

# Data-driven control of nonlinear systems from input-output data\*

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**Abstract**—The design of controllers from data for nonlinear systems is a challenging problem. In a recent paper, De Persis, Rotulo and Tesi, “Learning controllers from data via approximate nonlinearity cancellation,” *IEEE Transactions on Automatic Control*, 2023, a method to learn controllers that make the closed-loop system stable and dominantly linear was proposed. The approach leads to a simple solution based on data-dependent semidefinite programs. The method uses input-state measurements as data, while in a realistic setup it is more likely that only input-output measurements are available. In this note we report how the design principle of the above mentioned paper can be adjusted to deal with input-output data and obtain dynamic output feedback controllers in a favourable setting.

## I. INTRODUCTION

Learning controllers from data is of uttermost importance and a fascinating topic, with foundations in both control theory and data science. Several recent approaches have been proposed for data-driven control, initially focusing, as is natural, on linear systems, e.g. [1], [2], [3], [4]. For nonlinear systems, some results have appeared as well, mostly focusing on special classes of nonlinear systems, bilinear [5], [6], polynomial [7], [8], [9], rational [10] or with quadratic nonlinearities [11], [12]. Other approaches consist of approximating general nonlinear control systems to classes for which data-driven design is possible [13], [14] or expressing nonlinear systems via a dictionary of known functions, in which case the design can aim at making the closed-loop system dominantly linear [15] or prescribing a desired output signal [16].

The understanding of the topic is far from having reached a mature phase, even in the case full measurements of the state are available. Yet, it can be argued that the use of these data-dependent design schemes in practice very much rely on the possibility that they work with output measurements data only, which dispenses the designer from requiring to know the state of the system – a very restrictive prior in many cases. In this paper we report on some early results on using data-driven control techniques in conjunction with input/output data for discrete-time nonlinear systems.

*Related work.* Even when a model is known, output feedback control for nonlinear systems is a challenging open

problem [17, Section 8.7]. The certainty equivalence principle, which is valid for linear systems, is hard to extend to a nonlinear setting. Nonetheless, certain nonlinear discrete-time versions of the certainty equivalence principle have been obtained [18]. In [19], the state in a globally stabilizing state feedback (possibly generated by a finite horizon model predictive scheme) is replaced by an estimate provided by an observer under a uniform observability assumption to obtain a globally stabilizing output feedback controller.

The important uniform observability property [20], [21], [22] can be explored in different ways in the context of learning control from data. Since it guarantees the existence of an injective map from input/output sequences to the state, deep neural networks can be trained to approximate such a map and provide estimates of the state to be used in the given input-to-state stabilizing feedback, obtaining a locally asymptotically stable closed-loop system [23]. The injective map can also be used to define the regression relating the input/output sequences of the system and deep neural networks can be used to learn such a regression [24]. However, to the best of our knowledge there are very few other attempts at designing controllers for nonlinear system from input/output data.

*Contribution.* The aim of this note is to start the investigation of feedback design from input/output data for nonlinear discrete-time systems. We adopt the notion of uniform observability, which allows us to extend some of the design procedures introduced in [2]. Namely, we consider past inputs and outputs as fictitious state variables and obtain a form of the system for which the data-driven “state” feedback design techniques for nonlinear systems of [15] can be used. The implementation of the controller is then carried out by replacing the past input/output measurements with the quantities returned by a dead-beat observer of the output and a chain of integrators driven by the input. A formal analysis of the stability of the overall closed-loop system is then presented along with a discussion about the proposed solution.

In Section II we recall the notion of observability that we adopt for our analysis and introduce an auxiliary system that reproduces the input/output behaviour of the system to control. The auxiliary system is extended in Section III-A with a chain of integrators that provides the past inputs of the system to be used in the controller. The design of the output feedback dynamic controller based on input/output data is presented in Section III. The analysis of the closed-loop system to show the convergence of the system’s and the controller’s state to the origin is the topic of Section IV, along with a discussion of the result. The proofs are omitted

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due to space limitations and can be found in [25].

## II. PRELIMINARIES

We consider the single-input single-output nonlinear discrete-time system

$$\begin{aligned} x^+ &= f(x, u) \\ y &= h(x) \end{aligned} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}$ ,  $f(0, 0) = 0$  and  $h(0) = 0$ .  $f, h$  are continuous functions of their arguments with domains  $\mathbb{R}^n \times \mathbb{R}$  and  $\mathbb{R}^n$ . These functions are unknown. The dimension of the state-space  $n$  is not necessarily known.

