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Fully Distributed Adaptive Consensus of Multiple Manipulators with Elastic Joints under a Directed Graph

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***Abstract*— In this paper, the distributed leaderless consensus problem of multiple manipulators with elastic joints under a directed graph is investigated, which extends existing results on the coordination of multiple second-order Euler-Lagrange systems to fourth-order manipulators with elastic joints. We use the model reference adaptive consensus scheme to transform the consensus problem into two subproblems. A trajectory tracking algorithm is designed based on the command filter adaptive backstepping approach, and it is shown that the joint positions of the manipulators achieve ultimately bounded consensus. The proposed algorithm only requires the interaction of the relative joint position information.**

***Index Terms*— Multi-agent systems, directed graph, manipulator with elastic joints, command filter based adaptive backstepping approach, model reference adaptive consensus**

I. Introduction

In the past two decades, cooperative control of multi-agent systems has drawn a lot of attention in the control field. Current research mainly includes consensus, formation control, coordination exploration, and distributed optimization [1], [2]. As a fundamental problem, the consensus of multi-agent systems is deeply studied. The goal of consensus is to use local information interaction among agents to make the states of agents achieve the same final value. Furthermore, the consensus of multi-agent systems is closely related to formation, flocking and containment control.

The results of early research on multi-agent systems mainly focus on linear systems. However, many mechanical systems like the robotic manipulators are inherently nonlinear. Basically, the coordination on networked manipulators is based on Euler-Lagrange dynamics [3]–[5]. [3] proposes a distributed algorithm of multiple Euler-Lagrange systems under an undirected graph, and considers the situation of unknown velocity information and input saturation. For a directed graph, the nonlinearities in Euler-Lagrange dynamics bring a lot of difficulties for the convergence analysis. A widely used strategy is to employ distributed sliding mode variables and design a control algorithm to drive the agents' states toward the sliding surface. [4] presents a consensus algorithm without the need of relative velocity information. In a recent study [5], the problem of distributed adaptive consensus under switching directed graphs is examined.

The above results concentrate on second-order Euler-Lagrange dynamics. For manipulators with elastic joints, the dynamics become a fourth-order one with unmatched uncertainties. The benefits of using fourth-order dynamics include enhanced modeling accuracy, improved control precision, and better performance in compliance control. Meanwhile, unmatched uncertain high-order dynamic systems brings significant challenges to the consensus algorithm design. To cope with the unmatched uncertainties, backstepping method is commonly used in [6]–[8]. In [6], a tracking control algorithm based on the integrator backstepping is proposed. The authors in [8] present an output feedback tracking algorithm by adopting the link velocity observer and actuator velocity observer. A proportional-derivative type algorithm for the position tracking problem is proposed in [7]. However, one of the shortcomings of backstepping method is that the derivatives of the virtual control inputs become difficult to calculate with the increase in the number of uncertain parameters. An effective method to solve this problem is using command filter based backstepping approach [9]. In [10], the command filters are designed to approximate the virtual control inputs in the tracking problem of a manipulator with elastic joints. The results in [6]–[10] consider the control problem of one single manipulator. As far as we know, there are only a few relevant results about the coordination of multiple manipulators with elastic joints. In [11], the authors propose a consensus algorithm under an undirected graph. In [12], the consensus problem in task space under a directed graph is investigated. However, the interactive information among agents in [11], [12] is related to the dynamical parameters, which are difficult to obtain when dealing with parametric uncertainties. In [13], we study the consensus problem without the exchange of high-order derivatives under a directed graph where there are no parametric uncertainties.

In this paper, we study the consensus problem of multiple manipulators with elastic joints under a directed graph in the presence of parametric uncertainties. By utilizing the model reference adaptive scheme in [14], we assign each manipulator a fourth-order reference to track. A trajectory tracking algorithm is designed based on the command filter adaptive backstepping approach [10], and it is shown that the joint positions of the manipulators achieve ultimately bounded consensus.

Notations: Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n\times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{n\times n}$ denote the $n\times n$ matrix with all zeros and $\mathbf{I}_n$ denote the $n\times n$ identity matrix. Throughout the paper, we use $\|\cdot\|$ to denote the Euclidean norm. For a vector $\eta=(\eta_1,\dots,\eta_n)^T$, $\mathrm{diag}(\eta)$ denotes a diagonal matrix with $\eta_1,\dots,\eta_n$ on its diagonal.

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II. Background And Problem Statement

A. Graph Theory

We use a directed graph to describe the network topology among the N agents. Let $\mathscr{G}_N \triangleq (\mathscr{V}_N, \mathscr{E}_N)$ be a directed graph with the node set $\mathscr{V}_N \triangleq \{v_1, ..., v_N\}$ and the edge set $\mathscr{E}_N \subseteq \mathscr{V}_N \times \mathscr{V}_N$. An edge $(v_i, v_j) \in \mathscr{E}_N$ denotes that agent v_j can obtain information from agent v_i, but not vice versa. Here, node v_i is the parent node while node v_j is the child node. A directed path from node v_i to node v_j is a sequence of edges of the form (v_i, v_{i2}), (v_{i2}, v_{i3}), ..., (v_{ik}, v_j), in a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph.

The adjacency matrix $\mathscr{A}_N \triangleq [a_{ij}] \in \mathbb{R}^{N\times N}$ of a directed graph $(\mathscr{V}_N, \mathscr{E}_N)$ is defined such that $a_{ij} > 0$ if $(v_j, v_i) \in \mathscr{E}_N$, and $a_{ij} = 0$ if $(v_j, v_i) \notin \mathscr{E}_N$. In this paper, self-edges are not allowed, *i.e.*, $a_{ii} = 0$. The in-degree of node i is defined as follows: $\deg_{in}(v_i) = \sum_{j=1}^{N} a_{ij}, i = 1, ..., N$. Then, $\mathscr{D}_N = \operatorname{diag}\{\deg_{in}(v_1), \cdots, \deg_{in}(v_N)\}$ is called the in-degree matrix of $\mathscr{G}_N$. The (non-symmetric) Laplacian matrix of $\mathscr{G}_N$ is defined as $\mathscr{L}_N = \mathscr{D}_N - \mathscr{A}_N$, with $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. A directed graph associated with n agents is denoted by $\mathscr{G}_N \triangleq (\mathscr{V}_N, \mathscr{E}_N)$.

Assumption 1*:* The directed graph $\mathscr{G}$ contains a directed spanning tree.

Lemma 1*:* [15] Under Assumption 1, the eigenvalues of $\mathscr{L}_N + k_\alpha \mathbf{1}_N v_N^T$ possess positive real parts, where k_α is a positive constant, v_N represents the left eigenvector of the Laplacian matrix associated with the zero eigenvalue and meets the condition $v_N^T \mathbf{1}_N = 1$.

