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Signal Comparison Average Consensus Algorithm Under Binary-Valued Communications

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***Abstract*—The average consensus problem under binary-valued communications is investigated in the paper. Measurement noises and fixed quantizers are considered. A signal comparison algorithm is proposed for the problem. Neither the noise distribution information nor assumptions on the states' approximate locations are required for the algorithm design. The signal comparison algorithm is proved to achieve average consensus both in the almost sure and mean square sense. The algorithm's mean square convergence rate is also calculated. The efficiency of the algorithm is demonstrated by a numerical example.**

I. Introduction

Cooperative control and distributed consensus problems of multi-agent systems have attracted increasing attention due to the applications in distributed sensor networks, satellite internet constellations, and multiple unmanned aerial vehicle systems, etc. Many of these applications are based on digital networks. Digital networks only allow quantized communications. Therefore, in recent years, researchers pay interests in distributed consensus algorithms that rely on quantized communications.

Consensus algorithms based on infinite-level quantizers have been well-developed. [1] proposes the quantized gossip algorithm under the integer-valued state assumption. When the states are real-valued, [2] applies logarithm quantizers in the average consensus algorithm design. The algorithm achieves consensus exponentially. Besides, [3], [4] design quantized consensus algorithms based on probabilistic quantizers, whose quantization error is independent of the state values and unbiased [5].

Consensus problems under finite communication data rate is more important and difficult, since real digital networks can only transmit finite bits of information at each moment. Under the constraint of communication data rate, [6], [7] use the zooming-in technique to design quantized consensus algorithms. [8] considers measurement noises, and proposes an empirical measurement consensus algorithm. The algorithm only requires binary-valued communications. [9]–[11] give recursive projection-based consensus algorithms, whose mean square convergence rate can be $O(1/k)$ when the step-size is properly selected.

There are still challenging issues for the consensus problem under finite communication data rate. One of the issues is to achieve average consensus without any *a priori* information on the initial values. Among existing works, [6], [11] proposes average consensus algorithms under the assumption that all the agents' initial states are located in a known compact set. Using the assumption, they design control laws to constrain all the states in the compact set. The assumption is difficult to remove, because there is no upper bound for quantization errors of finite-level quantizers. [8] gives a consensus algorithm without any *a priori* information on the initial values. But, the algorithm cannot achieve average consensus, and the mean square convergence rate is slower compared with the algorithm in [9].

Another important issue is about the distribution of the measurement noises. In existing works considering measurement noises [8]–[11], the distribution of the measurement noises should be exactly known. This is because these algorithms are based on the set-valued system identification methods that require full knowledge of noise distribution [8], [12]. To avoid the limitation, we should consider a new consensus algorithm that does not rely on set-valued system identification methods.

In the paper, a signal comparison average consensus algorithm under binary-valued communications and measurement noises is designed. The assumption on the initial states required in [6], [7], [9]–[11] is removed. The key is to construct a stochastic process with averaged observations (SPAO) [13] to estimate the distribution tail. Besides, our algorithm does not rely on the knowledge of the noise distribution. Instead, our algorithm controls the system by directly comparing the binary-valued outputs of the agent itself and its neighbours.

The main contributions of the paper include

- A signal comparison average consensus algorithm is proposed. Measurement noises are considered. Only binary-valued communications are required.
- The algorithm's efficiency is verified. The almost sure and mean square average consensus is proved. It appears to be first to obtain almost sure consensus under finite communication data rate and measurement noises. Besides, when the step-size is properly selected, the mean square convergence rate can be $O(1/k)$.
- The consensus analysis is established under weak con-

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ditions. Firstly, any *a priori* information on the initial states is not required. Secondly, the noise distribution is not required to be known. This appears to be the first to be achieved for the consensus problem under binary-valued communications. Thirdly, the noise distribution is only assumed to be strictly increasing, in contrast to the normal distribution assumption for noises in [8]–[10].

The rest of the paper is organized as follows. Section II formulates the average consensus problem and designs the signal comparison algorithm. Section III analyzes the convergence properties of the algorithm. Section IV simulates a numerical example to demonstrate the efficiency of the algorithm. Finally, Section V gives a brief conclusion.