### A. Dataset

A dataset consisting of open-loop input-output measurements

$$\mathcal{D} := \{(u(k), y(k))\}_{k=0}^{N+T-1} \quad (2)$$

is available, where the positive integers  $N, T$  will be specified later. The samples in the dataset are obtained from off-line experiment(s) conducted on system (1), hence they satisfy the equations (1), namely

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)), \quad \forall k = 0, 1, \dots, N+T-1 \end{aligned}$$

For our purpose of designing an output feedback controller from  $\mathcal{D}$  it is not required that all the samples of the dataset are sequentially obtained in a single experiment. In fact, even multiple experiments collecting  $N+T$  samples suffice. This is useful especially when dealing with unstable dynamics.

### B. Uniform Observability

The problem of interest is to design an output feedback controller that stabilizes the nonlinear system, based on the dataset  $\mathcal{D}$ . To this purpose, we need to infer the behavior of the state  $x$  from input-output measurements, for which suitable ‘‘observability’’ conditions on the system (1) are required. Before stating them, we introduce some notation. We let

$$\begin{aligned} F^0(x) &:= x \\ F^1(x, v_0) &:= f(x, v_0) \\ F^{k+1}(x, v_0, \dots, v_k) &:= f(F^k(x, v_0, \dots, v_{k-1}), v_k), k \geq 1 \end{aligned} \quad (3)$$

Note that (3) gives  $x(k) = F^N(x(k-N), u_{[k-N, k-1]})$ . To reduce the notational complexity, we introduce  $v_{[0, k]}$ , which denotes the sequence of values  $v_0, \dots, v_k$ . Hence, the last identity above is rewritten as  $F^{k+1}(x, v_{[0, k]}) := f(F^k(x, v_{[0, k-1]}), v_k)$ . In what follows, we will use symbols like  $v_{[0, k]}$  also to denote the vector  $[v_0 \ v_1 \ \dots \ v_k]^\top$ .

The following is the main assumption on system (1).

*Assumption 1:* Let  $\mathcal{X} \subset \mathbb{R}^n$  and  $\mathcal{U} \subset \mathbb{R}$  be compact sets such that  $\mathcal{X} \times \mathcal{U}$  contains the origin of  $\mathbb{R}^{n+1}$ . There exists  $N \in \mathbb{Z}_{>0}$  such that, for any  $v_{[0, N-2]} \in \mathcal{U}^{N-1}$ , the mapping

$$\Phi_N(x, v_{[0, N-2]}) = \begin{bmatrix} h \circ F^0(x) \\ h \circ F^1(x, v_0) \\ \vdots \\ h \circ F^{N-1}(x, v_{[0, N-2]}) \end{bmatrix} \quad (4)$$

is injective as a function of  $x$  on  $\mathcal{X}$ .  $\square$

Following [22, Definition 1], we refer to the assumption above as a uniform observability on  $\mathcal{X}$  property. It is observed in [22] that, if  $f, h$  are continuously differentiable functions, uniform observability is not restrictive in the sense that a nonuniform distinguishability property and a nonuniform observability rank condition imply uniform observability. Since for any  $M \geq N$  the mapping  $\Phi_M$  remains injective, we do not need to know the smallest  $N$  for which Assumption 1 holds.

For any  $v_{[0, N-2]} \in \mathcal{U}^{N-1}$ , the function

$$\Phi_N(\cdot, v_{[0, N-2]}): \mathcal{X} \rightarrow \mathbb{R}^N$$

such that  $x \mapsto w = \Phi_N(x, v_{[0, N-2]})$ , is injective on  $\mathcal{X}$  and one can define a left inverse

$$\Psi_N(\cdot, v_{[0, N-2]}): \Phi_N(\mathcal{X}, v_{[0, N-2]}) \rightarrow \mathbb{R}^n$$

such that  $\Psi_N(\Phi_N(x, v_{[0, N-2]}), v_{[0, N-2]}) = x$  for all  $x \in \mathcal{X}$ .

### C. An auxiliary system

We introduce a system equivalent to (1) which is better suited for control design. By equivalent it is meant that the new system has the same input-output behavior of system (1) when properly initialized. We use this auxiliary system for control design purposes. Later on we show the effect of the designed controller on the actual system (1).