B. Agent Model

The dynamics of a manipulator with p elastic joints can be represented by the following equations [16]:

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = K_i(\theta_i - q_i), \tag{1}$$

$$J_i\ddot{\theta}_i + B_i\dot{\theta}_i + K_i(\theta_i - q_i) = u_i + d_i(t), \tag{2}$$

where $q_i \in \mathbb{R}^p$ and $\theta_i \in \mathbb{R}^p$ represent, respectively, the vectors of joint position and actuator position, $M_i(q_i) \in \mathbb{R}^{p\times p}$ represents the link inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p\times p}$ represents the Coriolis and centrifugal terms, $g_i(q_i) \in \mathbb{R}^p$ represents the gravitational terms, u_i represents the vector of actuator torques, $K_i = \operatorname{diag}\{K_{i1}, K_{i2}, \ldots, K_{ip}\}$, J_i, $B_i \in \mathbb{R}^{p\times p}$ are positive constant diagonal matrices that represent joint stiffness, actuator inertia, and actuator damping, respectively, and $d_i(t) \in \mathbb{R}^p$ is the external time-varying disturbance satisfying $\|d_i(t)\| \leq \bar{d}_i$, where $\bar{d}_i$ is assumed to be a known constant. Several fundamental properties can be employed to simplify the subsequent control algorithm design [17].

(P1) The link inertia matrix is symmetric, positive definite and satisfies the following inequality for any $x, y \in \mathbb{R}^p$

$$\underline{m}_i\|x\|^2 \leq x^T M_i(y)x \leq \bar{m}_i\|x\|^2,$$

where $\underline{m}_i$, $\bar{m}_i \in \mathbb{R}$ are known positive constants. Moreover, there exist positive constants k'_{Mi}, $k_{Ci,1}$, and k'_{gi} such that $k'_{Mi}\|y\| \leq \|M_i(x)y\|$, $\|C_i(x,y)z\| \leq k_{Ci,1}\|y\|\|z\|$, $\|g_i(x)\| \leq k'_{gi}$, $\forall x, y, z \in \mathbb{R}^p$.

(P2) Skew symmetric property: $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric.

(P3) Linear in parameters: $Y_{i,1}(q_i, \dot{q}_i, \ddot{q}_i)\theta_{i,1} = M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i)$, $\forall \ddot{q}_i$, $\dot{q}_i$, q_i, $\in \mathbb{R}^p$, where $Y_{i,1}(q_i, \dot{q}_i, \ddot{q}_i) \in \mathbb{R}^{p\times p_{\theta}i,1}$ is known as the regression matrix, and $\theta_{i,1} \in \mathbb{R}^{p_\theta i,1}$ is an unknown parameter vector associated with the ith agent satisfying $\|\theta_{i,1}\| \leq \bar{\theta}_{i,1}$, where $\bar{\theta}_{i,1}$ is known upper bound.

(P4) Joint stiffness matrix K_i satisfies

$$x^T \underline{K}_i x \leq x^T K_i x \leq x^T \bar{K}_i x, \quad \forall x \in \mathbb{R}^p, \tag{3}$$

where $\underline{K}_i = \operatorname{diag}\{\underline{K}_{i1}, \underline{K}_{i2}, \ldots, \underline{K}_{ip}\}$ and $\bar{K}_i = \operatorname{diag}\{\bar{K}_{i1}, \bar{K}_{i2}, \ldots, \bar{K}_{ip}\}$ are positive definite diagonal matrices. Define $\underline{k}_i$ as the minimum element of $\underline{K}_i$ and $\bar{k}_i$ as the maximum element of $\bar{K}_i$.

In this paper, we assume that the parameters $\theta_{i,1}$, K_i, J_i, and B_i are constant but unknown. Define $x_{i,1} = \theta_i$, $x_{i,2} = \dot{\theta}_i$. The above system can be described as

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_1)\dot{q}_i + g_i(q_i) + K_i q_i = K_i x_{i,1}, \tag{4}$$

$$\dot{x}_{i,1} = x_{i,2}, \tag{5}$$

$$\dot{x}_{i,2} = J_i^{-1}u_i + J_i^{-1}d_i - J_i^{-1}B_i x_{i,2} - J_i^{-1}K_i(x_{i,1} - q_i). \tag{6}$$

It can be seen from (4)-(6) that the dynamics of a manipulator with elastic joints is different from second-order Euler-Lagrange system. Therefore, we cannot directly design consensus algorithm for (4) due to the influence of the actuator dynamics.

The objective of this paper is to design a consensus algorithm for multiple manipulators with elastic joints under a directed graph in the sense that the relative joint position error $q_i(t) - q_j(t)$ and the joint velocity $\dot{q}_i(t)$ converge to certain bounded sets.

III. Main Results

Due to the nonlinearity and strong coupling of the manipulator with elastic joints, it is difficult to design consensus algorithm directly. Inspired by [14], by designing a linear reference model for each manipulator, the consensus problem of multiple manipulators with elastic joints can be transformed into two subproblems, namely, the consensus of the linear reference models and the tracking algorithm design of one single manipulator with elastic joints.

A. The Framework of The Consensus Algorithm

In this subsection, we first design a suitable reference model for each manipulator. Note that the manipulator with elastic joints has a fourth-order dynamics. Therefore, the reference model should also be fourth-order with the relative joint position error $\tau_i = -\sum_{j=1}^n a_{ij}(q_i - q_j)$ as input and $(z_i, \dot{z}_i, \ddot{z}_i, z_i^{(3)}) \in \mathbb{R}^{4p}$ as state. For agent i, the dynamics of the linear reference model can be designed as

$$z_i^{(4)} + a_{i,3} z_i^{(3)} + a_{i,2}\ddot{z}_i + a_{i,1}\dot{z}_i + a_{i,0} z_i = b_i \tau_i, \ i = 1, \ldots, n, \quad (7)$$

where $a_{i,0}, a_{i,1}, a_{i,2}, a_{i,3}, b_i$ are positive parameters to be determined later. The tracking error is defined as

$$e_i = q_i - z_i.$$

Then, we present a variable substitution for the fourth-order reference model as follows. Define the auxiliary variables $\xi_{i,1} = z_i$ and $\xi_{i,2}, \xi_{i,3}, \xi_{i,4} \in \mathbb{R}^p$ with

$$\xi_{i,l} = \frac{1}{k_i}\dot{\xi}_{i,l-1} + \xi_{i,l-1}, \quad l = 2, 3, 4, \quad (8)$$

where k_i is an arbitrarily positive constant. From (8), $\xi_{i,4}$ can be written as

$$\xi_{i,4} = z_i + \frac{3}{k_i}\dot{z}_i + \frac{3}{k_i^2}\ddot{z}_i + \frac{1}{k_i^3} z_i^{(3)}. \quad (9)$$

Taking the derivative of (9) with respect to time yields

$$\dot{\xi}_{i,4} = \dot{z}_i + \frac{3}{k_i}\ddot{z}_i + \frac{3}{k_i^2} z_i^{(3)} + \frac{1}{k_i^3} z_i^{(4)}. \quad (10)$$

