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Safe and Robust Stabilization of Uncertain Nonlinear Systems via Control Lyapunov-barrier Function and Disturbance Observer: A Preliminary Study

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Abstract— **In this paper, we address the problem of safe and robust stabilization for a class of uncertain nonlinear systems. The key idea is to employ the disturbance observer (DOB) to a nominal safety-critical controller designed for the control Lyapunov-barrier function (CLBF). The DOB estimates and compensates the lumped disturbance that represents all the effect of model uncertainty and disturbance to the system approximately but as accurately as possible. As a result, only a small perturbation remains in the control loop, which can be dealt with as long as the nominal closed-loop system is input-to-state safe (ISSf) in a sense. To ensure the ISSf property without restriction on the CLBF, we propose a modified version of the Sontag's universal formula as a nominal controller. This preliminary study verifies the validity of the proposed approach for 2nd-order nonlinear systems, but with mathematical analysis and simulations for the inverted pendulum on a cart.**

I. Introduction

As modern control systems such as quadrotors and intelligent robots usually operate in a complex and uncertain environment together with other objects or humans, safety has received a tremendous amount of attentions in the control society. A widely-used approach to ensuring the safety is to employ a Lyapunov function-like *measure* that represents how the system is safe, which is termed a *barrier function* (BF). The BF is used to characterize a set of unsafe states, on which the BF is set to have a specific sign or be infinitely large.

As a safety analogue of the control Lyapunov function (CLF), a lot of research efforts have been made for the *control barrier function* (CBF) [1]–[3]. Similar to the CLF, the CBF suggests an inequality condition for the control input, under which the system remains safe. When it comes to achieving both stability and safety simultaneously, some researchers have made several attempts to combine the CBF with the CLF, mainly in two directions. The first approach is to formulate an optimization problem (usually in the form of quadratic programming (QP)) for computing a control input, where the CLF- and CBF-related inequalites are added as hard constraints in the optimization [1], [2]. On the other hand, the authors of [3]–[6] directly sum the CLF and the CBF in a way that the resulting function has positive sign on the unsafe region, and the equilibrium point to stabilize is the (global) minima of the function: such a function is called the *control Lyapunov-barrier function* (CLBF). In the latter approach, the *safe* stabilization problem of a system is reinterpreted as the problem of enforcing the CLBF to decrease (which is similar to the conventional stabilization problem), which can be simply done by the Sontag's universal formula (SU) [4]–[7].

A remarkable point is that safety of a control system is fundamentally fragile against model uncertainties and unknown disturbances incurred by aging of machine, temporal physical effects or unexpected situations, etc. Despite its significance, however, relatively less attentions have been paid to the robustness issue of the safety, except some recent works [2], [8], [9]. In this stream of research (especially following the CLBF-based approach), we in this paper propose a CLBF-based robust stabilizing controller with safety guaranteed for a class of uncertain nonlinear systems. The underlying idea is to attach the inverse model-based disturbance observer (DOB) in [10] to the control loop with a modified version of the SU-based controller. As the DOB estimates and compensates a lumped signal that represents the effect of model uncertainty and disturbance to the plant, the inner loop containing the DOB behaves like a disturbance-free nominal model. It should be emphasized that compensation of the uncertain factors by the DOB is inherently approximate. It means that, possibly small but non-vanishing perturbation still affects the control loop, which requires the *input-to-state safety* (ISSf) of the SU-based controller [9]. A conventional way of ensuring the ISSf is to assume that the CBF has a quadratic Lyapunov function-like property, which may make the controller design too complicated. Alternatively, in this work we present a *modified Sontag's universal formula* (MSU) by adding a negative quantity to the conventional SU. In doing so, the CLBF with the MSU have the same desired property without any restriction on its structure, by which the ISSf is obtained and thus the overall safety (and also stability) is recovered.

It is important to note that, the idea of employing a robust control technique to the safety-critical controller is recently introduced in [8]. In the paper, the authors developed a QP-based robust safety-critical control law, in which the estimate of a disturbance is utilized in the CLF- and CBF-related constraints. While the controller presented in [8] is easier-to-implement than ours, it is not straightforward to prove whether or not safety and stability are achieved simultaneously, as a slack variable used to resolve the feasibility issue

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may perturb the CLF-related constraint. Compared with [8], with the help of CLBF and the modification of the SU, we provide a mathematical analysis for safe and robust stabilization of the system, provided that the uncertain quantities are (bounded but) arbitrarily large. For ease of explanation, we restrict ourselves to the class of the 2nd-order nonlinear affine systems, while extension of the proposed approach to the higher-order system seems not that difficult and will be presented in the upcoming journal paper.

Notations: The notation, $L_f V(x)$, is used for the Lie derivative $\frac{\partial V(x)}{\partial x} f(x)$ of $V(x)$ along the vector field $f(x)$. For a square matrix $P = P^\top$, $\lambda_{\rm m}(P)$ and $\lambda_{\rm M}(P)$ denote the smallest and largest eigenvalue of P, respectively. Given a set $\mathcal{D}$, denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$ and the closure of $\mathcal{D}$ by ${\rm cl}(\mathcal{D})$, respectively. For a vector x, $\|x\|_{\mathcal{D}} := \inf_{y\in\mathcal{D}} \|y-x\|$ represents the set distance between x and $\mathcal{D}$.