For any  $v_{[0, N-1]} \in \mathbb{R}^N$ , define the functions

$$\begin{aligned} \psi(w, v_{[0, N-1]}) &:= F^N(\Psi_N(w, v_{[0, N-2]}), v_{[0, N-1]}) \\ \tilde{h}(w, v_{[0, N-1]}) &:= h \circ \psi(w, v_{[0, N-1]}) \\ \tilde{f}(w, v_{[0, N-1]}) &:= A_c w + B_c \tilde{h}(w, v_{[0, N-1]}) \end{aligned} \quad (5)$$

with the pair  $(A_c, B_c) \in \mathbb{R}^{N \times N} \times \mathbb{R}^N$  in the Brunovsky form. The domain of  $\psi(\cdot, v_{[0, N-1]})$ ,  $\tilde{h}(\cdot, v_{[0, N-1]})$ ,  $\tilde{f}(\cdot, v_{[0, N-1]})$  is  $\Phi_N(\mathcal{X}, v_{[0, N-2]})$ . Under the standing assumptions on  $f, h$ , these functions are continuous and zero at  $(w, v) = (0, 0)$ .

In the result below, for a  $k \in \mathbb{Z}$ , we let  $u_{[k-N, k-1]}$  be an input sequence applied to system (1) and  $y_{[k-N, k-1]}$  its output response from some initial condition  $x(k-N)$ .

*Lemma 1:* Let system (1) satisfy Assumption 1. Consider arbitrary  $k_0 \in \mathbb{Z}$ ,  $x(k-N) \in \mathcal{X}$  and  $u_{[k-N, k-1]} \in \mathcal{U}^N$  for all  $k \in \mathbb{Z}_{\geq k_0}$ . Consider the system

$$\begin{aligned} w^+ &= \tilde{f}(w, v) \\ y_w &= \tilde{h}(w, v) \end{aligned} \quad (6)$$

with  $\tilde{f}, \tilde{h}$  defined in (5). If the input  $v(k)$  applied to (6) satisfies  $v(k) = u_{[k-N, k-1]}$  for all  $k \in \mathbb{Z}_{\geq k_0}$  and the initial condition of (6) is set to  $w(k_0) = y_{[k_0-N, k_0-1]}$ , then

$$w(k) = y_{[k-N, k-1]}, \quad y_w(k) = y(k), \quad \forall k \in \mathbb{Z}_{\geq k_0}.$$

Furthermore,  $x(k) = \psi(w(k), v(k))$ , for all  $k \in \mathbb{Z}_{\geq k_0}$ .  $\square$

### III. DESIGN OF AN OUTPUT FEEDBACK CONTROLLER FROM DATA

#### A. A dynamic extension

System (6) is driven by the past  $N$  samples of  $u$ , which is the input to (1). These past values are obtained by adding a chain of integrators to the dynamics (6)

$$\xi^+ = A_c \xi + B_c u \quad (7)$$

with the interconnection condition

$$v = \xi$$

which returns the system

$$\begin{aligned} w^+ &= \tilde{f}(w, \xi) \\ \xi^+ &= A_c \xi + B_c u \\ y &= \tilde{h}(w, \xi) \end{aligned} \quad (8)$$

Once the system's state satisfies  $(w(\bar{k}), \xi(\bar{k})) = (y_{[\bar{k}-n, \bar{k}-1]}, u_{[\bar{k}-n, \bar{k}-1]})$  for some  $\bar{k} \in \mathbb{Z}$ , the input-output behavior of this system matches the one of (1) for all  $k \geq \bar{k}$ . We will discuss later on the availability of such initial condition at a time  $\bar{k}$ .

#### B. Control input design

To obtain  $u$  that drives the chain of integrators making the dynamic controller, we argue as in [2], [15]. We first introduce the following:

*Assumption 2:* For any  $\xi \in \mathcal{U}^N$  and any  $w \in \Phi_N(\mathcal{X}, \xi_{[1, N-1]})$ , where  $\xi_{[1, N-1]}$  denotes the first  $N-1$  entries of  $\xi$ , it holds that  $\tilde{h}(w, \xi) = \alpha Z(w, \xi)$ , where  $Z(w, \xi) \in \mathbb{R}^S$  is a vector of known continuous functions and  $\alpha \in \mathbb{R}^{1 \times S}$  is an unknown vector.  $\square$

This is a technical assumption due to the need to give the nonlinearities of (8) a form for which the controller design is possible. Although it is restrictive, [15, Section VI.B] bypasses such an assumption by expressing  $\tilde{h}(w, \xi)$  as  $\alpha Z(w, \xi) + d(w, \xi)$ , where the term  $d(w, \xi)$  represents the nonlinearities that were excluded from  $Z(w, \xi)$ , and then analyzing the stability of the system in the presence of the neglected nonlinearity  $d(w, \xi)$ . This analysis goes beyond the scope of this paper.