By adding and subtracting $-k_i^{-1}\sum_{j=1}^n a_{ij}\xi_{i,4} - k_i^{-1}\sum_{j=1}^n a_{ij}(\xi_{i,1} - \xi_{j,1})$ into (10), we obtain

$$\begin{aligned}\dot{\xi}_{i,4} = & -[\frac{a_{i,1}}{k_i^3} - \frac{3}{k_i^2}\sum_{j=1}^n a_{ij} - 1]\dot{z}_i - [\frac{a_{i,2}}{k_i^3} - \frac{3}{k_i} - \frac{3}{k_i^3}\sum_{j=1}^n a_{ij}]\ddot{z}_i \\ & - [\frac{a_{i,3}}{k_i^3} - \frac{3}{k_i^2} - \frac{1}{k_i^4}\sum_{j=1}^n a_{ij}] z_i^{(3)} + \frac{a_{i,0}}{k_i^3} z_i - \frac{b_i}{k_i^3}\sum_{j=1}^n a_{ij}(q_i \\ & - q_j) - \frac{1}{k_i}\sum_{j=1}^n a_{ij}\xi_{i,4} + \frac{1}{k_i}\sum_{j=1}^n a_{ij}\xi_{j,1} + \frac{1}{k_i}\sum_{j=1}^n a_{ij}(\xi_{i,1} \\ & - \xi_{j,1}). \end{aligned} \quad (11)$$

Choose $a_{i,0} = 0$, $a_{i,1} = k_i^3 + 3k_i\sum_{j=1}^n a_{ij}$, $a_{i,2} = 3k_i^2 + 3\sum_{j=1}^n a_{ij}$, $a_{i,3} = 3k_i + k_i^{-1}\sum_{j=1}^n a_{ij}$, $b_i = k_i^2$. Then, (11) can be written as

$$\dot{\xi}_{i,4} = k_i^{-1}\sum_{j=1}^n a_{ij}\xi_{i,4} + k_i^{-1}\sum_{j=1}^n a_{ij}\xi_{j,1} + k_i^{-1}\sum_{j=1}^n a_{ij}(e_i - e_j). \quad (12)$$

Define the stack vectors $\xi = (\xi_{1,1}^T, \xi_{2,1}^T, \ldots, \xi_{n,1}^T, \ldots, \xi_{1,4}^T, \xi_{2,4}^T, \ldots, \xi_{n,4}^T)^T$, and $e_* = \left(e_1^T, e_2^T, \ldots, e_n^T\right)^T$. (8) and (12) can be written in the following vector form

$$\dot{\xi} = -(\widetilde{\mathscr{L}} \otimes \mathbf{I}_p)\xi + He_*, \quad (13)$$

where $\widetilde{\mathscr{L}}$ is defined as

$$\widetilde{\mathscr{L}} = \begin{pmatrix} K & -K & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & K & -K & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & K & K \\ -K^{-1}\mathscr{A} & 0_{n\times n} & 0_{n\times n} & K^{-1}\mathscr{D} \end{pmatrix},$$

where $H \triangleq (\mathbf{0}_{np\times np}, \mathbf{0}_{np\times np}, \mathbf{0}_{np\times np}, -(\mathscr{L}^T K^{-1}) \otimes \mathbf{I}_p)^T$, $\mathscr{A}$, $\mathscr{D}$, $\mathscr{L}$ are the adjacency matrix, the in-degree matrix, and the Laplacian matrix of $\mathscr{G}$, respectively, and $K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$. For the matrix $\widetilde{\mathscr{L}}$, it can be seen that its row sum is zero, the diagonal elements are all non-negative, and the off-diagonal elements are non-positive. Hence $\widetilde{\mathscr{L}}$ can be viewed as the Laplacian matrix of a directed graph with $4n$ nodes denoted by $\widetilde{\mathscr{G}} \triangleq (\widetilde{\mathscr{V}}, \widetilde{\mathscr{E}})$. Overall, the reference model (7) can be written as

$$\begin{aligned} & z_i^{(4)} + [3k_i + k_i^{-1}\sum_{j=1}^n a_{ij}] z_i^{(3)} + [3k_i^2 + 3\sum_{j=1}^n a_{ij}]\ddot{z}_i \\ & + [k_i^3 + 3\sum_{j=1}^n a_{ij}]\dot{z}_i = -k_i^2\sum_{j=1}^n a_{ij}(q_i - q_j), \end{aligned} \quad (14)$$

with the initial states being chosen as $z_i(0) = q_i(0)$, $\dot{z}_i(0) = \mathbf{0}_p$, $\ddot{z}_i(0) = \mathbf{0}_p$, $z_i^{(3)}(0) = \mathbf{0}_p$. (14) can be seen as the fourth-order linear system with $\sum_{j=1}^n a_{ij}(q_i - q_j)$ as input and $z_i, \dot{z}_i, \ddot{z}_i, z_i^{(3)}$ as output.

Under the above framework, the following analysis is divided into two parts. Firstly, we analyze the consensus convergence of the reference models, in which the key is the proof of the graph connectivity. Secondly, we design a trajectory tracking algorithm for each manipulator to track the output of the reference model. Under a fixed directed graph, consensus can be achieved for multiple first-order integrators if and only if the graph includes a spanning tree. And we have the following result.

Lemma 2: Graph $\widetilde{\mathscr{G}}$ contains a spanning tree if and only if $\mathscr{G}$ contains a spanning tree.

The proof follows a similar procedure to that in Theorem 4.6 of [14] and Lemma 3 of [13], and we omit it here.

B. Command Filter based Adaptive Backstepping Control Algorithm Design

In this subsection, we aim to design a trajectory tracking algorithm for each manipulator such that e_i converges to a bounded set. Since the parameters of the manipulator are unknown and a desired trajectory is given by the reference model, the actual control input can be designed using the command filter based adaptive backstepping approach. Define the following auxiliary variables

$$\begin{aligned} \dot{q}_{ri} &= \dot{z}_i - k_{di} e_i, \\ s_i &= \dot{q}_i - \dot{q}_{ri} = \dot{e}_i + k_{di} e_i, \end{aligned} \quad (15)$$

where k_{di} is a constant positive gain. Then (4) can be written as

$$\begin{aligned} M_i(q_i)\dot{s}_i = & -C_i(q_i, \dot{q}_i)s_i + K_i(x_{i,1} - q_i) - M_i(z_i)\ddot{z}_i \\ & - C_i(z_i, \dot{z}_i)\dot{z}_i - g_i(z_i) + h_i. \end{aligned} \quad (16)$$

where the residual term h_i is

$$\begin{aligned} h_i = & M_i(q_i)k_{di}(s_i - k_{di}e_i) + C_i(q_i, k_{di}e_i)(s_i - k_{di}e_i) \\ & + C_i(q_i, k_{di}e_i)\dot{z}_i + [M_i(z_i) - M_i(q_i)]\ddot{z}_i \\ & + [C_i(z_i, \dot{z}_i) - C_i(q_i, \dot{q}_i)]\dot{z}_i + g_i(z_i) - g_i(q_i), \end{aligned} \quad (17)$$

