*_{k+1} = \mathrm{col}\{\mathbf{x}^*_{k+1}, \mathbf{u}^*_k\}$ and $\mathcal{K}_k = \mathrm{col}\{\mathbf{L}_{\mathbf{x}_k}, \mathbf{K}_{\mathbf{u}_k}\}$, or equivalently,

$$\mathcal{K}_k = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathcal{Q}_k & \mathcal{E}_{k+1}\\ \mathcal{E}^\mathsf{T}_{k+1} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathcal{A}_k\\ \mathbf{0}\end{bmatrix}, \tag{21}$$

where $\mathcal{Q}_k = \mathcal{P}_k^{-1} \oplus \mathbf{0}$ with $\mathcal{P}_k = \mathbf{P}_k \oplus \mathbf{R}_k \oplus \mathbf{Q}_k$, and

$$\mathcal{A}_k = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ -\mathbf{I}\\ \mathbf{A}_k\end{bmatrix}, \qquad \mathcal{E}_{k+1} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{E}_{k+1} & -\mathbf{B}_k\end{bmatrix},$$

and

$$\mathbf{P}_k = \mathbf{L}^\mathsf{T}_{\mathbf{x}_k}\mathbf{P}_{k+1}\mathbf{L}_{\mathbf{x}_k} + \mathbf{K}^\mathsf{T}_{\mathbf{u}_k}\mathbf{R}_k\mathbf{K}_{\mathbf{u}_k} + \mathbf{Q}_k. \tag{22}$$

Proof: The recursive solution can be obtained through the steps indicated below and evolved backward at finite horizon N:

Step $i = N$: In this step, the minimization problem

$$\min_{\mathbf{x}_{N+1},\mathbf{u}_N} \{\mathbf{x}_{N+1}^\mathsf{T}\mathbf{P}_{N+1}\mathbf{x}_{N+1} + \mathbf{x}_N^\mathsf{T}\mathbf{Q}\mathbf{x}_N + \mathbf{u}_N^\mathsf{T}\mathbf{R}_N\mathbf{u}_N\} \quad \text{s.t.} \quad \mathbf{E}_{N+1}\mathbf{x}_{N+1} = \mathbf{A}_N\mathbf{x}_N + \mathbf{B}_N\mathbf{u}_N \tag{23}$$

will be solved. By hypothesis, we have $\mathbf{Q}_N$, $\mathbf{R}_N$, and $\mathbf{P}_{N+1}$ are symmetric PD matrices. Then, by considering the following substitutions

$$\begin{aligned} &\mathbf{x} \leftarrow \mathbf{x}_{N+1}, \quad \mathbf{u} \leftarrow \mathbf{u}_N, \quad \mathbf{z} \leftarrow \mathbf{x}_N, \\ &\mathbf{P} \leftarrow \mathbf{P}_{N+1}, \quad \mathbf{R} \leftarrow \mathbf{R}_N, \quad \mathbf{Q} \leftarrow \mathbf{Q}_N, \\ &\mathbf{E} \leftarrow \mathbf{E}_{N+1}, \quad \mathbf{A} \leftarrow \mathbf{A}_N, \quad \mathbf{B} \leftarrow \mathbf{B}_N, \end{aligned} \tag{24}$$

we can connect the problem in this step to the optimization problem in Lemma 4. Therefore,

$$\mathcal{U}_{N+1}^* = \mathcal{K}_N\mathbf{x}_N, \tag{25}$$

where $\mathcal{U}_{N+1}^* = \text{col}\{\mathbf{x}_{N+1}^*, \mathbf{u}_N^*\}$ and $\mathcal{K}_N = \text{col}\{\mathbf{L}_{\mathbf{x}_N}, \mathbf{K}_{\mathbf{u}_N}\}$, or equivalently,

$$\mathcal{K}_N = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathcal{Q}_N & \mathcal{E}_{N+1} \\ \mathcal{E}_{N+1}^\mathsf{T} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathcal{A}_N \\ \mathbf{0}\end{bmatrix}, \tag{26}$$

where $\mathcal{Q}_N = \mathcal{P}_N^{-1} \oplus \mathbf{0}$ with $\mathcal{P}_N = \mathbf{P}_N \oplus \mathbf{R}_N \oplus \mathbf{Q}_N$, and

$$\mathcal{A}_N = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{A}_N\end{bmatrix}, \qquad \mathcal{E}_{N+1} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{N+1} & -\mathbf{B}_N\end{bmatrix}.$$