II. Problem Formulation

Consider a 2nd-order nonlinear affine system

$$\dot{x}_1 = x_2 \tag{1a}$$
$$\dot{x}_2 = f(x) + g(x)(u+d) \tag{1b}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state that is measurable, $u \in \mathbb{R}$ is the input, and $d \in \mathbb{R}$ is the disturbance. The set of the initial state is defined by $\mathcal{X}_0 \subset \mathbb{R}^2$. We assume that both $\mathcal{X}_0$ and the region of interest $\mathcal{X} \supset \mathcal{X}_0$ are compact and contain the origin. The functions $f(x)$ and $g(x)$ are sufficiently smooth and are not exactly known: that is, the system (1) has model uncertainty. Particularly, it is supposed that the size of the model uncertainty and external disturbance can be arbitrarily large, while their bounds are known.

Assumption 1: There exist positive constants l_f, $l_{g,\rm m}$, $l_{g,\rm M}$ and l_d such that

$$\|f(x)\| \le l_f\|x\|, \quad 0 < l_{g,\rm m} \le g(x) \le l_{g,\rm M},$$
$$\|(d(t), \dot{d}(t))\| \le l_d$$

hold for all $(t,x) \in \mathbb{R}_{\ge 0} \times \mathcal{X}$. □

Suppose that we in this work are interested in enforcing the state $x(t)$ not to belong to an open set $\mathcal{D} \subset \mathcal{X}$ of unsafe states, which satisfies $0 \notin \mathcal{D}$. A formal definition of the safety of a system is stated below.

Definition 1: Given a system (1) and a set of unsafe states $\mathcal{D}$, the system is said to be *safe* if the state $x(t)$ satisfies

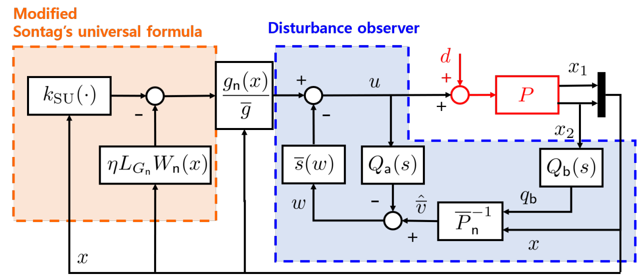
$$x(t) \notin {\rm cl}(\mathcal{D}), \quad \forall t \ge 0. \tag{2}$$

□

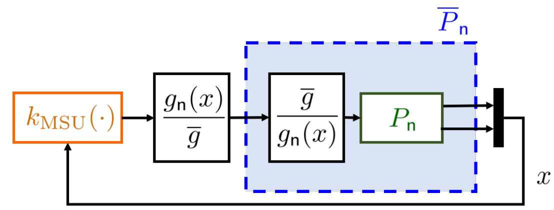
We are ready to introduce the problem that we will deal with in this work.

Problem of Interest: *For the uncertain nonlinear system* (1) *and the set $\mathcal{D}$ of unsafe states, find a control input u such that*

- (**Stability**) *the origin $x = 0$ of the closed-loop system is practically stable with ϵ-bound: that is, there exists $t^\star > 0$ such that $\|x(t)\| < \epsilon$ for all $t \ge t^\star$.*
- (**Safety**) *$x(t)$ satisfies* (2). □



(a) Block diagram of the overall system consisting of the plant (1), the $(q_{\sf a}, q_{\sf b})$-dynamics (21), the modified Sontag's universal formula (19), and the composite control law (26). $\overline{P}_{\sf n}$ is nominal model for P, which uses a constant $\overline{g}$ instead of $g_{\sf n}(x)$.



(b) Block diagram of the nominal closed-loop system: Here $g_{\sf n}(x)/\overline{g}$ is pre- and post-multiplied in the input channel for technical reasons, which does not affect the resulting output.

Fig. 1. Block diagrams of (a) the overall system and (b) the nominal closed-loop system

Remark 1: In this proof-of-concept work, we restrict ourselves to the class of single-input single-output, and 2nd-order nonlinear systems. At the expense of losing generality, this conference paper will focus on presenting the underlying principle of the proposed control scheme in a comprehensive manner, together with a rigorous analysis for stability and safety. Extension of the main results to the multi-input multi-output and higher-order systems is left to future works. □