II. Problem Statement

A. Preliminaries

Here, we denote $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ as the sets of n-dimensional real vectors and $n\times m$-dimensional matrices, respectively. The n-dimensional identity matrix is denoted by I_n. $\|v\|$ is the Euclidean norm for vector v. $\mathbf{1}_n, \mathbf{0}_n \in \mathbb{R}^n$ are column vectors with all ones and zeros, respectively. Then, we define a projection matrix $J_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. $\text{rank}(A)$ as the rank of the matrix A. Besides, we use $\text{diag}(\lambda_1,\dots,\lambda_n)$ to represent the diagonal matrix with i-th diagonal element being λ_i, and $\text{vol}(\lambda_1,\dots,\lambda_n)$ to represent the column vector with i-th element being λ_i.

The communications between agents can given by an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the agent set $\mathcal{V} = \{1,\dots,N\}$, and the edge set $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\}$. $\mathcal{A} = (a_{ij})_{N\times N}$ represents the symmetric weighted adjacency matrix of the graph whose elements are all non-negative, and $a_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$. Besides, $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$ is used to denote the agent i's the neighbour set. Define Laplacian matrix as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}\left(\sum_{i\in\mathcal{N}_1} a_{i1},\dots,\sum_{i\in\mathcal{N}_N} a_{iN}\right)$.

B. Average Consensus Problem and Algorithm Design

Consider a multi-agent system with an undirected and connected communication graph $\mathcal{G}$, whose dynamics can be written as

$$x_i(k) = x_i(k-1) + u_i(k),\ \forall i = 1,\dots,N,$$

where $x_i(k) \in \mathbb{R}$ is the agent i's state, and $u_i(k) \in \mathbb{R}$ is the input to be designed. The state values $x_i(k)$, including the initial value $x_i(0)$, are not necessarily directly measured by the agent i. The agent i quantizes the state measurement using a fixed quantizer with a single threshold $C \in \mathbb{R}$. Then, the binary-valued message can be represented by

$$s_i(k) = \begin{cases} 1, & \text{if } x_i(k) + d_i(k) < C; \\ 0, & \text{otherwise.} \end{cases}.$$

A necessary assumption for the noises is given below.

Assumption 1. Measurement noise sequence $\{d_i(k)\}$ is independent and identically distributed (i.i.d.) with a strictly positive distribution function $F(\cdot)$.

Remark 1. The strict positive assumption for the distribution function $F(\cdot)$ is necessary for the consensus problem. Otherwise, $F(C-\theta_1) = F(C-\theta_2)$ for some $\theta_1 > \theta_2$. Then, one cannot distinguish the cases of $x_i(k) = \theta_1$ and $x_i(k) = \theta_2$ through the stochastic properties of binary-valued output messages $s_i(k)$. The assumption is weak. Even the density function of the noises does not necessarily exist under Assumption 1. For comparison, [8]–[10] assume that measurement noises $d_i(k)$ should be normal variables.

Our goal is to have all the states converge to $\frac{1}{N}\sum_{i=1}^N x_i(0)$ based on binary-valued messages $s_i(k)$. To achieve the goal, we propose the signal comparison average consensus algorithm

$$x_i(k) = x_i(k-1) + q(k)\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i(k-1) - s_j(k-1)\right), \tag{1}$$

where step-size $q(k)$ satisfies

$$\sum_{k=1}^{\infty} q(k) = \infty,\ \sum_{k=1}^{\infty} q^2(k) < \infty.$$

Remark 2. Both the *a priori* information on the location of $x_i(0)$ and the distribution function $F(\cdot)$ are not required to be known for the algorithm design. For comparison, [6], [7], [9]–[11] assume that $\max_i |x_i(0)| \le M$ for known $M > 0$, and [8]–[11] assume that the noise distribution function $F(\cdot)$ should be known for the algorithm design.

III. Main Results

The section focuses on the consensus properties of the signal comparison algorithm (1), including the almost sure and mean square average consensus, as well as the mean square convergence rate.

A. Consensus Analysis

The subsection verifies the average consensus of the signal comparison algorithm (1).

The following lemmas are given first before proceeding the analysis.

Lemma 1. For the signal comparison algorithm (1), we have that $\sum_{i=1}^N x_i(k)$ keeps constant.

Proof. Note that

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i(k-1) - s_j(k-1)\right) = 0.$$

Then, the lemma is proved. □

Remark 3. By Lemma 1, the sum of agents' states keeps constant. Therefore, it suffices to prove that the algorithm can achieve consensus.

Here, we give a lemma for the construction of the Lyapunov function.