Also, by substituting the optimal solution (25) in (23), the minimum value can be calculated as $\mathbf{x}_N^\mathsf{T}\mathbf{S}_N\mathbf{x}_N$, where

$$\mathbf{S}_N = \mathbf{L}_{\mathbf{x}_N}^\mathsf{T}\mathbf{P}_{N+1}\mathbf{L}_{\mathbf{x}_N} + \mathbf{K}_{\mathbf{u}_N}^\mathsf{T}\mathbf{R}_N\mathbf{K}_{\mathbf{u}_N} + \mathbf{Q}_N. \tag{27}$$

Now, it is enough to consider $\mathbf{P}_N = \mathbf{S}_N$.

Step $i = N-1$: According to Lemma 5, the problem

$$\min_{\mathbf{x}_N,\mathbf{u}_{N-1}} \{\mathbf{L}_{N-1}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) + \mathbf{x}_N^\mathsf{T}\mathbf{S}_N\mathbf{x}_N\} \quad \text{s.t.} \quad \mathbf{E}_N\mathbf{x}_N = \mathbf{A}_{N-1}\mathbf{x}_{N-1} + \mathbf{B}_{N-1}\mathbf{u}_{N-1} \tag{28}$$

will be solved, where the term $\mathbf{x}_N^\mathsf{T}\mathbf{S}_N\mathbf{x}_N$ provides minimum solution carried out in the previous step. Again and by considering the following substitutions

$$\begin{aligned} &\mathbf{x} \leftarrow \mathbf{x}_N, \quad \mathbf{u} \leftarrow \mathbf{u}_{N-1}, \quad \mathbf{z} \leftarrow \mathbf{x}_{N-1}, \\ &\mathbf{P} \leftarrow \mathbf{P}_N, \quad \mathbf{R} \leftarrow \mathbf{R}_{N-1}, \quad \mathbf{Q} \leftarrow \mathbf{Q}_{N-1}, \\ &\mathbf{E} \leftarrow \mathbf{E}_N, \quad \mathbf{A} \leftarrow \mathbf{A}_{N-1}, \quad \mathbf{B} \leftarrow \mathbf{B}_{N-1}, \end{aligned} \tag{29}$$

we can connect the problem in this step to the optimization problem in Lemma 4. Hence,

$$\mathcal{U}_N^* = \mathcal{K}_{N-1}\mathbf{x}_{N-1}, \tag{30}$$

where $\mathcal{U}_N^* = \text{col}\{\mathbf{x}_N^*, \mathbf{u}_{N-1}^*\}$ and $\mathcal{K}_{N-1} = \text{col}\{\mathbf{L}_{\mathbf{x}_{N-1}}, \mathbf{K}_{\mathbf{u}_{N-1}}\}$, or equivalently,

$$\mathcal{K}_{N-1} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathcal{Q}_{N-1} & \mathcal{E}_N \\ \mathcal{E}_N^\mathsf{T} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathcal{A}_{N-1} \\ \mathbf{0}\end{bmatrix}, \tag{31}$$

where $\mathcal{Q}_{N-1} = \mathcal{P}_{N-1}^{-1} \oplus \mathbf{0}$ with $\mathcal{P}_{N-1} = \mathbf{P}_{N-1} \oplus \mathbf{R}_{N-1} \oplus \mathbf{Q}_{N-1}$, and

$$\mathcal{A}_{N-1} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{A}_{N-1}\end{bmatrix}, \qquad \mathcal{E}_N = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{E}_N & -\mathbf{B}_{N-1}\end{bmatrix}.$$

