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# Design of Q-filter-based Disturbance Observer for Differential Algebraic Equations and a Robust Stability Condition: Zero Relative Degree Case

Hamin Chang[1,2] and Stephan Trenn[1]

*Abstract*— **While the disturbance observer (DOB)-based controller is widely utilized in various practical applications, there has been a lack of extension of its use to differential algebraic equations (DAEs). In this paper, we introduce several lemmas that establish necessary and/or sufficient conditions for specifying the relative degree of DAEs. Using these lemmas, we also figure out that there is a class of DAEs which can be viewed as linear systems with zero relative degree. For the class of DAEs, we propose a design of Q-filter-based DOB as well as a robust stability condition for systems controlled by the DOB through time domain analysis using singular perturbation theory. The proposed stability condition is verified by an illustrative example.**

## I. Introduction

Differential algebraic equations (DAEs) have attracted considerable attention in recent years due to their widespread use in engineering and science. They arise naturally in many applications such as robotics, aerospace, and circuit theory, where the behavior of the system is constrained by algebraic equations. Because DAEs do not treat the algebraic equations as independent constraints along with ordinary differential equations (ODEs), they provide a more accurate representation of many physical systems and allow for the modeling of more complex behaviors.

In particular, robust control of DAEs whose goal is to compensate plant uncertainties and reject disturbances is becoming an important and challenging problem in control theory. For instance, disturbance decoupling has been studied for the discrete-time linear DAEs in [1] and in continuous time in a general behavioral framework in [2]. Other robust control schemes for DAEs have been investigated in [3]–[5].

A popular robust controller design method, which has not yet been applied for DAE systems is the Q-filter-based disturbance observer (DOB) which was first introduced by [6]. As a matter of fact, the robust stability condition of the DOB-based control systems has been extensively investigated. For example, [7] and [8] incorporated singular perturbation theory to analyze the DOB-based control systems and discovered a necessary and sufficient condition for robust stability, which was further studied in [9]. This condition makes it possible to systematically design the Q-filter-based DOB for plants, which have a positive relative degree, guaranteeing robust stability.

[1]Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands.

[2]ASRI, Department of Electrical and Computer Engineering, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul, 08826, Korea. Email: h.chang@rug.nl; s.trenn@rug.nl

One of the main features of the Q-filter-based DOB is that it is designed based on the inverse model of a given plant. For this reason, it is essential to have some information on the relative degree, zero dynamics, and normal form of the plant. In fact, there are several recent studies on the notion of the relative degree of DAEs. For instance, [10]–[12] developed the notions of relative degree, zero dynamics, and normal form of nonlinear DAEs (corresponding to Byrnes-Isidori normal form of nonlinear ODEs) and made use of the notions for stabilization and funnel control of DAEs. In particular, the notion of the vector relative degree of muti-input multi-output (MIMO) systems was generalized later on in [13]. More recently, [14] established a connection between the differentiation index of a nonlinear DAE and the relative degree of an associated control system. However, the design of the Q-filter-based DOB for DAEs has yet to be investigated even for the simplest case, where DAEs can be regarded as biproper linear ODEs.

The contributions of this paper are as follows. First, we provide a lemma that explains a necessary and sufficient condition for regular DAEs to be equivalent to ODEs in the sense of input and output behavior. Additionally, we present another lemma that enables us to find the relative degree of a given regular DAE. On the other hand, observing the fact that a certain class of DAEs can be represented by linear systems with a direct feed-through term, we propose a design method of the Q-filter-based DOB for linear systems with zero relative degree. It is also noted that the DOB can reject external disturbances as well as compensate model uncertainties. Finally, a necessary and sufficient condition for robust stability of the closed-loop system with the Q-filter-based DOB is provided.

The rest of this paper is organized as follows. Section II provides a background on DAEs and presents two lemmas dealing with the relative degree of DAEs. Section III introduces a DOB design for linear systems with zero relative degree and proposes a robust stability condition for the closed-loop system with the DOB. In Section IV, we demonstrate the effectiveness of the proposed DOB in rejecting the effects of input disturbances on the system output through a simulation result. Finally, Section V concludes this paper and suggests future research directions.