Under (P1), the residual term h_i satisfies

$$\|s_i^T h_i\| \leq (\zeta_{i,1} + \zeta_{i,2}\|\dot{z}_i\|^2 + \zeta_{i,3}\|\dot{z}_i\|^4 + \zeta_{i,4}\|\ddot{z}_i\|^2 + \zeta_{i,5}\|e_i\|^2)\|s_i\|^2 + \zeta_{i,6}\|e_i\|^2. \quad (18)$$

where $\zeta_{i,j}$ are positive constants associated with the ith agents parameters, $j = 1, \ldots, 6$. Here we define the linearly parameterizable function $Y_{i,1}(z_i, \dot{z}_i, \ddot{z}_i)\theta_{i,1} = M_i(z_i)\ddot{z}_i + C_i(z_i, \dot{z}_i)\dot{z}_i + g_i(z_i)$. Let $\theta_{ki} \in \mathbb{R}^p$ denote the vector consisting of the diagonal elements of K_i^{-1}, and let $\hat{\theta}_i \in \mathbb{R}^{p\theta i,1}$ be the estimate of θ_i, $\hat{\theta}_{ki} = (\hat{\theta}_{ki1}, \ldots, \hat{\theta}_{kip})^T \in \mathbb{R}^p$ present the estimate of θ_{ki}. Additionally, $\hat{K}_i^{-1} = \mathrm{diag}(\hat{\theta}_{ki}) = \mathrm{diag}\{\hat{\theta}_{ki1}, \ldots, \hat{\theta}_{kip}\}$. Choose $x_{i,1}$ as the virtual control input of (16) and define the virtual control law as

$$\begin{aligned} x_{i,1d} &= -k_{qi}s_i + q_i + \mathrm{diag}(Y_{i,1}\hat{\theta}_{i,1})\hat{\theta}_{ki} \\ &= -k_{qi}s_i + q_i + \hat{K}_i^{-1}Y_{i,1}\hat{\theta}_{i,1}, \end{aligned} \quad (19)$$

where $k_{qi} = \zeta_{i,0} + \underline{k}^{-1}(\zeta_{i,1} + k_{di}^{-2}\zeta_{i,6} + \zeta_{i,2}\|\dot{z}_i\|^2 + \zeta_{i,3}\|\dot{z}_i\|^4 + \zeta_{i,4}\|\ddot{z}_i\|^2 + \zeta_{i,5}\|e_i\|^2)$ is time-varying control gain with $\zeta_{i,0}$ being a positive constant to be determined later. Differentiating (19), we obtain

$$\dot{x}_{i,1d} = -k_{qi}\dot{s}_i - \dot{k}_{qi}s_i + \dot{q}_i + \frac{\mathrm{d}(\mathrm{diag}(Y_{i,1}\hat{\theta}_{i,1})\hat{\theta}_{ki})}{\mathrm{d}t}. \quad (20)$$

From (20), the calculation of $\dot{s}_i$ is difficult and hinders the backstepping. Here we use a second-order command filter to approximate the derivatives of $x_{i,1d}$. The dynamics of the second-order command filter can be written as

$$\begin{aligned} \dot{\bar{x}}_{i,1} &= -k_{xi}(\bar{x}_{i,1} - \bar{x}_{i,2}),\ \bar{x}_{i,1}(0) = x_{i,1d}(0), \\ \dot{\bar{x}}_{i,2} &= -k_{xi}(\bar{x}_{i,2} - x_{i,1d}),\ \bar{x}_{i,2}(0) = x_{i,1d}(0), \end{aligned} \quad (21)$$

where $k_{xi}(t) > 0$ is a continuous time-varying positive gain to be determined later, and $\bar{x}_{i,1}, \dot{\bar{x}}_{i,1} \in \mathbb{R}^p$ are the outputs of the command filter. (21) can be written as

$$\ddot{\bar{x}}_{i,1} = -(2k_{xi} - k_{xi}^{-1}\dot{k}_{xi})\dot{\bar{x}}_{i,1} - k_{xi}^2(\bar{x}_{i,1} - x_{i,1d}). \quad (22)$$

where $k_{xi}(t) > 0$ is a continuous differential time-varying positive gain associated with the ith agent to be determined later, $\bar{x}_{i,1}, \dot{\bar{x}}_{i,1} \in \mathbb{R}^p$ are the outputs of the command filter.

Define $\tilde{x}_{i,1} = \bar{x}_{i,1} - x_{i,1d},\ \tilde{x}_{i,2} = \bar{x}_{i,2} - x_{i,1d}$. The dynamics of $\tilde{x}_{i,1}, \tilde{x}_{i,2}$ can be written as follows

$$\begin{aligned} \dot{\tilde{x}}_{i,1} &= -k_{xi}(\tilde{x}_{i,1} - \tilde{x}_{i,2}) - \dot{x}_{i,1d}, \\ \dot{\tilde{x}}_{i,2} &= -k_{xi}\tilde{x}_{i,2} - \dot{x}_{i,1d}. \end{aligned} \quad (23)$$

Define $w_{i,1} = (w_{i,11}, w_{i,12}, \ldots, w_{i,1p})^T \in \mathbb{R}^p$ with

$$w_{i,1} = x_{i,1} - \bar{x}_{i,1}. \quad (24)$$

Define the parameter estimate errors $\tilde{\theta}_{ki} = \hat{\theta}_{ki} - \theta_{ki}$ and $\tilde{\theta}_{i,1} = \hat{\theta}_{i,1} - \theta_{i,1}$. Then (16) can be rewritten as

$$\begin{aligned} M_i(q_i)\dot{s}_i = &- C_i(q_i, \dot{q}_i)s_i - k_{qi}K_i s_i + Y_{i,1}\tilde{\theta}_{i,1} + h_i \\ &+ K_i\mathrm{diag}(Y_{i,1}\hat{\theta}_{i,1})\tilde{\theta}_{ki} + K_i w_{i,1} + K_i\tilde{x}_{i,1}. \end{aligned} \quad (25)$$