Lemma 2. The set

$$[\mathbf{1}]_N^\perp = \left\{B \in \mathbb{R}^{N\times N} \middle| B\mathbf{1}_N = B^\top\mathbf{1}_N = \mathbf{0}_N, \text{rank}(B) = N-1\right\}$$

is a group whose identity element is J_N, and for any $B \in [\mathbf{1}]_N^{\perp}$, the inverse of B in the group $[\mathbf{1}]_N^{\perp}$ is the B's pseudo-inverse B^+.

Proof. Since $B\mathbf{1}_N = B^\top \mathbf{1}_N = \mathbf{0}_N$ for all $B \in [\mathbf{1}]_N^{\perp}$, one can get $BJ_N = J_N B = B$. Therefore, J_N is the identity element of $[\mathbf{1}]_N^{\perp}$.

Then, we prove $B^+ \in [\mathbf{1}]_N^{\perp}$ and $B^+B = J_N$. There exist mutually independent unit vectors $e_1, \dots, e_{N-1} \in \mathbb{R}^N$ such that $\mathbf{1}_N^\top e_i = 0$ for any $i = 1, \dots, N-1$. Since $B\mathbf{1}_N = \mathbf{0}_N$, it holds that

$$B = \sum_{i=1}^{N-1} Be_i e_i^\top = BHH^\top,$$

where

$$H = \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix} \in \mathbb{R}^{N \times (N-1)}.$$

Therefore, as pointed out in Section 4.4 of [14], the pseudo-inverse B^+ can be written as

$$\begin{aligned} B^+ &= H(H^\top H)^{-1}(H^\top B^\top BH)^{-1}H^\top B^\top \\ &= H(H^\top B^\top BH)^{-1}H^\top B^\top. \end{aligned}$$

By $H^\top \mathbf{1}_N = \mathbf{0}_{N-1}$ and $B^\top \mathbf{1}_N = \mathbf{0}_N$, we have $B^+ \in [\mathbf{1}]_N^{\perp}$. Besides, one can get

$$\begin{aligned} B^+B &= H(H^\top B^\top BH)^{-1}H^\top B^\top BHH^\top \\ &= HH^\top = J_N. \end{aligned}$$

The lemma is thereby proved. □

Remark 4. Note that $\mathcal{L} \in [\mathbf{1}]_N^{\perp}$. Then, from Lemma 2, one can get $\mathcal{L}^+ \in [\mathbf{1}]_N^{\perp}$ and $\mathcal{L}^+\mathcal{L} = \mathcal{L}\mathcal{L}^+ = J_N$. Here, we use the Lyapunov function $V(x) = x^\top \mathcal{L}^+ x$ for the consensus analysis.

For the consensus analysis of the signal comparison algorithm (1), it is important to analyze the stochastic properties of $s_i(k) - s_j(k)$. Especially, given

$$\mathcal{F}_{k-1} = \sigma(\{d_i(t) : i \in \mathcal{V},\ 1 \le t \le k-1\}), \tag{2}$$

The following lemma is for

$$\mathbb{E}\left[s_i(k) - s_j(k) | \mathcal{F}_{k-1}\right] = F(C - x_i(k)) - F(C - x_j(k)).$$

Lemma 3. Given a strictly monotonic function $F(\cdot)$ and two real sequences $\phi_1(k),\ \phi_2(k)$ satisfying

$$\begin{aligned} &\inf_{k\in\mathbb{N}} \max\{\phi_1(k), \phi_2(k)\} > -\infty, \\ &\sup_{k\in\mathbb{N}} \max\{\phi_1(k), \phi_2(k)\} < \infty, \end{aligned}$$

then $\lim_{k\to\infty} F(\phi_1(k)) - F(\phi_2(k)) = 0$ implies $\lim_{k\to\infty} \phi_1(k) - \phi_2(k) = 0$.

Proof. Without loss of generality, $F(\cdot)$ is assumed to be strictly monotonically increasing and bounded. Otherwise, consider

$$F'(\phi) = \int_{-\infty}^{F(\phi)(F(1)-F(0))} e^{-t^2} \mathrm{d}t.$$

We now prove the lemma by contradiction.