## II. Differential Algebraic Equations and Relative Degree

Let us consider the following single-input single-output (SISO) system based on a linear time-invariant (LTI) differ-

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ential algebraic equation (DAE):

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $E, A \in \mathbb{R}^{n\times n}$, $u(t) \in \mathbb{R}$ is the smooth input, $x(t) \in \mathbb{R}^n$ is the state, and $y(t) \in \mathbb{R}$ is the output. The matrix $E$ is not invertible and for that reason (1a) can not be represented as an ordinary differential equation (ODE). The matrix pair $(E, A)$ is assumed to be regular, namely, $\det(sE - A)$ is not the zero polynomial. For the regular pair $(E, A)$, it is known in [15, Theorem 2.6] that there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$SET = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad SAT = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix},$$

where $J \in \mathbb{R}^{n_1\times n_1}, 0 \le n_1 \le n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}, n_2 = n - n_1$, is nilpotent. Let us define the following matrices

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$
$$\Pi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \qquad \Pi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S,$$
$$A^{\text{diff}} := \Pi^{\text{diff}} A, \qquad E^{\text{imp}} := \Pi^{\text{imp}} E,$$
$$B^{\text{diff}} := \Pi^{\text{diff}} B, \qquad B^{\text{imp}} := \Pi^{\text{imp}} B.$$

It is important to note that these matrices (viewed as linear maps) are still acting on the original linear space and are therefore independent from the chosen coordinate system (i.e. they do not depend on $T$ and $S$, but only on the original linear maps $E$, $A$, and $B$). According to [16, Remark 6.4.5], the initial value problem (1a) with $x(0) \in \mathbb{R}^n$ has a solution if and only if $x(0) \in \mathcal{I}(u(0), u^{(1)}(0), \ldots, u^{(n-1)}(0))$, where

$$\mathcal{I}(u(0), u^{(1)}(0), \ldots, u^{(n-1)}(0)) := \left\{ \zeta \in \mathbb{R}^n \middle| \zeta + \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(0) \in \operatorname{im} \Pi \right\}.$$

Taking this into account, the following lemma provides a necessary and sufficient condition for system (1) to be externally equivalent to an ODE-based system.

*Lemma 1:* Consider system (1) with smooth inputs and an initial value $x(0) \in \mathcal{I}(u(0), u^{(1)}(0), \ldots, u^{(n-1)}(0))$. Then, there exists an ODE-based LTI system

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t) \end{aligned} \tag{2}$$

with suitable initial condition $\hat{x}(0) \in \mathbb{R}^n$ (depending on the input), such that

$$y(t) = \hat{y}(t), \quad \forall u(\cdot),\ t \ge 0$$

if and only if

$$C(E^{\text{imp}})^i B^{\text{imp}} = 0, \quad i = 1, \ldots, n-1. \tag{3}$$

*Proof:* Since $x(0) \in \mathcal{I}(u(0), u^{(1)}(0), \ldots, u^{(n-1)}(0))$, there exists $\xi \in \mathbb{R}^n$ such that

$$x(0) = \Pi\xi - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(0).$$

By [15, Theorem 2.7], a unique solution of (1a) with the initial value $x(0)$ is given by

$$x(t) = e^{A^{\text{diff}} t}\Pi\xi + \int_0^t e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau) d\tau - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \tag{4}$$

Now we first prove the "if" part. Since $C(E^{\text{imp}})^i B^{\text{imp}} = 0$ for $i = 1, \ldots, n-1$, output (1b) can be written as

$$y(t) = Ce^{A^{\text{diff}} t}\Pi\xi + C\int_0^t e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau) d\tau + (D - CB^{\text{imp}})u(t), \tag{5}$$

which is in fact the same as $\hat{y}(t)$, the output of system (2), for every input $u(\cdot)$ if $\hat{x}(0) = \Pi\xi = x(0) + \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(0)$ and the system matrices are chosen as

$$\hat{A} = A^{\text{diff}},\ \hat{B} = B^{\text{diff}},\ \hat{C} = C,\ \hat{D} = D - CB^{\text{imp}}.$$