Also, by substituting the optimal solution (30) in (28), the minimum value can be obtained as $\mathbf{x}_{N-1}^\mathsf{T}\mathbf{S}_{N-1}\mathbf{x}_{N-1}$, where

$$\mathbf{S}_{N-1} = \mathbf{L}_{\mathbf{x}_{N-1}}^\mathsf{T}\mathbf{P}_N\mathbf{L}_{\mathbf{x}_{N-1}} + \mathbf{K}_{\mathbf{u}_{N-1}}^\mathsf{T}\mathbf{R}_{N-1}\mathbf{K}_{\mathbf{u}_{N-1}} + \mathbf{Q}_{N-1}. \tag{32}$$

Again and in this step, we consider $\mathbf{P}_{N-1} = \mathbf{S}_{N-1}$.

Step $i = j$**,** ($j = N-2, \ldots, 0$): By continuing the same procedure, decreasing step i, and applying the optimality principle on the minimization problem

$$\min_{\mathbf{x}_{j+1},\mathbf{u}_j} \{\mathbf{L}_j(\mathbf{x}_j, \mathbf{u}_j) + \mathbf{x}_{j+1}^\mathsf{T}\mathbf{S}_{j+1}\mathbf{x}_{j+1}\} \quad \text{s.t.} \quad \mathbf{E}_{j+1}\mathbf{x}_{j+1} = \mathbf{A}_j\mathbf{x}_j + \mathbf{B}_j\mathbf{u}_j, \tag{33}$$

the result will be

$$\mathcal{U}_{j+1}^* = \mathcal{K}_j\mathbf{x}_j, \tag{34}$$

where $\mathcal{U}_{j+1}^* = \text{col}\{\mathbf{x}_{j+1}^*, \mathbf{u}_j^*\}$ and $\mathcal{K}_j = \text{col}\{\mathbf{L}_{\mathbf{x}_j}, \mathbf{K}_{\mathbf{u}_j}\}$, or equivalently,

$$\mathcal{K}_j = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathcal{Q}_j & \mathcal{E}_{j+1} \\ \mathcal{E}_{j+1}^\mathsf{T} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathcal{A}_j \\ \mathbf{0}\end{bmatrix}, \tag{35}$$

where $\mathcal{Q}_j = \mathcal{P}_j^{-1} \oplus \mathbf{0}$ with $\mathcal{P}_j = \mathbf{P}_j \oplus \mathbf{R}_j \oplus \mathbf{Q}_j$, and

$$\mathcal{A}_j = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{A}_j\end{bmatrix}, \qquad \mathcal{E}_{j+1} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{j+1} & -\mathbf{B}_j\end{bmatrix}.$$

Also, by substituting the optimal solution (34) in (33), the minimum value can be calculated as $\mathbf{x}_j^\mathsf{T}\mathbf{S}_j\mathbf{x}_j$, where

$$\mathbf{S}_j = \mathbf{L}_{\mathbf{x}_j}^\mathsf{T}\mathbf{P}_{j+1}\mathbf{L}_{\mathbf{x}_j} + \mathbf{K}_{\mathbf{u}_j}^\mathsf{T}\mathbf{R}_j\mathbf{K}_{\mathbf{u}_j} + \mathbf{Q}_j. \tag{36}$$