The adaptation law for $\hat{\theta}_{i,1}$ with σ-modification is formulated as

$$\dot{\hat{\theta}}_{i,1} = -\Gamma_{i,1}Y_{i,1}^T s_i - \sigma_i\Gamma_{i,1}\hat{\theta}_{i,1}, \quad (26)$$

where $\Gamma_{i,1} = \mathrm{diag}\{\Gamma_{i,11}, \Gamma_{i,12}, \ldots, \Gamma_{i,1p\theta_{i,1}}\}^T \in \mathbb{R}^{p\theta i,1 \times p\theta i,1}$ is a positive diagonal matrix satisfying $0 < \underline{\Gamma}_i \preceq \Gamma_{i,1} \preceq \bar{\Gamma}_i$. Note that $\hat{\theta}_{ki}$ should satisfy $\bar{K}_{ij}^{-1} \leq \hat{\theta}_{kij} \leq \underline{K}_{ij}^{-1}$. A projection-based adaptation law of $\hat{\theta}_{kij},\ j = 1, \ldots, p$, is used

$$\dot{\hat{\theta}}_{kij} = \begin{cases} \Gamma_{Kij}y_{ij} & \text{if } \bar{K}_{ij}^{-1} < \hat{\theta}_{kij} < \underline{K}_{ij}^{-1} \text{ or } \hat{\theta}_{kij} = \underline{K}_{ij}^{-1} \\ & \text{and } y_{ij} \leq 0, \text{ or } \hat{\theta}_{kij} = \bar{K}_{ij}^{-1} \\ & \text{and } y_{ij} \geq 0. \\ 0 & \text{if } \hat{\theta}_{kij} = \underline{K}_{ij}^{-1} \text{ and } y_{ij} > 0, \\ & \text{or } \hat{\theta}_{kij} = \bar{K}_{ij}^{-1} \text{ and } y_{ij} < 0. \end{cases} \quad (27)$$

where $\Gamma_{Ki} = \mathrm{diag}\{\Gamma_{Ki1}, \Gamma_{Ki2}, \ldots, \Gamma_{Kip}\}^T$ is a positive diagonal matrix satisfying $0 < \underline{\Gamma}_{Ki} \preceq \Gamma_{Ki} \preceq \bar{\Gamma}_{Ki}$, $y_i = (y_{i1}, \ldots, y_{ip})^T = -\mathrm{diag}(Y_{i,1}\hat{\theta}_{i,1})s_i$, the initial value $\hat{\theta}_{ki}(0)$ is chosen such that $\bar{K}_{ij}^{-1} \leq \hat{\theta}_{kij}(0) \leq \underline{K}_{ij}^{-1}$. Note that $x_{i,1}$ is not the actual control input of the manipulator with elastic joints. Therefore, we assume that $x_{i,2}$ in the subsystem (5) is the virtual control input. Taking the derivative of $w_{i,1}$ yields

$$\dot{w}_{i,1} = \dot{x}_{i,1} - \dot{\bar{x}}_{i,1} = x_{i,2} - \dot{\bar{x}}_{i,1}. \quad (28)$$

A virtual control law for $x_{i,2}$ for (28) would be

$$x_{i,2d} = -\Lambda_{i,1}w_{i,1} + \dot{\bar{x}}_{i,1}, \quad (29)$$

where $\Lambda_{i,1}$ is a positive definite diagonal matrix and the smallest diagonal element of $\Lambda_{i,1}$ is $\underline{\Lambda}_{i,1} > \frac{1}{4}\bar{k}_i$. Then (28) can be rewritten as

$$\dot{w}_{i,1} = (x_{i,2} - x_{i,2d}) - \Lambda_{i,1}w_{i,1}. \quad (30)$$

Define

$$w_{i,2} = x_{i,2} - x_{i,2d}. \quad (31)$$

Substituting (31) into (30) yields

$$\dot{w}_{i,1} = w_{i,2} - \Lambda_{i,1}w_{i,1}, \quad (32)$$

However, $x_{i,2}$ is not the actual control input. We then use the backstepping method and take the derivative of $w_{i,2}$ to obtain the control input u_i. Note that

$$\dot{w}_{i,2} = \dot{x}_{i,2} - \dot{x}_{i,2d}. \quad (33)$$

Multiplying both sides of (33) by J_i yields

$$J_i\dot{w}_{i,2} = -K_i(x_{i,1} - q_i) - B_i x_{i,2} + u_i - J_i\dot{x}_{i,2d}, \quad (34)$$

Note that (34) can be written as

$$J_i\dot{w}_{i,2} = u_i - Y_{i,2}\theta_{i,2}, \quad (35)$$

where $Y_{i,2}\theta_{i,2} = K_i(x_{i,1} - q_i) + B_i x_{i,2} + J_i\dot{x}_{i,2d}$. Let $\hat{\theta}_{i,2}$ be the estimate of $\theta_{i,2}$, and define $\tilde{\theta}_{i,2} = \hat{\theta}_{i,2} - \theta_{i,2}$. For (35), we propose the following control algorithm

$$u_i = -\Lambda_{i,2}w_{i,2} - w_{i,1} + Y_{i,2}\hat{\theta}_{i,2} - \frac{\bar{d}_i w_{i,2}}{\|w_{i,2}\| + e^{-t}}, \quad (36)$$

where $\Lambda_{i,2}$ is a positive definite diagonal matrix and the smallest diagonal element of $\Lambda_{i,2}$ is $\underline{\Lambda}_{i,2}$. Substituting (36) into (35) yields

$$J_i\dot{w}_{i,2} = -\Lambda_{i,2}w_{i,2} - w_{i,1} + Y_{i,2}\tilde{\theta}_{i,2} - \frac{\bar{d}_i w_{i,2}}{\|w_{i,2}\| + e^{-t}}. \quad (37)$$

The parameter adaptation law for $\hat{\theta}_{i,2}$ with σ-modification is designed as

$$\dot{\hat{\theta}}_{i,2} = -\Gamma_{i,2}Y_{i,2}^T w_{i,2} - \sigma_i\Gamma_{i,2}\hat{\theta}_{i,2}, \tag{38}$$

where $\Gamma_{i,2} = \mathrm{diag}\{\Gamma_{i,21},\Gamma_{i,22},\ldots,\Gamma_{i,2p\theta_{i,2}}\}^T \in \mathbb{R}^{p\theta i,2\times p\theta i,2}$ is a positive diagonal matrix satisfying $0 < \underline{\Gamma}_i \preceq \Gamma_{i,2} \preceq \bar{\Gamma}_i$.

Now, we have the following result.

Lemma 3*:* For (4), (5), and (6) with unknown parameters, under the control algorithm (36) and adaptation law (26), (27), (38), the tracking error e_i is ultimately bounded.

Proof. Consider the following Lyapunov function candidate

$$V_{i,1} = \frac{1}{2}s_i^T M_i(q_i)s_i + \frac{1}{2}\tilde{\theta}_{i,1}^T\Gamma_{i,1}^{-1}\tilde{\theta}_{i,1} + \frac{1}{2}\tilde{\theta}_{ki}^T\Gamma_{Ki}^{-1}K_i\tilde{\theta}_{ki} + \zeta_{i,6}k_{di}^{-1}e_i^T e_i + \frac{1}{2}w_{i,1}^T w_{i,1} + \frac{1}{2}w_{i,2}^T J_i w_{i,2} + \frac{1}{2}\tilde{\theta}_{i,2}^T\Gamma_{i,2}^{-1}\tilde{\theta}_{i,2}.$$