Due to the boundedness of $F(\cdot)$, if $\overline{\lim_{k\to\infty}}\ \phi_1(k) - \phi_2(k) > 0$, then there would exist subsequences $\{\phi_1(k_s), \phi_2(k_s)\}_{s\in\mathbb{N}}$ such that $\phi_1(k_s) - \phi_2(k_s)$ converges to a positive number or diverges to infinity, and $F(\phi_1(k_s))$ and $F(\phi_2(k_s))$ converge to the same value. Note that $F(\phi_1(k_s))$ and $F(\phi_2(k_s))$ converges to 0 or 1 contradicts $\inf_{k\in\mathbb{N}} \phi_1(k) > -\infty$ and $\sup_{k\in\mathbb{N}} \phi_2(k) < \infty$, and $F(\phi_1(k_s))$ and $F(\phi_2(k_s))$ converges to a finite number contradicts $\lim_{s\to\infty} \phi_1(k_s) - \phi_2(k_s) > 0$. The lemma is thereby proved. □

Corollary 3.1. Under the condition of Lemma 3, one can get $\lim_{k\to\infty} (\phi_1(k) - \phi_2(k)) (F(\phi_1(k)) - F(\phi_2(k))) = 0$ implies $\lim_{k\to\infty} \phi_1(k) - \phi_2(k) = 0$.

Lemma 4. Given a Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$ with $\operatorname{rank}(\mathcal{L}) = N-1$, a strictly monotonic distribution function $F(\cdot)$, a fixed threshold $C \in \mathbb{R}$, and the function $\mathbb{F}(\cdot)$ defined as

$$\mathbb{F}(x) = \operatorname{vol}\{F(C - x_1), \dots, F(C - x_N)\}, \tag{3}$$

where $x = [x_1, \dots, x_N]^\top$, we have the following assertions:

1) $x\mathcal{L}\mathbb{F}(x) < 0$ if $J_N x \ne \mathbf{0}_N$;
2) $\lim_{J_N x \to \mathbf{0}_N} x\mathcal{L}\mathbb{F}(x) = 0$;
3) $\lim_{k\to\infty} x(k)\mathcal{L}\mathbb{F}(x(k)) = 0$, $\inf_{k\in\mathbb{N}} \max_i x_i(k) > -\infty$ and $\sup_{k\in\mathbb{N}} \min_i x_i(k) < \infty$ imply $\lim_{k\to\infty} J_N x(k) = \mathbf{0}_N$;
4) $\lim_{k\to\infty} \frac{x(k) J_N \mathbb{F}(x(k))}{\|J_N x(k)\|} = 0$, $\inf_{k\in\mathbb{N}} \max_i x_i(k) > -\infty$ and $\sup_{k\in\mathbb{N}} \min_i x_i(k) < \infty$ imply $\lim_{k\to\infty} J_N x(k) = \mathbf{0}_N$.

Proof. Because

$$x\mathcal{L}\mathbb{F}(x) = \sum_{(i,j)\in\mathcal{E}} a_{ij}(x_i - x_j)(F(C - x_i) - F(C - x_j)), \tag{4}$$

we have $x\mathcal{L}\mathbb{F}(x) < 0$ if $J_N x \ne \mathbf{0}_N$.

Besides, by (4) one can get

$$\begin{aligned} &- \lim_{J_N x \to \mathbf{0}_N} x\mathcal{L}\mathbb{F}(x) \\ &\le \lim_{J_N x \to \mathbf{0}_N} \sum_{(i,j)\in\mathcal{E}} a_{ij}|x_i - x_j| \\ &\le \lim_{J_N x \to \mathbf{0}_N} N(\max_{i,j} a_{ij}) x^\top J_N x \\ &= 0. \end{aligned}$$

The third assertion of the lemma can be obtained by Corollary 3.1 and (4).

Note that J_N is a Laplacian matrix. By (4), when $J_N x \ne \mathbf{0}_N$, one can get

$$\begin{aligned} &\frac{xJ_N\mathbb{F}(x)}{\|J_N x\|} \\ =&\frac{\sum_{i=1}^N \sum_{j=1}^N (F(C - x_i) - F(C - x_j))(x_i - x_j)}{\sqrt{2N \sum_{i=1}^N \sum_{j=1}^N (x_i - x_j)^2}} \\ <&\frac{(\max_i x_i - \min_i x_i)(F(C - \max_i x_i) - F(C - \min_i x_i))}{\sqrt{2N^3}(\max_i x_i - \min_i x_i)} \\ =&\frac{1}{\sqrt{2N^3}} \left(F\left(C - \max_i x_i\right) - F\left(C - \min_i x_i\right)\right). \end{aligned}$$

Then, since $\lim\limits_{k\to\infty} \frac{x(k)J_N\mathbb{F}(x(k))}{\|J_N x(k)\|}=0$,

$$\lim_{k\to\infty} F\left(C-\max_i x_i(k)\right)-F\left(C-\min_i x_i(k)\right)=0,$$

which together with Lemma 3 implies the fourth assertion. □

Following theorem gives the almost sure average consensus of the signal comparison algorithm.