By considering $\mathbf{P}_j = \mathbf{S}_j$, the proof is completed. ■

Now, reconsider the solution in (20). According to Lemma 2, $\mathcal{U}_k^*$ must satisfy the following system of equations

$$\begin{bmatrix}\mathcal{Q}_k & \mathcal{E}_{k+1} \\ \mathcal{E}_{k+1}^\mathsf{T} & \mathbf{0}\end{bmatrix} \begin{bmatrix}\lambda \\ \mathcal{U}_k^*\end{bmatrix} = \begin{bmatrix}\mathcal{A}_k \\ \mathbf{0}\end{bmatrix}\mathbf{x}_k, \tag{37}$$

where $\lambda = \text{col}\{\lambda_i\}$, $i = 1, \cdots, 4$. This matrix equation provides the following system of equations

$$\mathbf{P}_{k+1}^{-1}\lambda_1 + \mathbf{I}\mathbf{x}_{k+1}^* = \mathbf{0}, \tag{38}$$

$$\mathbf{R}_k^{-1}\lambda_2 + \mathbf{I}\mathbf{u}_k^* = \mathbf{0}, \tag{39}$$

$$\mathbf{Q}_k^{-1}\lambda_3 = -\mathbf{I}\mathbf{x}_k, \tag{40}$$

$$\mathbf{E}_{k+1}\mathbf{x}_{k+1}^* - \mathbf{B}_k\mathbf{u}_k^* = \mathbf{A}_k\mathbf{x}_k, \tag{41}$$

$$\mathbf{I}\lambda_1 + \mathbf{E}_{k+1}^\mathsf{T}\lambda_4 = \mathbf{0}, \tag{42}$$

$$\mathbf{I}\lambda_2 - \mathbf{B}_k^\mathsf{T}\lambda_4 = \mathbf{0}. \tag{43}$$

From (38), (41), and (42), we have

$$\mathbf{E}_{k+1}\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^\mathsf{T}\lambda_4 - \mathbf{B}_k\mathbf{u}_k^* = \mathbf{A}_k\mathbf{x}_k. \tag{44}$$

Also, from (39) and (43), we have

$$\mathbf{u}_k^* = -\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}}\lambda_4. \tag{45}$$

Therefore, to derive the control law $\mathbf{u}_k^*$, it is enough to substitute (45) in (44), to get λ_4 as

$$\begin{aligned}\lambda_4 &= (\mathbf{E}_{k+1}\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}} + \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}})^{-1}\mathbf{A}_k\mathbf{x}_k \\ &= -\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{P}_{k+1} & \mathbf{0} & \mathbf{E}_{k+1}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{R}_k & -\mathbf{B}_k^{\mathsf{T}} \\ \mathbf{E}_{k+1} & -\mathbf{B}_k & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ \mathbf{A}_k\end{bmatrix}\mathbf{x}_k.\end{aligned} \tag{46}$$

So, the final control law can be rewritten as

$$\mathbf{u}_k^* = -\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}}(\mathbf{E}_{k+1}\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}} + \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}})^{-1}\mathbf{A}_k\mathbf{x}_k. \tag{47}$$

Also, from (38), (42), and (46), one has

$$\begin{aligned}\mathbf{x}_{k+1}^* = \mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}}&(\mathbf{E}_{k+1}\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}} + \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}})^{-1} \\ &\times \mathbf{A}_k\mathbf{x}_k.\end{aligned} \tag{48}$$

According to (20), the Riccati equation can be expressed as (22) with the following feedback gain, which is the coefficient of $\mathbf{x}_k$ in (47) and (48):

$$\begin{aligned}\mathcal{K}_k = \begin{bmatrix}\mathbf{L}_{\mathbf{x}_k} \\ \mathbf{K}_{\mathbf{u}_k}\end{bmatrix} &= \begin{bmatrix}\mathbf{0} & \mathbf{0} & -\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}}\end{bmatrix} \\ &\times \begin{bmatrix}\mathbf{P}_{k+1} & \mathbf{0} & \mathbf{E}_{k+1}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{R}_k & -\mathbf{B}_k^{\mathsf{T}} \\ \mathbf{E}_{k+1} & -\mathbf{B}_k & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ \mathbf{A}_k\end{bmatrix}.\end{aligned} \tag{49}$$