Taking the derivative of $V_{i,1}$ along with the solutions of (37) and (38) yields

$$\begin{aligned}\dot{V}_{i,1} \le & -\zeta_{i,0}s_i^T K_i s_i - w_{i,1}^T\Lambda_{i,1}w_{i,1} + w_{i,1}^T w_{i,2} + s_i^T K_i\tilde{x}_{i,1}\\ & - s_i^T Y_{i,1}\tilde{\theta}_{i,1} + \tilde{\theta}_{i,1}^T\Gamma_{i,1}^{-1}\dot{\tilde{\theta}}_{i,1} - w_{i,2}^T\Lambda_{i,2}w_{i,2} - w_{i,1}^T w_{i,2}\\ & - w_{i,1}^T Y_{i,2}\tilde{\theta}_{i,2} + \tilde{\theta}_{i,2}^T\Gamma_{i,2}^{-1}\dot{\tilde{\theta}}_{i,2} - \frac{\bar{d}_i w_{i,2}^T w_{i,2}}{\|w_{i,2}\| + e^{-t}} + w_{i,2}^T d_i\\ \le & -(\zeta_{i,0}-1)s_i^T K_i s_i - (\underline{\Lambda}_{i,1} - \frac{1}{4}\bar{k}_i)w_{i,1}^T w_{i,1} - w_{i,2}^T\Lambda_{i,2}w_{i,2}\\ & - \frac{1}{2}\sigma_i\tilde{\theta}_{i,1}^T\tilde{\theta}_{i,1} + \frac{1}{2}\sigma_i\theta_{i,1}^T\theta_{i,1} - \frac{1}{2}\sigma_i\tilde{\theta}_{i,2}^T\tilde{\theta}_{i,2} + \frac{1}{2}\sigma_i\theta_{i,2}^T\theta_{i,2}\\ & + \frac{1}{2}s_i^T K_i^2 s_i + \frac{1}{2}\tilde{x}_{i,1}^T\tilde{x}_{i,1} + \bar{d}_i e^{-t}.\end{aligned} \tag{39}$$

Then, we design the following Lyapunov function candidate for (23)

$$V_{i,2} = \frac{1}{2}\underline{k}_{xi}^{-1}\tilde{x}_{i,1}^T\tilde{x}_{i,1} + \frac{1}{2}\underline{k}_{xi}^{-1}\tilde{x}_{i,2}^T\tilde{x}_{i,2}. \tag{40}$$

Taking the derivative of (40) yields

$$\begin{aligned}\dot{V}_{i,2} = & \underline{k}_{xi}^{-1}\tilde{x}_{i,1}^T\dot{\tilde{x}}_{i,1} + \underline{k}_{xi}^{-1}\tilde{x}_{i,2}^T\dot{\tilde{x}}_{i,2}\\ \le & -\frac{1}{4}\tilde{x}_{i,1}^T\tilde{x}_{i,1} - \frac{1}{4}\tilde{x}_{i,2}^T\tilde{x}_{i,2} + 2\underline{k}_{xi}^{-1}k_{xi}^{-1}\dot{x}_{i,1d}^T\dot{x}_{i,1d}.\end{aligned} \tag{41}$$

Note that $\dot{x}_{i,1d}$ satisfies $\|\dot{x}_{i,1d}\|^2 \le \lambda_i(\frac{17}{32}\|s_i\|^2 + \frac{1}{512}\|s_i\|^4 + 2)$, where a detailed analysis about calculation of λ_i and $\underline{\lambda}_i$ is given in Appendix V-A. Therefore, we choose $k_{xi} = \underline{\lambda}_i^{-\frac{1}{2}}\lambda_i$, and $\underline{k}_{xi} = \underline{\lambda}_i^{\frac{1}{2}}$. Substituting k_{xi} into (41) yields

$$\dot{V}_{i,2} \le -\frac{1}{4}\tilde{x}_{i,1}^T\tilde{x}_{i,1} - \frac{1}{4}\tilde{x}_{i,2}^T\tilde{x}_{i,2} + \frac{17}{16}\|s_i\|^2 + \frac{1}{256}\|s_i\|^4 + 4.$$

Define the following Lyapunov function candidate as

$$V_{i,3} = 16V_{i,1} + V_{i,2}. \tag{42}$$

The derivative of $V_{i,3}$ satisfies

$$\begin{aligned}\dot{V}_{i,3} \le & -(\zeta_{i,0}-1)s_i^T K_i s_i - (\underline{\Lambda}_{i,1} - \frac{1}{4}\bar{k}_i)w_{i,1}^T w_{i,1} - w_{i,2}^T\Lambda_{i,2}w_{i,2}\\ & - \frac{1}{2}\sigma_i\tilde{\theta}_{i,1}^T\tilde{\theta}_{i,1} + \frac{1}{2}\sigma_i\theta_{i,1}^T\theta_{i,1} - \frac{1}{2}\sigma_i\tilde{\theta}_{i,2}^T\tilde{\theta}_{i,2} + \frac{1}{2}\sigma_i\theta_{i,2}^T\theta_{i,2}\\ & + \frac{1}{12}s_i^T K_i^2 s_i - \tilde{x}_{i,1}^T\tilde{x}_{i,1} - 4\tilde{x}_{i,2}^T\tilde{x}_{i,2} + 17\|s_i\|^2 + \frac{1}{16}\|s_i\|^4\\ & + 64 + \bar{d}_i e^{-t}.\end{aligned} \tag{43}$$

Let $\bar{V}_{i,3}$ be the upper bound of $V_{i,3}(0)$. Define a constant $c_i = \min(2\underline{\Lambda}_{i,1} - \frac{1}{2}\bar{k}_i, 2\underline{\Lambda}_{i,2}, \frac{1}{8}\underline{k}_{xi}, \sigma_i\underline{\Gamma}_i)$. Here we choose $\zeta_{i,0} = 1 + \frac{1}{12}\bar{k}_i + 17\underline{k}_i^{-1} + \underline{k}_i^{-1}(\frac{1}{4}\underline{m}_i^{-2}\bar{V}_{i,3} + \frac{1}{2}\bar{m}_i c_i + \beta_i)$, where β_i is an arbitrarily positive constant.

$$\begin{aligned}\dot{V}_{i,3} \le & -(\underline{\zeta}_{i,0} - \frac{1}{2}\bar{m}_i c_i - \frac{1}{4}\underline{m}_i^{-2}V_{i,3})\|s_i\|^2 + \frac{1}{2}\sigma_i\bar{\theta}_{i,1}^2 + \frac{1}{2}\sigma_i\bar{\theta}_{i,2}^2\\ & - c_iV_{i,3} + \frac{1}{2}\sigma_i\bar{\Gamma}_i\underline{\Gamma}_{Ki}^{-1}\sum_{j=1}^{p}\bar{K}_{ij}(\bar{K}_{ij}^{-1} - \underline{K}_{ij}^{-1})^2 + 64 + \bar{d}_i e^{-t}\\ \le & -(\beta_i + \frac{1}{2}c_i\underline{m}_i)\|s_i\|^2 + \frac{1}{2}\sigma_i\bar{\theta}_{i,1}^2 + \frac{1}{2}\sigma_i\bar{\theta}_{i,2}^2 + 64 + \bar{d}_i e^{-t}\\ & + \frac{1}{2}\sigma_i\bar{\Gamma}_i\underline{\Gamma}_{Ki}^{-1}\sum_{j=1}^{p}\bar{K}_{ij}(\bar{K}_{ij}^{-1} - \underline{K}_{ij}^{-1})^2.\end{aligned} \tag{44}$$