To prove the "only if" part, suppose that $y(t) = \hat{y}(t)$ for all $u(\cdot)$ and $t \ge 0$. In other words,

$$\begin{aligned} &Ce^{A^{\text{diff}} t}\Pi\xi + C\int_0^t e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau) d\tau \\ &\quad - C\sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t) + Du(t) \\ &= \hat{C}e^{\hat{A}t}\hat{x}(0) + \hat{C}\int_0^t e^{\hat{A}(t-\tau)}\hat{B}u(\tau)d\tau + \hat{D}u(t) \end{aligned} \tag{6}$$

holds for all $u(\cdot)$ and $t \ge 0$. Substituting $u(t) \equiv 0$, we have $Ce^{A^{\text{diff}} t}\Pi\xi = \hat{C}e^{\hat{A}t}\hat{x}(0)$ which hence cancels out in (6). Let us now consider the smooth function

$$f_k(t) = \frac{1}{k!} t^k, \quad k = 1, 2, \ldots$$

that satisfies

$$f_k^{(j)}(0) = \begin{cases} 0 & j \ne k, \\ 1 & j = k. \end{cases}$$

Again by substituting $u(t) = f_i(t)$ for $i = 1, \ldots, n-1$ and evaluating (6) at $t = 0$, we obtain $C(E^{\text{imp}})^i B^{\text{imp}} = 0$ which completes the proof. ■

We also present the following lemma that explains necessary and/or sufficient condition for specifying the relative degree of system (1).

*Lemma 2:* Consider system (1) with smooth inputs and an initial value $x(0) \in \mathcal{I}(u(0), u^{(1)}(0), \ldots, u^{(n-1)}(0))$. Let us denote the (possibly negative) relative degree of system (1) as $\mathrm{r} \in \mathbb{Z}$, i.e. $\mathrm{r} = \deg(q(s)) - \deg(p(s))$, where $\frac{p(s)}{q(s)} = C(sE - A)^{-1}B$.

(a) If $C(E^{\mathrm{imp}})^i B^{\mathrm{imp}} = 0,\ i = 1, \dots, n-1$, then $\mathrm{r} \geq 0$. Additionally, $\mathrm{r} = 0$ if and only if $D - CB^{\mathrm{imp}} \neq 0$.

(b) If $C(E^{\mathrm{imp}})^i B^{\mathrm{imp}} \neq 0$ for some $i \in \{1, \dots, n-1\}$, then

$$\mathrm{r} = -\max\{i \in \{1, \dots, n-1\} \,|\, C(E^{\mathrm{imp}})^i B^{\mathrm{imp}} \neq 0\}.$$

*Proof:* It is straightforward that both (a) and (b) hold according to (4) and (5). ■

*Example 1:* Consider an electric circuit in Fig. 1 which is a part of the circuit of a transistor network in [17, Figure 1].

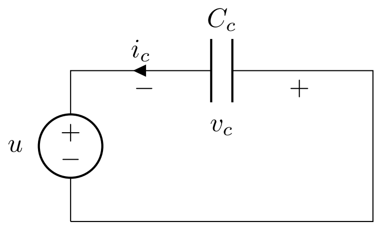


Fig. 1. A voltage source connected with a capacitor.