Remark 1: To ensure the existence of optimal recursive solution for the constrained optimization problem (4) subject to the linear singular model (1), the following matrix

$$\mathbf{E}_{k+1}\mathbf{P}_{k+1}^{-1}\mathbf{E}_{k+1}^{\mathsf{T}} + \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathsf{T}} \tag{50}$$

will be invertible. Matrix (50), can be rewritten as

$$\begin{bmatrix}\mathbf{E}_{k+1} & -\mathbf{B}_k\end{bmatrix}\begin{bmatrix}\mathbf{P}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_k\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{E}_{k+1} & -\mathbf{B}_k\end{bmatrix}^{\mathsf{T}}. \tag{51}$$

Therefore, because $\mathbf{P}_{k+1}$ and $\mathbf{R}_k$ are PD matrices, and the matrix inverse of a PD matrix is also PD, to ensure the existence of LQR algorithm, the matrix $\begin{bmatrix}\mathbf{E}_{k+1} & -\mathbf{B}_k\end{bmatrix}$ will be of full row rank.

V. SIMULATION RESULTS

Case 1: Consider a time-invariant version of the system described by (1) with parameter matrices given as

$$\mathbf{E} = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}, \mathbf{A} = \begin{bmatrix}1.5 & 1 \\ 5.3 & 5.2\end{bmatrix}, \mathbf{B} = \begin{bmatrix}0 \\ 1\end{bmatrix}, \tag{52}$$

and the respective weighting matrices as

$$\mathbf{Q} = \begin{bmatrix}5 & -4.5 \\ -4.5 & 4.5\end{bmatrix}, \quad \mathbf{R} = 1.1. \tag{53}$$

The matrix

$$\begin{bmatrix}\mathbf{E} & -\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1\end{bmatrix} \tag{54}$$

is of full row rank, and according to Remark 1, the LQR algorithm exists. Hence, according to the derived LQR algorithm in Theorem 1, the control $\mathbf{u}_k^*$ must be optimal to the LQR problem (4) subject to the singular system (1) with the given parameters (52). Trajectories of the states $\mathbf{x}_{1,k}^*$, $\mathbf{x}_{2,k}^*$, and the control $\mathbf{u}_k^*$ are illustrated in Fig. 1. In this case, the solution of Riccati equation (22) converges to

$$\mathbf{P}_k = \begin{bmatrix}91.0218 & 62.6617 \\ 62.6617 & 58.9031\end{bmatrix}. \tag{55}$$

Case 2: Let's repeat the simulation with a time varying version of (1) with same parameters (52) and (53) except

$$\mathbf{E}_{k+1} = \begin{bmatrix}k & 0 \\ 0 & 0\end{bmatrix}, \quad \mathbf{A}_k = \begin{bmatrix}1.5 & 1 \\ -\sin(k) & 5.2\end{bmatrix}. \tag{56}$$

Again, the matrix

$$\begin{bmatrix}\mathbf{E}_{k+1} & -\mathbf{B}_k\end{bmatrix} = \begin{bmatrix}k & 0 & 0 \\ 0 & 0 & -1\end{bmatrix}, 0 < k \leq N \tag{57}$$

is of full row rank, and according to Remark 1, the LQR algorithm exists. Hence, the control $\mathbf{u}_k^*$ must be optimal to the LQR problem (4) for the time varying system (1) with the given parameters. Trajectories of the states $\mathbf{x}_{1,k}^*$, $\mathbf{x}_{2,k}^*$, and the control $\mathbf{u}_k^*$ are depicted in Fig. 2. In this case, the solution of Riccati equation (22) converges to

$$\mathbf{P}_k = \begin{bmatrix}5.7817 & -9.3308 \\ -9.3308 & 34.5432\end{bmatrix}. \tag{58}$$

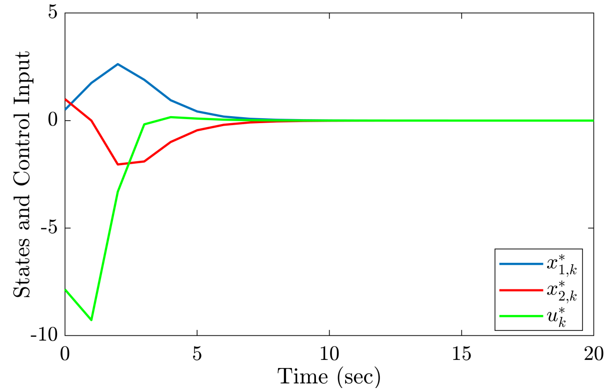

Fig. 1: Regulated states and optimal control signal (case 1).