We consider the case in which the function  $Z(w, \xi)$  comprises both a linear part and a nonlinear part  $Q(w, \xi)$ , i.e.

$$Z(w, \xi) = \begin{bmatrix} w \\ \xi \\ Q(w, \xi) \end{bmatrix}$$

The system (8) can then be written as

$$\begin{aligned} \begin{bmatrix} w^+ \\ \xi^+ \end{bmatrix} &= A \begin{bmatrix} w \\ \xi \end{bmatrix} + B_1 u + B_2 \alpha Z(w, \xi) \\ y &= \alpha Z(w, \xi) \end{aligned} \quad (9)$$

where

$$A := \begin{bmatrix} A_c & 0 \\ 0 & A_c \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \quad B_2 := \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

and the pair  $(A_c, B_c)$  is in the Brunovsky canonical form.

We focus on the case in which the input  $u$  is designed as a function of  $Z(w, \xi)$ , i.e.

$$u = \kappa Z(w, \xi) \quad (10)$$

where  $\kappa \in \mathbb{R}^{1 \times S}$  is the control gain. Write the closed-loop system (9)-(10) as

$$\begin{aligned} \begin{bmatrix} w^+ \\ \xi^+ \end{bmatrix} &= A \begin{bmatrix} w \\ \xi \end{bmatrix} + B_1 \kappa Z(w, \xi) + B_2 \alpha Z(w, \xi) \\ y &= \alpha Z(w, \xi) \end{aligned} \quad (11)$$

The system is defined for any  $\xi \in \mathcal{U}^N$  and any  $w \in \Phi_N(\mathcal{X}, \xi_{[1, N-1]})$ .

#### C. Data-dependent representation of the closed-loop system

Preliminary to the design of the controller is a data-dependent representation of the closed-loop system. We first introduce some notation. Recall the dataset in (2) and introduce, for  $i = 0, \dots, T-1$ ,

$$U(i) := \begin{bmatrix} u(i) \\ u(i+1) \\ \vdots \\ u(i+N-1) \end{bmatrix}, \quad Y(i) := \begin{bmatrix} y(i) \\ y(i+1) \\ \vdots \\ y(i+N-1) \end{bmatrix}$$

We assume that the samples of the dataset evolve in the domain of definition of (11).

*Assumption 3:* For any  $i = 0, \dots, T-1$ ,  $U(i) \in \mathcal{U}^N$  and  $Y(i) \in \Phi_N(\mathcal{X}, u_{[i, i+N-2]})$ .  $\square$

We let:

$$\begin{aligned} Y_0 &:= [Y(0) \quad Y(1) \quad \dots \quad Y(T-1)] \\ V_0 &:= [U(0) \quad U(1) \quad \dots \quad U(T-1)] \\ Y_1 &:= [Y(1) \quad Y(2) \quad \dots \quad Y(T)] \\ V_1 &:= [U(1) \quad U(2) \quad \dots \quad U(T)] \\ Q_0 &:= [Q(0) \quad Q(1) \quad \dots \quad Q(T-1)] \\ U_0 &:= [u(N) \quad u(N+1) \quad \dots \quad u(N+T-1)] \end{aligned} \quad (12)$$

In the definition of  $Q_0$ , we are using the shorthand notation  $Q(i)$  for  $Q(Y(i), U(i))$ . As shown in Section V, these matrices can also be built from data obtained from multiple experiments. Under Assumption 3, bearing in mind the dynamics (9), the dataset-dependent matrices introduced in (12) satisfy

$$\begin{bmatrix} Y_1 \\ V_1 \end{bmatrix} = A \begin{bmatrix} Y_0 \\ V_0 \end{bmatrix} + B_1 U_0 + B_2 \alpha \begin{bmatrix} Y_0 \\ V_0 \\ Q_0 \end{bmatrix} \quad (13)$$

We establish the following:

*Lemma 2:* Let Assumptions 1, 2 and 3 hold. Consider any matrices  $\kappa \in \mathbb{R}^{1 \times S}$ ,  $G \in \mathbb{R}^{T \times S}$  that satisfy the relation

$$\begin{bmatrix} \kappa \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Y_0 \\ V_0 \\ Q_0 \end{bmatrix} G \quad (14)$$

and partition  $G$  as

$$G = [G_1 \quad G_2]$$

where  $G_1 \in \mathbb{R}^{T \times 2N}$ ,  $G_2 \in \mathbb{R}^{T \times (S-2N)}$ . Then the closed-loop system (11) can be written as

$$\begin{bmatrix} w^+ \\ \xi^+ \end{bmatrix} = M \begin{bmatrix} w \\ \xi \end{bmatrix} + NQ(w, \xi)$$

where

$$M = \mathcal{X}_1 G_1, \quad N = \mathcal{X}_1 G_2, \quad \mathcal{X}_1 = \begin{bmatrix} Y_1 \\ V_1 \end{bmatrix}. \quad \square \quad (15)$$

Let the set of real-valued symmetric matrices of dimension  $n \times n$  be denoted by  $\mathbb{S}^{n \times n}$ . This data-dependent representation leads to the following local stabilization result:

*Proposition 1:* Let Assumptions 1, 2 and 3 hold. Consider the following SDP in the decision variables  $\mathcal{P}_1 \in \mathbb{S}^{2N \times 2N}$ ,  $\mathcal{Y}_1 \in \mathbb{R}^{T \times 2N}$ , and  $G_2 \in \mathbb{R}^{T \times (S-2N)}$ :

$$\text{minimize}_{\mathcal{P}_1, \mathcal{Y}_1, G_2} \quad \|\mathcal{X}_1 G_2\| \quad (16a)$$

$$\text{subject to} \quad \begin{bmatrix} Y_0 \\ V_0 \\ Q_0 \end{bmatrix} \mathcal{Y}_1 = \begin{bmatrix} \mathcal{P}_1 \\ 0_{(S-2N) \times 2N} \end{bmatrix}, \quad (16b)$$

$$\begin{bmatrix} \mathcal{P}_1 & (\mathcal{X}_1 \mathcal{Y}_1)^\top \\ \mathcal{X}_1 \mathcal{Y}_1 & \mathcal{P}_1 \end{bmatrix} \succ 0, \quad (16c)$$

$$\begin{bmatrix} Y_0 \\ V_0 \\ Q_0 \end{bmatrix} G_2 = \begin{bmatrix} 0_{2N \times (S-2N)} \\ I_{S-2N} \end{bmatrix}. \quad (16d)$$

Assume that

$$\lim_{|(w, \xi)| \rightarrow 0} \frac{|Q(w, \xi)|}{|(w, \xi)|} = 0. \quad (17)$$

If the SDP is feasible then

$$\xi^+ = A_c \xi + B_c u \quad (18)$$

with

$$u = \kappa Z(w, \xi) \quad (19)$$

and  $\kappa$  as in

$$\kappa = U_0 [\mathcal{Y}_1 \quad G_2] \begin{bmatrix} \mathcal{P}_1^{-1} & 0_{2N \times (S-2N)} \\ 0_{(S-2N) \times 2N} & I_{S-2N} \end{bmatrix} \quad (20)$$

renders the origin  $(\bar{w}, \bar{\xi}) = (0, 0)$  an asymptotically stable equilibrium of

$$\begin{aligned} w^+ &= \tilde{f}(w, \xi) \\ \xi^+ &= A_c \xi + B_c \kappa Z(w, \xi) \\ y &= \tilde{h}(w, \xi). \end{aligned} \quad (21)$$

□

#### D. Region of Attraction

Proposition 1 provides a local stabilization result. Following [15], Proposition 1 can be extended to provide an estimate of the Region of Attraction (ROA) of the system (21). First we recall the following definitions.

*Definition 1:* [26, Definition 13.2] Suppose that  $\bar{x} = 0$  is an asymptotically stable equilibrium for  $x^+ = f(x)$ . Then the ROA of  $x^+ = f(x)$  is given by

$$\mathcal{A}_0 = \{x_0 : \lim_{k \rightarrow \infty} s_k(x_0) = 0\}$$

where  $s_k(x_0)$  is the solution to  $x^+ = f(x)$  at time  $k \geq k_0$  from the initial condition  $x_0$ . □

*Definition 2:* [26, Definition 13.4] A set  $\mathcal{M} \subset \mathbb{R}^n$  is a positively invariant set for  $x^+ = f(x)$  if  $s_k(\mathcal{M}) \subseteq \mathcal{M}$  for all  $k \geq k_0$ , where  $s_k(\mathcal{M}) = \{s_k(x_0) : x_0 \in \mathcal{M}\}$ . □

Recall the Lyapunov difference

$$\begin{aligned} & V(w^+, \xi^+) - V(w, \xi) \\ &= \left( M \begin{bmatrix} w \\ \xi \end{bmatrix} + NQ(w, \xi) \right)^\top \mathcal{P}_1^{-1} \left( M \begin{bmatrix} w \\ \xi \end{bmatrix} + NQ(w, \xi) \right) \\ &\quad - \begin{bmatrix} w \\ \xi \end{bmatrix}^\top \mathcal{P}_1^{-1} \begin{bmatrix} w \\ \xi \end{bmatrix} =: \mathcal{W}(w, \xi) \end{aligned}$$

with  $M, N$  as in (15).