Defining the state $x := [v_c, i_c]^\top$ and the output $y := v_c$, we have a DAE that models the circuit as

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{7}$$

where

$$E = \begin{bmatrix} C_c & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $C_c > 0$ represents the capacitance. It is easy to see that the matrix pair $(E, A)$ is regular. Choosing invertible matrices

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

yields

$$SET = \begin{bmatrix} 0 & C_c \\ 0 & 0 \end{bmatrix} = N, \quad SAT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Here, the nilpotent matrix $N$ satisfies $N \neq 0$ and $N^2 = 0$ since $C_c > 0$ which directly means that the matrix pair $(E, A)$ has index 2 [18, Definition 2.9]. By definition we have

$$\Pi^{\mathrm{imp}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E^{\mathrm{imp}} = \begin{bmatrix} 0 & 0 \\ C_c & 0 \end{bmatrix}, \quad B^{\mathrm{imp}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, $CE^{\mathrm{imp}}B^{\mathrm{imp}} = 0$ and $D - CB^{\mathrm{imp}} = -1 \neq 0$ hold which means that system (7) has relative degree 0 according to Lemma 2-(a). (In fact, it is trivial to see that the input and output of system (7) have the relation $y = -u$ and this can also be deduced by using Lemma 1.)

From Lemmas 1 and 2, we observe that there is a certain class of linear systems based on DAEs which can be considered as ordinary linear systems with a direct feed-through term; specifically, this is the case when system (1) satisfies

$$\begin{aligned} C(E^{\mathrm{imp}})^i B^{\mathrm{imp}} &= 0, \quad i = 1, \dots, n-1, \\ D - CB^{\mathrm{imp}} &\neq 0. \end{aligned}$$

Motivated by this observation, we present a design of Q-filter-based disturbance observer (DOB) for linear systems with zero relative degree and provide a robust stability condition of the closed-loop system in the following section.

*Remark 1:* We point out two sufficient conditions for (3).

(i) $E^{\mathrm{imp}} = 0$. In that case we have

$$0 = E^{\mathrm{imp}} = \Pi^{\mathrm{imp}} E = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} SE.$$

Since $T$ is invertible, this implies that

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} SET = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix},$$

and thus, $N = 0$. In fact, the converse implications are also true so that $E^{\mathrm{imp}} = 0$ if and only if $N = 0$. Therefore, regular DAEs whose index is 1 are contained in the class of systems that satisfies (3).

(ii) $CE^{\mathrm{imp}} = 0$. In that case, the DAE can have an arbitrary high index, but possible Dirac impulses in the state induced by inconsistent initial values as well as dependencies on higher derivatives of the input are inside $\ker C$ and not visible in the output. In other words the higher index parts of the DAE are "hidden" inside the kernel of the output matrix. In fact, if we would consider arbitrary initial values, the condition $CE^{\mathrm{imp}} = 0$ would be necessary for an equivalence between (1) and (2), because otherwise there would exist inconsistent initial values leading to Dirac impulses in the output, which ODE (2) of course is not able to reproduce.

## III. Design of Disturbance Observer for Relative Degree Zero Systems

In this section, we introduce a design of Q-filter-based DOB for LTI systems with zero relative degree. Furthermore, we present a robust stability condition for the closed-loop system with the proposed DOB through a time domain analysis. A classical configuration of the closed-loop system with Q-filter-based DOB is depicted in Fig. 2.

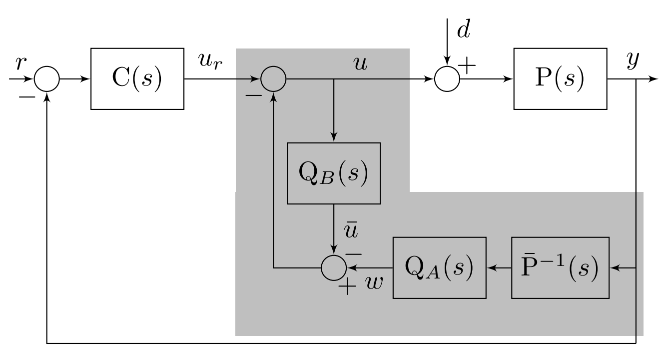


Fig. 2. Block diagram of the closed-loop system with Q-filter-based DOB (gray block).