From (44), we have the semi-global ultimately bounded stability of the error dynamics. Therefore, $s_i, w_{i,1}, w_{i,2}, \tilde{x}_{i,1}, \tilde{x}_{i,2}, \tilde{\theta}_{i,1}, \tilde{\theta}_{Ki}, \tilde{\theta}_{i,2} \in \mathbb{L}_\infty$. From (44), the term $\bar{d}_i e^{-t}$ eventually converges to zero and we have $\lim_{t\to\infty} s_i(t) \in \Omega_{s_i} = \{\|s_i\| \le \bar{s}_i\}$, where $\bar{s}_i = (\beta_i + \frac{1}{2}c_i\underline{m}_i)^{-\frac{1}{2}}(\frac{1}{2}\sigma_i\theta_{i,1}^T\theta_{i,1} + \frac{1}{2}\sigma_i\bar{\Gamma}_i\underline{\Gamma}_{Ki}^{-1}\sum_{j=1}^{p}\bar{K}_{ij}(\bar{K}_{ij}^{-1} - \underline{K}_{ij}^{-1})^2 + \frac{1}{2}\sigma_i\theta_{i,2}^T\theta_{i,2} + 64)^{\frac{1}{2}}$. According to (15), we obtain that $\lim_{t\to\infty} e_i(t) \in \Omega_{e_i} = \{\|e_i\| \le k_{di}^{-1}\bar{s}_i\}$, $i = 1,\ldots,n$. □

C. Consensus Analysis

The following theorem demonstrates the main result of this paper.

Theorem 1*:* Under Assumption 1, using (26), (27), (36) and (38) for (4), (5), and (6), the agents will achieve the ultimately bounded consensus, *i.e.*, $\|q_i(t) - q_j(t)\|$ converge to set $\Omega_{qij} = \{\|q_i - q_j\| \le 2c\sum_{i=1}^{n}k_{di}^{-1}\bar{s}_i + k_{di}^{-1}\bar{s}_i + k_{dj}^{-1}\bar{s}_j\}$, and $\|\dot{q}_i(t)\|$ converge to set $\Omega_{\dot{q}_i} = \{\|\dot{q}_i\| \le [\lambda_{\max}(P)c + \lambda_{\max}(K^{-1}\mathscr{L})]\sum_{i=1}^{n}k_{di}^{-1}\bar{s}_i + k_{di}^{-1}\bar{s}_i + \bar{s}_i\}$, $\forall i, j = 1,\ldots,n$.

Proof. Assume that the eigenvector corresponding to the zero eigenvalue of $\mathscr{L}$ and $\widetilde{\mathscr{L}}$ are $v_0 \in \mathbb{R}^n$ $\tilde{v} = \{\tilde{v}_1,\ldots,\tilde{v}_{4n}\} \in \mathbb{R}^{4n}$, respectively. From $\widetilde{\mathscr{L}}^T\tilde{v} = \mathbf{0}_{4n}$ and $\mathbf{1}_{4n}^T\tilde{v} = 1$, we have $\tilde{v} = (\mathbf{1}_{4n}^T\tilde{v}^*)^{-1}\tilde{v}^*$, where $\tilde{v}^* = (v_0^T\mathscr{D}K^{-1}, v_0^T\mathscr{D}K^{-1}, v_0^T\mathscr{D}K^{-1}, v_0^T)^T$. Left multiplying both sides of (13) by $\tilde{v}^T \otimes \mathbf{I}_p$ yields we have $(\tilde{v}^T \otimes \mathbf{I}_p)\dot{\xi} = \mathbf{0}_{4p}$. Therefore, $(\tilde{v}^T \otimes \mathbf{I}_p)\xi(t) = (\tilde{v}^T \otimes \mathbf{I}_p)\xi(0)$, $\forall t > 0$, which implies that $\sum_{j=1}^{4n}\tilde{v}_j\xi_j(t) = \sum_{j=1}^{4n}\tilde{v}_j\xi_j(0)$ and $\sum_{j=1}^{4n}\tilde{v}_j\xi_j \in \mathbb{L}_\infty$. Define $\tilde{\xi} = \xi - (\mathbf{1}_{4n}\tilde{v}^T \otimes \mathbf{I}_p)\xi$. Noted that $(\mathbf{1}_{4n}\tilde{v}^T \otimes \mathbf{I}_p)\tilde{\xi} = \mathbf{0}_{4np}$. Then, (13) can be written as

$$\dot{\tilde{\xi}} = P\tilde{\xi} + He_*, \tag{45}$$

where $P = -(\widetilde{\mathscr{L}} + k_\alpha\mathbf{1}_{4n}\tilde{v}^T) \otimes \mathbf{I}_p$. From Lemma 1, P is a Hurwitz matrix. Note that there exist positive constants ω, δ such that $\|e^{Pt}\| \le \delta e^{-\omega t}$. Integrating both sides of (45), we can get $\|\tilde{\xi}(t)\| \le \delta e^{-\omega t}\tilde{\xi}(t_0) + c\sup_{t_0\le\tau\le t}\|e_*(\tau)\|$, where $c = \omega^{-1}\delta\|H\|$. Therefore, we conclude that the consensus error of the reference models $\tilde{\xi}$ converges to the set $\Omega_{\tilde{\xi}} = \{\|\tilde{\xi}\| \le$

$c\sum_{i=1}^{n} k_{di}^{-1}\bar{s}_i\}$. The relative position errors among reference models $\|z_i - z_j\|$ are ultimately bounded with the ultimate bounds $\Omega_{zij} = \{\|z_i - z_j\| \le 2c\sum_{i=1}^{n} k_{di}^{-1}\bar{s}_i\}$, $\forall i,j = 1,\ldots,n$. From the boundedness of e_i, the relative joint position errors among agents $\|q_i - q_j\|$ are ultimately bounded with the ultimate bounds $\Omega_{qij} = \{\|q_i - q_j\| \le 2c\sum_{i=1}^{n} k_{di}^{-1}\bar{s}_i + k_{di}^{-1}\bar{s}_i + k_{dj}^{-1}\bar{s}_j\}$. The velocity of the reference models $\dot{\xi}$ converges to the set $\Omega_{\dot{\xi}} = \{\|\dot{\xi}\| \le [\lambda_{\max}(P)c + \lambda_{\max}(K^{-1}\mathscr{L})]\sum_{i=1}^{n} k_{di}^{-1}\bar{s}_i\}$. Finally, the joint velocities $\dot{q}_i$ converge to the set $\Omega_{\dot{q}_i} = \{\|\dot{q}_i\| \le [\lambda_{\max}(P)c + \lambda_{\max}(K^{-1}\mathscr{L})]\sum_{i=1}^{n} k_{di}^{-1}\bar{s}_i + k_{di}^{-1}\bar{s}_i + \bar{s}_i\}$, $i,j = 1,\ldots,n$, which completes the proof. □

IV. Conclusion

This paper has studied the distributed consensus problem of multiple manipulators with elastic joints under a directed graph. Through the model reference adaptive consensus method, a linear reference model has been constructed and the ultimately bounded consensus of the reference models has been proved. Then, for the trajectory tracking problem of a single manipulator with elastic joints, a tracking control algorithm has been designed using the command filter based adaptive backstepping method. Under the proposed distributed control law, the joint positions of the manipulators eventually achieve consensus with bounded error.