Theorem 1. Given Assumption 1, the signal comparison algorithm (1) achieves average consensus almost surely.

Proof. By Lemma 1, it suffices to prove the almost sure consensus of the algorithm.

For ease of notation, denote

$$x(k)=\mathrm{vol}\{x_1(k),\dots,x_N(k)\},$$
$$w(k)=\mathrm{vol}\{w_1(k),\dots,w_N(k)\},$$

where

$$w_i(k)=\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i(k)-F(C-x_i(k))\right.$$
$$\left.-s_j(k)+F(C-x_j(k))\right).$$

Given σ-algebra $\mathcal{F}_k$ that is defined in (2), one can get $x(k)$ is $\mathcal{F}_k$-measurable. Therefore, $\mathbb{E}\left[s_i(k)|\mathcal{F}_{k-1}\right]=F(C-x_i(k))$, which implies

$$\mathbb{E}\left[w(k)|\mathcal{F}_{k-1}\right]=\mathbf{0}_N.$$

By (1),

$$x(k)=x(k-1)+q(k)\left(\mathcal{L}\mathbb{F}(x(k-1))+w(k-1)\right),\quad (5)$$

where $\mathbb{F}(\cdot)$ is defined in (3). Set Lyapunov function $V(\cdot)$ as

$$V(x)=x^\top\mathcal{L}^+x,\quad (6)$$

where $\mathcal{L}^+$ is the pseudo-inverse of $\mathcal{L}$. Then, by Lemma 2, we have

$$\begin{aligned}&\mathbb{E}\left[V(x(k))|\mathcal{F}_{k-1}\right]\\ =&V(x(k-1))+2q(k)x^\top(k-1)J_N\mathbb{F}(x(k-1))\\ &+O\left(q^2(k)\right).\end{aligned}\quad (7)$$

Since $\sum_{k=1}^\infty q^2(k)<\infty$, by Theorem 1 of [15], it holds that the limit of $V(x(k))$ exists almost surely, and

$$\overline{\lim_{k\to\infty}}\, x^\top(k)J_N\mathbb{F}(x(k))=0,\ \text{a.s.}\quad (8)$$

Therefore, by Lemma 4, there is a subsequence $\{x(k_s)\}_{s\in\mathbb{N}}$ satisfying $\lim\limits_{s\to\infty} J_Nx(k_s)=\mathbf{0}_N$ almost surely, which together with Lemma 2 implies $\lim\limits_{s\to\infty} V(x(k_s))=0$. Then, due to the almost sure convergence of $V(x(k))$, we have $\lim\limits_{k\to\infty} V(x(k))=0$ almost surely. This proves the theorem. □

Theorem 2. Given Assumption 1, the signal comparison algorithm (1) achieves mean square average consensus.

Proof. By (7) one can get

$$\mathbb{E}V(x(k))\le\mathbb{E}V(x(k-1))+O\left(q^2(k)\right),$$

which together with $\sum_{k=1}^\infty q^2(k)<\infty$ implies $\mathbb{E}V(x(k))=O(1)$. Therefore, by Lemma 4,

$$\mathbb{E}\left[V(x(k-1))x^\top(k-1)J_N\mathbb{F}(x(k-1))\right]\le 0,$$

which together with (5) implies

$$\mathbb{E}[V(x(k))]^2\le\mathbb{E}[V(x(k-1))]^2+O\left(q^2(k)\right).$$

Then, we have $\mathbb{E}[V(x(k))]^2=O(1)$, which together with de La Vallée Poussin criterion [16] implies that $V(x(k))$ is uniformly integrable.

Note that by Theorem 1 and Corollary 3.3.1 in [17], $V(x(k))$ converges to 0 in probability. By Corollary 4.2.4 in [17] and the uniform integrability, one can get the theorem. □

B. Convergence Rate

The subsection calculates the mean square convergence rate. For an accurate convergence rate analysis, we set $q(k)=\frac{q_0}{k^p}$ for some $p\in(\frac{1}{2},1]$. Besides, the following continuity assumption of the distribution function $F(\cdot)$ is required.