*Corollary 1:* Consider the same setting as Proposition 1. Let<sup>1</sup>  $\mathcal{V} := \{(w, \xi) : \mathcal{W}(w, \xi) < 0\}$ . Any sublevel set  $\mathcal{R}_\gamma = \{(w, \xi) : V(w, \xi) \leq \gamma\}$  contained in  $\mathcal{V} \cup \{0\}$  is positively invariant for system (21) and defines an estimate of the ROA of system (21). □

As the function  $\mathcal{W}(w, \xi)$  is known from the data, the estimate of the ROA  $\mathcal{R}_\gamma$  is computable.

#### IV. MAIN RESULT

To draw conclusions on the convergence of system (1), we first observe that the dynamical controller (18) uses its own state  $\xi$  and the state  $w$  to generate the control action  $u = \kappa Z(\xi, w)$ . At time  $k$  the state  $w(k)$  contains the past  $N$  output measurements from the process (1), from which we only measure  $y(k)$ . To make the past measurements in  $w(k)$  available to the controller, we extend it with the dynamics

$$\eta^+ = A_c \eta + B_c y \quad (22)$$

Then, for any  $k_0 \in \mathbb{Z}$  and any  $\eta(k_0) \in \mathbb{R}^N$ , we have that  $\eta(k) = y_{[k-N, k-1]} = w(k)$  for all  $k \geq k_0 + N$ , that is, independently of the initialization of (22), its state  $\eta(k)$  provides the vector  $w(k)$  of the past output measurements from time  $N$  onward. Similarly, for any  $\xi(k_0) \in \mathbb{R}^N$ , system (18) is such that  $\xi(k) = u_{[k-N, k-1]}$  for all  $k \geq k_0 + N$ . See [19] for the same structure of the controller (18), (22).

*Remark 1:* System (22) is the so-called deadbeat observer, since for  $k \geq k_0 + N$ , the mapping  $\psi(\eta(k), \xi(k))$  would return  $x(k)$ . If both  $\psi$  and a state-feedback stabilizer for system (1) were known, one could obtain a dynamic output feedback controller for the system (1). Here we are interested to the case in which this knowledge is not available and we design a dynamic output feedback controller under a suitable assumption on the nonlinearity  $\tilde{h}$  (Assumption 2). ■

The following statement transfers the result obtained for the system (21) to the actual closed-loop system that includes the process (1).

*Proposition 2:* Let Assumptions 1, 2 and 3 hold. Consider the SDP (16), assume that it is feasible and let condition (17) hold. For any  $(x_0, \xi_0, \eta_0) \in \mathcal{X} \times \mathbb{R}^N \times \mathbb{R}^N$  for which there exists  $v = (v_{[0, N-2]}, v_{N-1}) \in \mathcal{U}^N$  such that

<sup>1</sup>Although not indicated explicitly,  $\mathcal{V}$  is a subset of the domain of definition of  $V(w, \xi)$ .

$(\Phi_N(x_0, v_{[0, N-2]}), v) \in \mathcal{R}_\gamma$ , the solution of the system (1) in closed-loop with the time-varying controller comprised by (18), (22) and

$$u(k) = \begin{cases} v_{k-k_0} & k_0 \leq k \leq k_0 + N - 1 \\ \kappa Z(\eta(k), \xi(k)) & k \geq k_0 + N \end{cases} \quad (23)$$

that starts from  $(x_0, \xi_0, \eta_0)$ , asymptotically converges to the origin.  $\square$

*Proof.* First note that, by definition of the mapping  $\Phi_N$  and since  $f(0, 0) = 0$  and  $h(0) = 0$ , each entry of  $\Phi_N$  is a continuous function of its arguments which is zero when these are zero, hence there exists a neighbourhood of the origin  $(x, v) = (0, 0)$  such that any point  $(x, v)$  in the neighbourhood satisfies  $(\Phi_N(x, v_{[0, N-2]}), v) \in \mathcal{R}_\gamma$ .