In the figure, $\mathrm{P}(s)$ and $\bar{\mathrm{P}}(s)$ represent a real plant and its nominal model, respectively, $\mathrm{C}(s)$ is an outer-loop controller which is typically designed in advance for $\bar{\mathrm{P}}(s)$. In particular, $\mathrm{Q}_A(s)$ and $\mathrm{Q}_B(s)$ are stable low-pass filters usually called

Q-filter with a parameter $\tau$ that determines the bandwidth. Here, we only consider the case where both the real plant and the nominal model have zero relative degree. Thus, the inverse of the nominal model $\bar{\mathrm{P}}^{-1}(s)$ is still a proper transfer function and it can be directly implemented unlike the case in, for example, [7, Remark 1], where $\bar{\mathrm{P}}^{-1}(s)$ is improper so that an additional Q-filter $\mathrm{Q}_A(s)$, that makes the block $\mathrm{Q}_A(s)\bar{\mathrm{P}}^{-1}(s)$ proper, must be employed. Therefore, we discover a robust stability condition for the closed-loop system with a fixed $\mathrm{Q}_A(s) = 1$. Another Q-filter $\mathrm{Q}_B(s)$ is conventionally designed as

$$\mathrm{Q}_B(s) = \frac{c_k(\tau s)^k + c_{k-1}(\tau s)^{k-1} + \cdots + c_0}{(\tau s)^l + a_{l-1}(\tau s)^{l-1} + \cdots + a_0},$$

where $k$ and $l$ are non-negative integers such that $l > k$ and $a_0 = c_0 \neq 0$ for unity dc gain. A parametric uncertainty of the real plant $\mathrm{P}(s)$ is taken into account as follows.

*Assumption 1:* The real plant $\mathrm{P}(s)$ belongs to the set of uncertain plants:

$$\mathcal{P} := \left\{ \frac{\beta_n s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0} : \alpha_i \in \left[\alpha_i^l, \alpha_i^u\right], \beta_i \in [\beta_i^l, \beta_i^u] \right\},$$

where $n$ is a positive integer and $\alpha_i^l$, $\alpha_i^u$, $\beta_i^l$, and $\beta_i^u$ are known constants. The nominal plant $\bar{\mathrm{P}}(s)$ also has the form

$$\bar{\mathrm{P}}(s) = \frac{\bar{\beta}_n s^n + \bar{\beta}_{n-1}s^{n-1} + \cdots + \bar{\beta}_0}{s^n + \bar{\alpha}_{n-1}s^{n-1} + \cdots + \bar{\alpha}_0},$$

where $\bar{\alpha}_i$ and $\bar{\beta}_i$ are known. In addition, it is known that $\beta_n, \bar{\beta}_n \neq 0$, and

$$\beta_n + \bar{\beta}_n \neq 0.$$

Suppose that a state space representation of $\mathrm{P}(s)$ is given as

$$\dot{x} = Ax + B(u + d) \tag{8}$$
$$y = Cx + D(u + d), \tag{9}$$

where $x \in \mathbb{R}^n$ is the state and $D = \beta_n \neq 0$. Observing that its inverse model can be realized as

$$\dot{x} = Ax + B(u + d) = (A - BD^{-1}C)x + BD^{-1}y$$
$$u + d = -D^{-1}Cx + D^{-1}y,$$

the inverse nominal model $\bar{\mathrm{P}}^{-1}(s)$ is represented as

$$\dot{\bar{x}} = (\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})\bar{x} + \bar{B}\bar{D}^{-1}y$$
$$w = -\bar{D}^{-1}\bar{C}\bar{x} + \bar{D}^{-1}y, \tag{10}$$

where $\bar{x} \in \mathbb{R}^n$ is the state and $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D} = \bar{\beta}_n \neq 0$ are the nominal system matrices. For the Q-filter $\mathrm{Q}_B(s)$, we use a realization of

$$\tau\dot{q} = A_1 q + B_1 u \tag{11}$$
$$\bar{u} = C_1 q, \tag{12}$$

where $q \in \mathbb{R}^l$ is the state and

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{l-1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} c_0 & c_1 & \cdots & c_k & 0 & \cdots & 0 \end{bmatrix}.$$