Assumption 2. Measurement noise sequence $\{d_i(k)\}$ is i.i.d. Besides, the density function $f(\cdot)$ exists and for any compact set Ξ,

$$\inf_{x\in\Xi} f(x)>0.$$

Theorem 3. Set $q(k)=\frac{q_0}{k^p}$ in the signal comparison algorithm (1). Then, under Assumption 2,

$$\begin{aligned}&\mathbb{E}\left\|x(k)-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top x(0)\right\|^2\\ =&\begin{cases}O\left(\frac{1}{k}\right), & \text{if } p=1,\ 2q_0\underline{f}\lambda_2(\mathcal{L})>1;\\ O\left(\frac{\ln k}{k}\right), & \text{if } p=1,\ 2q_0\underline{f}\lambda_2(\mathcal{L})=1;\\ O\left(\frac{1}{k^{2q_0\underline{f}\lambda_2(\mathcal{L})}}\right), & \text{if } p=1,\ 2q_0\underline{f}\lambda_2(\mathcal{L})<1;\\ O\left(\frac{1}{k^p}\right), & \text{if } p\in(\frac{1}{2},1),\end{cases}\end{aligned}$$

where $\underline{f}=f\left(C-\frac{1}{N}\sum_{i=1}^N x_i(0)\right)$, and $\lambda_2(\mathcal{L})$ is the minimum positive eigenvalue of the Laplacian matrix $\mathcal{L}$.

Proof. Here we just sketch the proof.

Firstly, we estimate the distribution tail using the technique of [13]. We construct a stochastic process with averaged observations (SPAO) as

$$\psi(k)=x(k)-A_W(k),$$

where $A_W(k)=\frac{q_0\sum_{t=1}^{k-1}w(t)}{k^p}$. Similar to the proofs of Theorems 1 and 7 in [13], one can get that for any given $m>0$ and $\varepsilon\in(0,p-\frac{1}{2})$, there exist $\delta,\nu\in(0,1)$ such that when k is sufficiently large,

$$\bigcup_{t=\lfloor k^\delta\rfloor}^{\infty}\left\{\|A_W(t)\|<mt^{-\varepsilon}\right\}\subseteq\left\{\|J_N\psi(k)\|^2<k^{-\nu}\right\}.$$

Then, one can get

$$\mathbb{P}\left\{\|J_Nx(k)\|^2\ge k^{-\nu}\right\}=O\left(\exp(-k^\gamma)\right)\quad (9)$$

for some $\gamma, \nu \in (0,1)$ similar to the proof of Theorem 2 in [13].

Secondly, for all $i \neq j$, define

$$\check{f}_{ij}(k) = \begin{cases} f(C - x_i(k)), & \text{if } x_i(k) = x_j(k), \\ -\frac{F(C-x_i(k))-F(C-x_j(k))}{x_i(k)-x_j(k)}, & \text{otherwise,} \end{cases}$$

and $\mathcal{L}_f(k) = (l_{ij}^f)_{N\times N}$, where $l_{ij}^f = -a_{ij}\check{f}_{ij}(k)$ if $i \neq j$, and $l_{ii}^f = \sum_{j\in\mathcal{N}_i} a_{ij}\check{f}_{ij}(k)$. Then,

$$\mathcal{L}\mathbb{F}(x(k)) = -\mathcal{L}_f(k)x(k). \tag{10}$$

By (9), there exist $\gamma, \nu \in (0,1)$ such that

$$\mathbb{P}\left\{\|\underline{f}\mathcal{L} - \mathcal{L}_f\|^2 \geq k^{-\nu}\right\} = O\left(\exp(-k^{\gamma})\right). \tag{11}$$

Thirdly, by (5) and (10), it holds that

$$\begin{aligned} &\mathbb{E}\left\|J_N x(k)\right\|^2 \\ =& \left(1 - \frac{2q_0}{k^p}\underline{f}\lambda_2(\mathcal{L}) + O\left(\frac{1}{k^{p+\frac{\nu}{2}}}\right)\right)\mathbb{E}\left\|J_N x(k-1)\right\|^2 \\ &+ O\left(\frac{1}{k^{2p}}\right). \end{aligned}$$