By definition of the mapping  $\Phi_N$  in Assumption 1 and (23),  $\Phi_N(x_0, v_{[0, N-2]}) = y_{[k_0, k_0+N-1]}$ , where  $y$  denotes the output response of the closed-loop system from the initial condition  $(x_0, \xi_0, \eta_0)$ .

By the dynamics of the controller (18), (22), we have  $\eta(k) = y_{[k-N, k-1]}$ ,  $\xi(k) = u_{[k-N, k-1]}$  for all  $k \geq k_0 + N$  and  $(\eta(k_0 + N), \xi(k_0 + N)) = (\Phi_N(x, v_{[0, N-2]}), v) \in \mathcal{R}_\gamma$ . Hence, by Lemma 1, the solution of (18), (22) are the same as those of system (21) initialized at  $(w(k_0 + N), \xi(k_0 + N)) = (y_{[k_0, k_0+N-1]}, u_{[k_0, k_0+N-1]})$ . As  $(\eta(k_0 + N), \xi(k_0 + N)) \in \mathcal{R}_\gamma$ , by Proposition 1 and Corollary 1,  $(\eta(k), \xi(k))$  converges to the origin. By Lemma 1, for all  $k \geq k_0 + N$ ,  $x(k) = \psi(\eta(k), \xi(k))$ , which implies convergence of  $x(k)$  to the origin by continuity of  $\psi$ .  $\blacksquare$

The particular form of  $u(k)$  in (23) is due to the fact that, during the first  $N$ -steps, the controller state does not provide an accurate value of the past input-output measurements of the system, hence the choice to apply an open-loop input sequence. After  $N$  time steps, when such past measurements become available through the controller states  $\eta(k), \xi(k)$ ,  $u(k)$  is set to the feedback  $\kappa Z(\eta(k), \xi(k))$ .

We also remark that in the result above if the initial condition  $x_0$  is sufficiently close to the origin and the initial sequence of control values  $v_0, \dots, v_{n-2}, v_{n-1}$  does not drive the output response of (1) outside the set  $\mathcal{R}_\gamma$ , then the designed controller (23) steers the state of the overall closed-loop system to the origin. Note that  $\mathcal{R}_\gamma$  is known thanks to Corollary 1, hence the designer can check whether the initial control sequence and the corresponding measured output response are in  $\mathcal{R}_\gamma$ . For the design of the initial control sequence, the designer could take advantage of some expert knowledge.

*Remark 2: (Prior on input/output measurements)* The controller is designed under the assumption that the input/output measurements collected during the experiment range over some specified sets – see Assumption 3 – where the measurements provide meaningful information about the system’s internal state. These sets are not known, hence, the feature that the evolution of the system during the experiments remain in the sets of interest must be considered as one of the priors under which the design is possible.

## V. NUMERICAL EXAMPLE

We consider [15, Example 5]

$$x_1^+ = x_1 + T_s x_2 \quad (24a)$$

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m \ell^2}\right) x_2 + \frac{T_s}{m \ell} (\cos x_1) u, \quad (24b)$$

with the output  $y = x_1$ . The system parameters are  $T_s = 0.1$ ,  $m = 1$ ,  $\ell = 1$ ,  $g = 9.8$  and  $\mu = 0.01$ . We compute

$$\Phi_2(x, v_0) = \begin{bmatrix} x_1 \\ x_1 + T_s x_2 \end{bmatrix}$$

which is globally invertible with  $N = 2$  (Assumption 1 holds). Following the analysis in Section II, we compute the auxiliary system in the form of (6) with  $w = \Phi_2(x, v_0)$ ,  $\xi_1 = v_0$  and

$$\tilde{h}(w, \xi_1) = \left(-1 + \frac{T_s \mu}{m \ell^2}\right) w_1 + \left(2 - \frac{T_s \mu}{m \ell^2}\right) w_2 + \frac{T_s g}{\ell} \sin w_1 + \frac{T_s^2}{m \ell} (\cos w_1) \xi_1$$

Next, following [15, Example 5], we choose

$$Z(w, \xi) = \text{col}(w, \xi, \sin w_1 - w_1, \xi_1 \cos w_1 - \xi_1).$$

Note that Assumption 2 and (17) hold.