By substituting

$$u = u_r + \bar{u} - w = u_r + C_1 q - w \tag{13}$$

from Fig. 2 and (12) into output equations (9) and (10), we have

$$Cx + D(u_r + C_1 q - w + d) = y = \bar{C}\bar{x} + \bar{D}w.$$

Thus, it holds that

$$w = g(Cx - \bar{C}\bar{x} + DC_1 q + D(u_r + d)), \tag{14}$$

where $g := (D + \bar{D})^{-1} = (\beta_n + \bar{\beta}_n)^{-1}$ is well-defined by Assumption 1. Therefore, the overall system is obtained as

$$\begin{aligned} \dot{x} &= Ax + B(u_r + C_1 q - w + d) \\ &= (A - gBC)x + gB\bar{C}\bar{x} \\ &\qquad + (BC_1 - gBDC_1)q + (B - gBD)(u_r + d) \\ \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}w \\ &= g\bar{B}Cx + (\bar{A} - g\bar{B}\bar{C})\bar{x} + g\bar{B}DC_1 q + g\bar{B}D(u_r + d) \\ \tau\dot{q} &= A_1 q + B_1(u_r + C_1 q - w) \\ &= -gB_1 Cx + gB_1\bar{C}\bar{x} + (A_1 + B_1 C_1 - gB_1 DC_1)q \\ &\quad - gB_1 D(u_r + d) + B_1 u_r \\ y &= \bar{C}\bar{x} + \bar{D}w \\ &= g\bar{D}Cx + (\bar{C} - g\bar{D}\bar{C})\bar{x} + g\bar{D}DC_1 q + g\bar{D}D(u_r + d). \end{aligned} \tag{15}$$

Let us analyze the overall system using singular perturbation theory [19] as in [7]. Firstly, consider the extreme case where $\tau = 0$. Then the $q$-dynamics (11) combined with (13) results in

$$0 = (A_1 + B_1 C_1)q + B_1 u_r - B_1 w. \tag{16}$$

According to the structure of matrices $A_1, B_1$, and $C_1$, it is clear that the solution $q = q^* := [q_1^*, \cdots, q_l^*]^\top$ of (16) satisfies $q_2^* = \cdots = q_l^* = 0$ and $u_r = w$. Then, we have $C_1 q^* = c_0 q_1^*$ so that from (14), it holds that

$$u_r = w = g(Cx - \bar{C}\bar{x} + Dc_0 q_1^* + D(u_r + d))$$

from which it follows that

$$q_1^* = \frac{1}{Dc_0}\left(-Cx + \bar{C}\bar{x} + \bar{D}u_r - Dd\right). \tag{17}$$

Therefore, we can conclude that the solution $q^*$ of (16) is an isolated root of the $q$-dynamics when $\tau = 0$, which ensures that overall system (15) is in standard form of singular perturbation [20, Chapter 11]. Consequently, when $\tau > 0$ is sufficiently small, it is possible to segregate the variables of (15) into two distinct categories, namely fast variables and slow variables. In fact, it is obvious that $q$ becomes the only fast variable while $x$, $\bar{x}$, $u_r$, and $d$ are considered to be slow variables. Imposing $\tau = 0$ and $q = q^*$ (which also imply that $u_r = w$) to the overall system (15), we obtain the following quasi-steady-state subsystem that depends only on the slow variables, also so-called slow model:

$$\begin{aligned}\dot{x} &= Ax + B(u_r + C_1 q^* - w + d)\\ &= Ax + Bc_0 q_1^* + Bd\\ &= (A - D^{-1}BC)x + D^{-1}B\bar{C}\bar{x} + D^{-1}\bar{D}Bu_r\\ \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u_r\\ y &= \bar{C}\bar{x} + \bar{D}u_r.\end{aligned} \quad (18)$$