Then, by Lemmas 3.2 and 3.3 in [18], we have

$$\begin{aligned} &\mathbb{E}\left\|J_N x(k)\right\|^2 \\ =& \begin{cases} O\left(\frac{1}{k}\right), & \text{if } p = 1,\ 2q_0\underline{f}\lambda_2(\mathcal{L}) > 1; \\ O\left(\frac{\ln k}{k}\right), & \text{if } p = 1,\ 2q_0\underline{f}\lambda_2(\mathcal{L}) = 1; \\ O\left(\frac{1}{k^{2q_0\underline{f}\lambda_2(\mathcal{L})}}\right), & \text{if } p = 1,\ 2q_0\underline{f}\lambda_2(\mathcal{L}) < 1; \\ O\left(\frac{1}{k^p}\right), & \text{if } p \in (\frac{1}{2}, 1). \end{cases} \end{aligned}$$

By Lemma 1, the theorem is proved. □

Remark 5. From Theorem 3, one can figure out when the step-size $q(k)$ is properly selected, the mean square convergence rate of the signal comparison algorithm can be $O(1/k)$. For comparison, the mean square convergence rate of the empirical measurement algorithm [8] is $O(1/k^{\zeta})$ for some $\zeta \in (0, \frac{1}{2})$, and that of the recursive projection-based algorithms [9], [10] can be $O(1/k)$ when the noise distribution function $F(\cdot)$ and the upper bound of $\max_i|x_i(0)|$ are known as *a priori*.

IV. Simulation

This section considers a four-agent system. The undirected communication graph is shown in Figure 1, which also gives the weights a_{ij} of the graph.

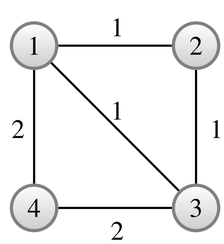

Fig. 1. Communication topology.

In the four-agent system, the agents' initial values are

$$x_1(0) = 3,\ x_2(0) = 1,\ x_3(0) = -1,\ x_4(0) = -3.$$

Then, $\sum_{i=1}^{4} x_i(0) = 0$. In the algorithm (1), the step-size is set to be $q(k) = 2/k$. The measurement noise $d_i(k)$ is unbiased and normal with variance 1. The threshold C is set to be 1.

Figure 2 gives a trajectory of the algorithm (1). The trajectory shows that the four-agent system achieves average consensus. Figure 3 illustrates a trajectory of $kx^{\top}(k)J_N x(k)$ in 200 repeated experiments. The trajectory is bounded, which demonstrates that the mean square convergence rate can be $O(1/k)$.

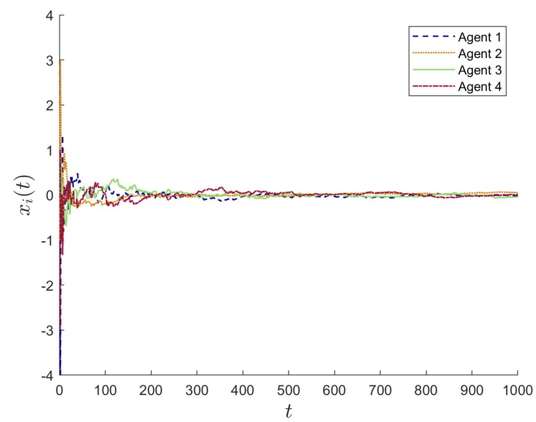

Fig. 2. The trajectories of the agents' states.

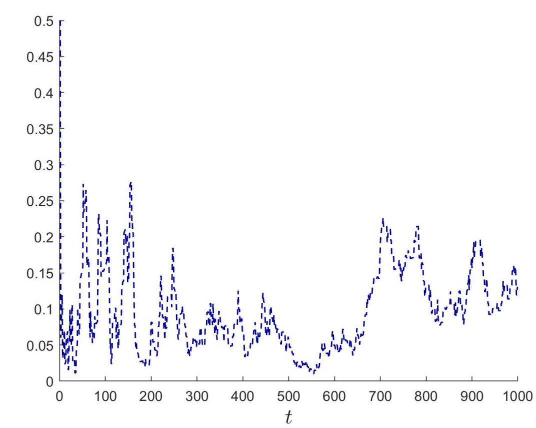

Fig. 3. The average trajectory of $kx^{\top}(k)J_N x(k)$ in 200 repeated experiments.