V. Appendix

A. Boundedness of $\dot{x}_{i,1d}$

From (20), $\dot{x}_{i,1d}$ has upper bound as follows

$$\begin{aligned}\|\dot{x}_{i,1d}\| =& \| -\dot{k}_{qi}s_i - k_{qi}\dot{s}_i + \dot{q}_i - \dot{\hat{K}}_i\hat{K}_i^{-2}Y_{i,1}\hat{\theta}_i - \hat{K}_i^{-1}\dot{Y}_{i,1}\hat{\theta}_i \\ & - \hat{K}_i^{-1}Y_{i,1}\dot{\hat{\theta}}_i\|. \end{aligned} \tag{46}$$

Under property (P1), $\ddot{q}_i$ satisfies

$$\|\ddot{q}_i\| \le \zeta_{i,7}\|s_i + k_{di}e_i + \dot{z}_i\|^2 + \zeta_{i,8}\|K_i(x_{i,1} - q_i)\| + \zeta_{i,9}. \tag{47}$$

where $\zeta_{i,7} = k_{Mi}^{\prime -1}k_{ci,1}$, $\zeta_{i,8} = k_{Mi}^{\prime -1}$, $\zeta_{i,9} = k_{Mi}^{\prime -1}k_{gi}^{\prime}$. Then (46) can be written as

$$\begin{aligned}\|\dot{x}_{i,1d}\| \le & \|\underline{k}^{-1}(2\zeta_{i,2}\dot{z}_i^T\ddot{z}_i + 4\zeta_{i,3}\|\dot{z}_i\|^2\dot{z}_i^T\ddot{z}_i + 2\zeta_{i,4}\ddot{z}_i^T z_i^{(3)} \\ & + k_{qi} + 1 + \bar{\Gamma}_{ki}\underline{k}_i^{-2}\|Y_{i,1}\hat{\theta}_{i,1}\|^2 + \underline{k}_i^{-1}\bar{\Gamma}_i\bar{Y}_{i,1}^2)\|s_i\| \\ & + \|(k_{qi}k_{di}^2 - k_{di})e_i + \underline{k}_i^{-1}k_{qi}\ddot{z}_i - \hat{K}_i^{-1}\dot{Y}_{i,1}\hat{\theta}_{i,1} \\ & + \sigma_i\hat{K}_i^{-1}Y_{i,1}\Gamma_{i,1}\hat{\theta}_i\| + 2\underline{k}_i^{-1}\zeta_{i,5}\|e_i\|\|s_i\|^2 + k_{qi}\|\ddot{q}_i\|.\end{aligned}$$

$Y_{i,1}$ satisfies $\|Y_{i,1}\| \le \bar{Y}_{i,1} = \zeta_{i,10}(\|\ddot{z}_i\| + \|\dot{z}_i\|^2 + 1)$, where $\zeta_{i,10}$ is a positive constant. Then $\dot{x}_{i,1d}$ satisfies

$$\begin{aligned}\|\dot{x}_{i,1d}\| \le & \lambda_{i,1}\|s_i\| + \lambda_{i,2}\|s_i\|^2 + \lambda_{i,3} \\ \le & (2\lambda_{i,1} + 32\lambda_{i,2} + \lambda_{i,3})(\frac{1}{2}\|s_i\| + \frac{1}{32}\|s_i\|^2 + 1),\end{aligned}$$

where $\lambda_{i,1} = \underline{k}_i^{-1}(\zeta_{i,2}\|\dot{z}_i\|^2 + \zeta_{i,2}\|\ddot{z}_i\|^2 + 2\zeta_{i,3}\|\dot{z}_i\|^6 + 2\zeta_{i,3}\|\ddot{z}_i\|^2) + \zeta_{i,4}\|\ddot{z}_i\|^2 + \zeta_{i,4}\|z_i^{(3)}\|^2 + k_{qi}k_{di} + 2\underline{k}_i^{-1}\zeta_{i,5}k_{di}\|e_i\|^2 + 1 + \bar{\Gamma}_i\|Y_{i,1}\hat{\theta}_i\|^2 + \underline{k}_i^{-1}\bar{\Gamma}_i\bar{Y}_{i,1}^2 + \zeta_{i,7}k_{qi}(k_{di}^2\|e_i\|^2 + \|\dot{z}_i\|^2 + 2)$, $\lambda_{i,2} = \underline{k}_i^{-1}\zeta_{i,5}(\|e_i\|^2 + 1) + k_{qi}\zeta_{i,7}$, $\lambda_{i,3} = \frac{1}{2} + \frac{1}{2}\|(k_{qi}k_{di}^2 - k_{di})e_i - k_{qi}\ddot{z}_i - \mathrm{diag}(\dot{Y}_{i,1}\hat{\theta}_i)\hat{\theta}_{Ki} + \sigma_i\mathrm{diag}(Y_{i,1}\hat{\theta}_i)\hat{\theta}_{Ki}\|^2 + k_{qi}\Big[2\zeta_{i,7}k_{di}^2\|e_i\|^2 + 2\zeta_{i,7}\|\dot{z}_i\|^2 + \frac{1}{2}\xi_{i,8}\zeta_{i,9}^{-1}\|\bar{K}_i(x_i - q_i)\|^2 + \frac{3}{2}\zeta_{i,9}\Big]$. Therefore, by using Young's inequality, we have

$$\begin{aligned}\|\dot{x}_{i,1d}\|^2 \le & (2\lambda_{i,1} + 32\lambda_{i,2} + \lambda_{i,3})^2(\frac{17}{32}\|s_i\|^2 + \frac{2}{1024}\|s_i\|^4 + 2) \\ \le & \lambda_i(\frac{17}{32}\|s_i\|^2 + \frac{1}{512}\|s_i\|^4 + 2),\end{aligned} \tag{48}$$

where $\lambda_i = (2\lambda_{i,1} + 32\lambda_{i,2} + \lambda_{i,3})^2$. From the definition of λ_i, the lower bound of λ_i can be calculated as

$$\begin{aligned}\underline{\lambda}_i = & \Big[\frac{5}{2} + 2\underline{k}_i^{-1}\bar{\Gamma}_i\zeta_{i,10}^2 + 32\underline{k}_i^{-1}\zeta_{i,5} + (\zeta_{i,0} + \underline{k}_i^{-1}\zeta_{i,1} \\ & + \underline{k}_i^{-1}k_{di}^{-2}\zeta_{i,6})(2k_{di} + 36\zeta_{i,7} + \frac{3}{2}\zeta_{i,9})\Big]^2.\end{aligned} \tag{49}$$

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