On the other hand, let us define a stretched time scale $\tilde{t} := t/\tau$ and a change of variable $\tilde{q} := q - q^*$. Then, the boundary-layer subsystem with respect to (15) is obtained as

$$\begin{aligned}\frac{d\tilde{q}}{d\tilde{t}} &= -gB_1Cx + gB_1\bar{C}\bar{x} - gB_1D(u_r + d) + B_1u_r\\ &\quad + (A_1 + B_1C_1 - gB_1DC_1)(\tilde{q} + q^*)\\ &= (A_1 + B_1C_1 - gB_1DC_1)\tilde{q}\end{aligned} \quad (19)$$

according to (16) and (17). Here, the slow variables $x$, $\bar{x}$, $u_r$, and $d$ are treated as a fixed (frozen) constant in the new time scale $\tilde{t}$. By singular perturbation theory, the slow variables rapidly converge to the solution of the quasi-steady-state subsystem (18) after a short transient as long as the boundary-layer subsystem (19) is exponentially stable. It is noted that the quasi-steady-state subsystem (18) does not contain any term associated with the disturbance $d$ and preserves the nominal plant inside. Therefore, the output $y$ of (18) is the one from nominal plant with the input $u_r$ generated by the outer-loop controller $\mathrm{C}(s)$. These are summarized in the following theorem as in [7, Theorem 1] without proof.

*Theorem 1:* Under Assumption 1, there exists a $\tau^* > 0$ such that, for all $0 < \tau \leq \tau^*$, the closed-loop system is robustly stable if

(a) the quasi-steady-state subsystem (18) with the outer-loop controller $\mathrm{C}(s)$ is exponentially stable and
(b) the boundary-layer subsystem (19) is exponentially stable, i.e., the system matrix

$$A_1 + B_1C_1 - gB_1DC_1 = A_1 + \frac{\bar{D}}{D + \bar{D}}B_1C_1 \quad (20)$$

is Hurwitz for all $D = \beta_n \in [\beta_n^l, \beta_n^u]$.

Conversely, if (a) or (b) is not satisfied in the way that the quasi-steady-state subsystem (18) with the outer-loop controller $\mathrm{C}(s)$ has an unstable pole with a positive real part or matrix (20) has an eigenvalue having a positive real part, then there exists a $\tau^* > 0$ such that for all $0 < \tau \leq \tau^*$, the closed-loop system is not robustly stable.

In order to guarantee condition (a) in Theorem 1, we need the matrix $A - D^{-1}BC$ to be Hurwitz which means the zero dynamics of the plant $\mathrm{P}(s)$ is stable for all variations allowed in Assumption 1 because state equation (8) can be rewritten as

$$\dot{x} = (A - D^{-1}BC)x + D^{-1}By.$$

It is in fact well-known in [7] and [9] that it is inevitable to have such minimum phase assumption also on strictly proper plants for designing the Q-filter-based DOB. On the other hand, condition (b) in Theorem 1 restricts the choice of coefficients of the Q-filter $a_0, \ldots, a_{l-1}$ and $c_0, \ldots, c_k$. Since the characteristic polynomial of matrix (20) is

$$s^l + a_{l-1}s^{l-1} + \cdots + a_0 - \frac{\bar{D}}{D + \bar{D}}(c_k s^k + \cdots + c_0),$$

one might use the design methodology of coefficients of the Q-filter proposed in [9, Section 2.3] which is based on root locus technique. Furthermore, we have no information on the value $\tau^*$ in Theorem 1 yet. Still, one can follow the algorithm proposed in [21], where a frequency domain approach is studied to find the exact value of $\tau^*$ considering strictly proper plants.

## IV. Illustrative Example

In this section, an illustrative example is presented to describe the utility of the proposed condition for robust stability of the closed-loop system with the DOB.