V. Concluding Remarks

The paper investigates the average consensus problem under finite communication data rate and measurement noises. A signal comparison algorithm is proposed for the problem. Only binary-valued communications are required for the algorithm. The algorithm design does not require the knowledge on the upper bound of $\max_i|x_i(0)|$. Besides, under finite communication data rate and measurement noises, our algorithm appears to be the first to allow the noise distribution to be unknown. The algorithm is proved to achieve the almost sure and mean square average consensus. The mean square convergence rate of the algorithm is also calculated. When the step-size $q(k)$ is properly selected, the mean square convergence rate can be $O(1/k)$. For the future work, we can try to extend the signal comparison algorithm into the directed graph case, and can also apply the signal comparison algorithm to other consensus problem, such as distributed optimization problems. We can also consider the privacy-preserving issue for the average consensus problem [19].

References

[1] A. Kashyap, T. Basar, and R. Srikant, "Consensus with quantized information updates," in *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, USA, Dec. 2006, pp. 2728–2733.

[2] R. Carli, F. Bullo, and S. Zampieri, "Quantized average consensus via dynamic coding/decodingschemes," *International Journal of Nonlinear and Robust Control*, vol. 20, no. 2, pp. 156–175, 2010.

[3] S. Zhu and B. Chen, "Quantized consensus by the ADMM: probabilistic versus deterministic quantizers," *IEEE Transactions on Signal Processing*, vol. 64, no. 7, pp. 1700–1713, 2016.

[4] M. Xiong, B. Zhang, D. Yuan, and S. Xu, "Distributed quantized mirror descent for strongly convex optimization over time-varying directed grap,"*Science China Information Sciences*, vol. 65, no. 10, pp. 202202, 2022.

[5] L. Schuchman, "Dither signals and their effect on quantization noise," *IEEE Transactions on Communication Technology*, vol. 12, no. 4, pp. 162–165, 1964.

[6] T. Li, M. Fu, L. Xie, and J. F. Zhang, "Distributed consensus with limited communication data rate," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 279–292, 2011.

[7] M. Ran, S. Feng, J. Li, and L. Xie, "Quantized consensus under data-rate constraints and DoS attacks: a zooming-in and holding approach," *IEEE Transactions on Automatic Control*, *DOI:10.1109/TAC.2022.3223277*, 2022.

[8] Y. L. Zhao, T. Wang, and W. Bi, "Consensus protocol for multiagent systems with undirected topologies and binary-valued communications," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 206–221, 2019.

[9] T. Wang, H. Zhang, and Y. L. Zhao, "Consensus of multi-agent systems under binary-valued measurements and recursive projection algorithm," *IEEE Transactions on Automatic Control*, vol. 65, no. 6, pp. 2678–2685, 2020.

[10] M. Hu, T. Wang, and Y. L. Zhao, "Consensus of switched multi-agent systems with binary-valued communications," *Science China Information Sciences*, vol. 65, no. 6, pp. 162207, 2022.

[11] B. Chen, J. Yue, W. Li, K. Qin, M. Shi, and B. Lin, "Average consensus control of multi-agent system under binary-valued observations with external disturbance and measurement noise", in *2021 IEEE 4th International Conference on Electronics Technology (ICET)*, Chengdu, China, May 2021, pp. 920-927.

[12] J. Guo and Y. L. Zhao, "Recursive projection algorithm on FIR system identification with binary-valued observations," *Automatica*, vol. 49, no. 11, pp. 3396–3401, 2013.

[13] J. M. Ke, Y. Wang, Y. L. Zhao, and J. F. Zhang, "Recursive identification of set-valued systems under uniform persistent excitations," *arXiv preprint arXiv:2212.01777*, 2022.

[14] L. Guo. *Introduction to control theory: from basic concepts to research frontier (Chinese)*. Science Press. Beijing, 2005.

[15] H. Robbins and D. Siegmund, "A convergence theorem for non negative almost supermartingales and some applications," *Optimizing Methods in Statistics*, pp. 233–257, 1971.

[16] T. C. Hu and A. Rosalsky "A note on the de La Vallée Poussin criterion for uniform integrability," *Statistics & Probability Letters*, vol. 81, no. 1, pp. 169–174, 2011.

[17] Y. S. Chow and H. Teicher (1997). *Probability Theory: Independence, Interchangeability, Martingales (3rd ed.)*. Springer Science & Business Media. New York, 1997.

[18] J. M. Wang, J. M. Ke, and J. F. Zhang, "Differentially private bipartite consensus over signed networks with time-varying noises," *arXiv preprint arXiv:2212.11479*, 2022.

[19] J. M. Ke, J. M. Wang, and J. F. Zhang, "Differentiated output-based privacy-Preserving average consensus", *IEEE Control Systems Letters*, vol. 7, pp. 1369-1374, 2023.