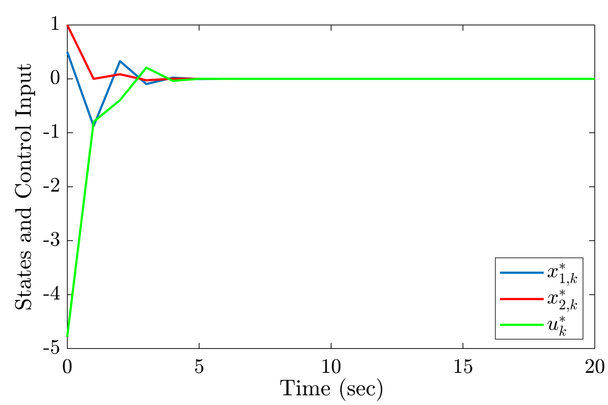

Fig. 2: Regulated states and optimal control signal (case 2).

Case 3: To verify the result in non-singular case, lets consider our time-invariant system again and consider $\mathbf{E} = \mathbf{I}$. We try to test the result in non-singular case and compare it with the function `dlqr` in MATLAB, which design the LQR model for standard discrete-time systems. According to the simulation results, the solution of Riccati equation and the state-feedback law are completely identical given as

$$\mathbf{P}_k = \begin{bmatrix} 112.4657 & 79.8327 \\ 79.8327 & 72.2204 \end{bmatrix}, \quad \mathbf{K}_{u_k} = \begin{bmatrix} 6.8527 \\ 6.2099 \end{bmatrix}^{\mathsf{T}}, \quad (59)$$

which shows that the LQR algorithm designed in this study reduces to the classical LQR algorithm for standard systems if $\mathbf{E} = \mathbf{I}$ in Theorem 1. The regulated states and optimal control are depicted in Fig. 3.

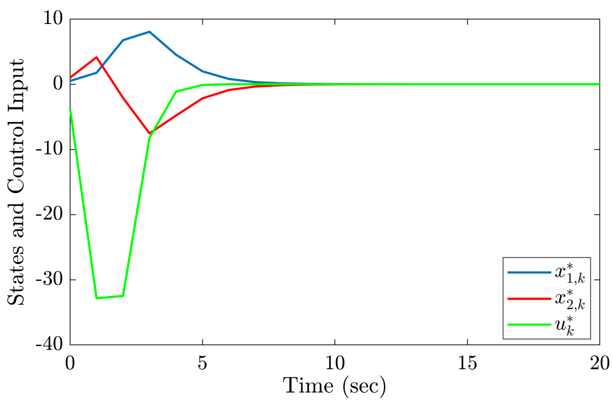

Fig. 3: Regulated states and optimal control signal (case 3).

VI. CONCLUSIONS

This technical note presents a novel approach for the derivation of a LQR algorithm for a time varying discrete linear singular model. The proposed solution eliminates the need for regularity assumptions of the system and is based on a penalized weighted least-squares formulation. By using Bellman's principle of optimality and dynamic programming, we have derived a recursive regulator algorithm associated with the existence conditions. Due to the derivation of the algorithm in a direct way and without transformation, its optimal performance will not be compromised under violation of constraints in the transformed non-singular version—a technique which was exploited in the literature. In future work, we plan to address the convergence and stability analysis of the developed algorithm. Also, the matrix array frameworks of the control law, feedback gain, and Riccati equation motivate us to develop an algorithm for the robust LQR problems in future research.

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