Suppose that components in Fig. 2 are given in the form of transfer functions as follows:

- $\mathrm{C}(s) = 2/(s + 4)$,
- $\bar{\mathrm{P}}(s) = (s + 4)/(s + 2)$,
- $\mathrm{P}(s) = (\beta_1 s + \beta_0)/(s + \alpha_0)$, where $\alpha_0 \in [2, 4]$, $\beta_0 \in [5, 6]$, and $\beta_1 \in [1, 2]$.
- $\mathrm{Q}_A(s) = 1$,
- $\mathrm{Q}_B(s) = 1/(\tau s + 1)$,
- $r(t) = 1(t)$ (Heaviside step function),
- $d(t) = 0.1 \sin(2t)$.

The parametric uncertainty of real plant $\mathrm{P}(s)$ follows the one in Assumption 1. Also, it is easily checked that these components satisfy conditions (a) and (b) in Theorem 1. When $\tau = 0.5$ and $\tau = 0.01$, the output of the closed-loop system with DOB for the real plant $\mathrm{P}(s) = (s + 5)/(s + 3)$ is shown in Fig. 3 and Fig. 4, respectively, in which all the initial conditions are set as zero. The output is also compared with that of the nominal closed-loop system, $\bar{\mathrm{P}}(s)\mathrm{C}(s)/(1 + \bar{\mathrm{P}}(s)\mathrm{C}(s))$, without any disturbance and that of the closed-loop system without DOB but with the same disturbance $d(t)$. It is observed that with DOB, the closed-loop system is stable and the smaller the $\tau$, the better the approximation performance, as expected in the analysis. This capability of DOB to approximate the real closed-loop system to the nominal closed-loop system is often called *nominal performance recovery* [22].

## V. Concluding Remarks

In this paper, a couple of lemmas regarding necessary and/or sufficient conditions for specifying the relative degree of DAEs were presented. Focusing on the class of DAEs which can be interpreted as linear systems with zero relative degree, we proposed a design of Q-filter-based DOB and a robust stability condition of the system controlled by the DOB. While this work figured out how to apply Q-filter-based DOB on this particular class of DAEs, further research

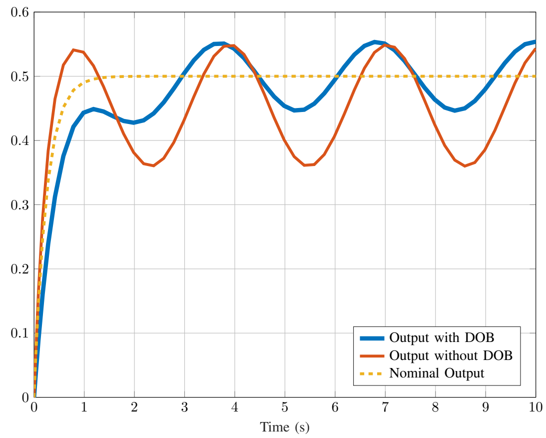


Fig. 3. Nominal performance recovery with $\tau = 0.5$

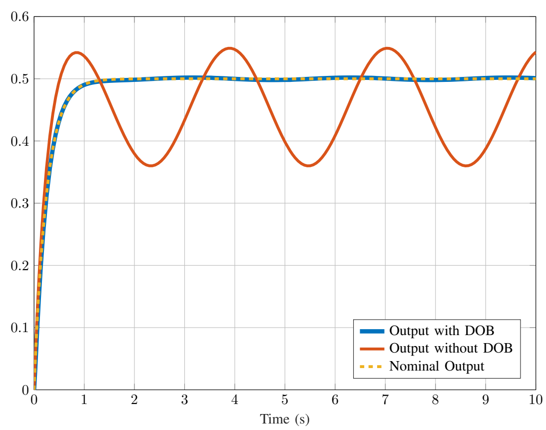


Fig. 4. Nominal performance recovery with $\tau = 0.01$

is needed to cover DAEs that are nonregular or have a higher index. Since this paper is using the existing singular perturbation theory-based approach, it may be feasible to apply the same approach to nonlinear DAEs with index 1 [23].

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