The problem is to learn an output feedback controller for (24) from input-output data that renders the origin of the closed-loop system locally asymptotically stable. We collect data by running  $T = 7$ ,  $N = 2$ -long experiments with input uniformly distributed in  $[-0.5, 0.5]$  and with an initial state in  $[-0.5, 0.5] \times [-0.5, 0.5] \subset \mathbb{R}^2$ . For each experiment  $j = 0, 1, \dots, T-1$ , we collect the samples  $\{w^j(k), y^j(k)\}_{k=0}^T$  in

$$\begin{aligned} Y_0^j &:= \text{col}(y^j(0), y^j(1)), & Y_1^j &:= \text{col}(y^j(1), y^j(2)), \\ V_0^j &:= \text{col}(u^j(0), u^j(1)), & V_1^j &:= \text{col}(u^j(1), u^j(2)), \\ U_0^j &:= u^j(2), & Q_0^j &:= Q(Y_0^j, V_0^j). \end{aligned}$$

Then we construct data matrices  $Y_1, Y_0, V_1, V_0, U_0, Q_0$  in (12) as

$$\begin{aligned} Y_0 &:= [Y_0^0 & Y_0^1 & \dots & Y_0^{T-1}], \\ Y_1 &:= [Y_1^0 & Y_1^1 & \dots & Y_1^{T-1}], \\ V_0 &:= [V_0^0 & V_0^1 & \dots & V_0^{T-1}], \\ V_1 &:= [V_1^0 & V_1^1 & \dots & V_1^{T-1}], \\ Q_0 &:= [Q_0^0 & Q_0^1 & \dots & Q_0^{T-1}], \\ U_0 &:= [U_0^0 & U_0^1 & \dots & U_0^{T-1}]. \end{aligned}$$

The program (16) is feasible and we obtain the controller gain with

$$\kappa = [52.4412 \quad -76.1179 \quad -0.5782 \quad -0.4467 \quad 0 \quad 0] \quad (25)$$

using the YALMIP toolbox [27], MOSEK solver [28]. To assess the effectiveness of the designed controller, instead of computing  $\mathcal{R}_\gamma$ , which for this example provides a conservative estimate of the ROA, we depict in Fig. 1 the set of initial conditions  $x_0$  for which, choosing  $v_{k-k_0} = 0$  for  $k_0 \leq k \leq k_0 + N - 1$  in (23), the state  $(x(k), \eta(k), \xi(k))$  converges to zero. Note that the choice of  $\eta_0, \xi_0$  is inessential.

The set is obtained by letting the closed-loop system evolve for 200 time steps and then checking whether or not the norm  $\|(x(k), \eta(k), \xi(k))\|_\infty$  is smaller than  $10^{-6}$  on the interval  $195 \leq k \leq 200$ .

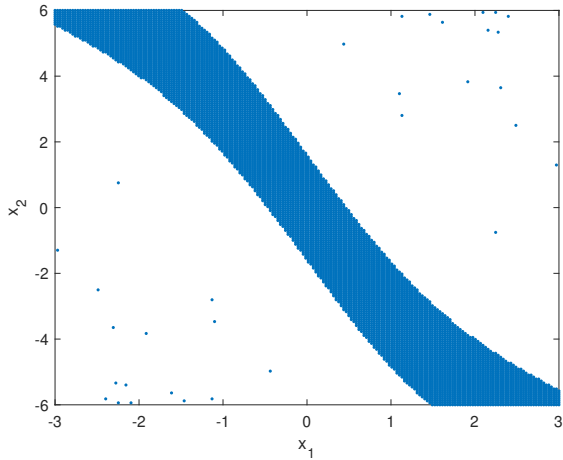


Fig. 1. The blue area represents the estimate of the ROA of system (24) in closed-loop with the controller (18), (22), (23), where  $v_{k-k_0} = 0$  for  $k_0 \leq k \leq k_0 + N - 1$  and  $\kappa$  is given in (25).

## VI. CONCLUSIONS

We have examined a design of dynamic output feedback controllers for nonlinear systems from input/output data. The uniform observability property of the system, a priori in the approach, is instrumental to define a new set of coordinates, from which a data-driven “state”-feedback design can be conducted. The result is local and the size of the region of attraction is limited by the free evolution of the system during the first  $N$  steps during which the dead-beat observer reconstructs the past input/output values that feed the controller. The design and analysis have been carried out in the favourable setting in which measurements are noise-free and the nonlinearities can be expressed via a dictionary of known functions. Regarding the future work, besides going beyond the favourable setting, we would like to explore either a more sophisticated observer design or a different data-driven control design method. An option is to express the function  $\psi$  via a dictionary of functions, perform a data-driven design of an observer and follow a certainty equivalence principle in the analysis of the closed-loop system.

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