III. Controller Design

In this section, we propose a robust safety-critical control law for the uncertain system (1) that solves Problem of Interest stated in the previous section. The key idea is to robustify a (less robust) safety-critical control law constructed for the disturbance-free nominal model for (1):

$$\dot{x} = F_{\sf n}(x) + G_{\sf n}(x)u = \begin{bmatrix} x_2 \\ f_{\sf n}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ g_{\sf n}(x) \end{bmatrix} u, \tag{3}$$

where $f_{\sf n}(x)$ and $g_{\sf n}(x)$ are nominal counterparts of $f(x)$ and $g(x)$, which do not have any uncertain factor. Notice that the plant (1) to be controlled can be viewed as a perturbed system

$$\dot{x} = F_{\sf n}(x) + G_{\sf n}(x)u + \Delta(t,x,u)$$

of (3) with a perturbation term $\Delta(t,x,u)$ which includes disturbance $d(t)$ and uncertainty of $f(x), g(x)$. With this in mind, we employ the *disturbance observer* (DOB) that estimates the perturbation Δ approximately but as accurately as we desire, and that compensates Δ by subtracting the estimate $\hat{\Delta}$ of Δ to the control input. If the quantity of the remaining perturbation $\Delta - \hat{\Delta}$ is small enough, safety and stability of the system can be ensured with a simple static control law constructed for the nominal model (3) with little consideration on robustness issue. In design of such a static control law, the *control Lyapunov-barrier function* (CLBF) plays a key role. The overall configuration of the proposed controller is depicted in Fig. 1(a), and each component will be presented in the following subsections.

A. Construction of Control Lyapunov Barrier Function

We begin by introducing the basic notion of the *control Lyapunov function* (CLF) and *control barrier function* (CBF) from [3], [5], [11].

Definition 2: For a nonlinear affine system (3) and a set of unsafe states $\mathcal{D}$,

(a) a proper and positive definite $V_{\mathsf{n}} : \mathbb{R}^2 \to \mathbb{R}$ is called a CLF if it satisfies

$$L_{F_{\mathsf{n}}} V_{\mathsf{n}}(x) < 0, \quad \forall x \in \{\mathsf{x} \in \mathbb{R}^2 \setminus \{0\} : L_{G_{\mathsf{n}}} V_{\mathsf{n}}(\mathsf{x}) = 0\}. \tag{4}$$

(b) a function $B_{\mathsf{n}} : \mathbb{R}^2 \to \mathbb{R}$ is called a CBF if it satisfies

$$B_{\mathsf{n}}(x) > 0, \quad \forall x \in \mathcal{D} \tag{5a}$$

$$L_{F_{\mathsf{n}}} B_{\mathsf{n}}(x) \le 0, \quad \forall x \in \{\mathsf{x} \in \mathcal{X} \setminus \mathcal{D} : L_{G_{\mathsf{n}}} B_{\mathsf{n}}(\mathsf{x}) = 0\} \tag{5b}$$

$$\mathcal{C}_B := \{\mathsf{x} \in \mathcal{X} : B_{\mathsf{n}}(\mathsf{x}) \le 0\} \neq \emptyset. \tag{5c}$$

□

The CLBF proposed in [5] has mixed properties of the CLF and the CBF in Definition 2, as stated below.

Definition 3: For a nonlinear affine system (3) and a set of unsafe states $\mathcal{D}$, a proper and lower-bounded function $W_{\mathsf{n}} : \mathbb{R}^2 \to \mathbb{R}$ is called a CLBF if

$$W_{\mathsf{n}}(x) > 0, \quad \forall x \in \mathcal{D} \tag{6a}$$

$$L_{F_{\mathsf{n}}} W_{\mathsf{n}}(x) < 0, \quad \forall x \in \{\mathsf{x} \in \mathcal{X} \setminus (\mathcal{D} \cup \{0\}) : L_{G_{\mathsf{n}}} W_{\mathsf{n}}(\mathsf{x}) = 0\} \tag{6b}$$

$$\mathcal{C}_W := \{\mathsf{x} \in \mathcal{X} : W_{\mathsf{n}}(\mathsf{x}) \le 0\} \neq \emptyset \tag{6c}$$

$$\mathrm{cl}(\mathcal{X} \setminus (\mathcal{D} \cup \mathcal{C}_W)) \cap \mathrm{cl}(\mathcal{D}) = \emptyset. \tag{6d}$$

□

From now on, we construct a CLBF for the nominal model (3) by following the procedure of [5]. To this end, it is needed to select a CLF and a CBF sequentially. Notice that since the system (3) is feedback linearizable, a quadratic function

$$V_{\mathsf{n}}(x) = \frac{1}{2} x^T P x \tag{7}$$

can serve as a CLF for (3), in which $P = P^T > 0$ is the solution of the Lyapunov equation

$$A^T P + PA = -Q, \quad A = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix} \tag{8}$$

for some $Q = Q^T > 0$ and $K_1 > 0$, $K_2 > 0$. This can be readily seen by applying the feedback linearization-based control law

$$u = k_{\mathrm{FL}}(x) := -g_{\mathsf{n}}^{-1}(x)(Kx + f_{\mathsf{n}}(x)) \tag{9}$$

with $K = [K_1 \ K_2] \in \mathbb{R}^{1\times 2}$ to the nominal model (3). It readily follows that

$$\frac{1}{2}\lambda_{\mathrm{m}}(P)\|x\|^2 \le V_{\mathsf{n}}(x) \le \frac{1}{2}\lambda_{\mathrm{M}}(P)\|x\|^2 \tag{10a}$$

$$\frac{\partial V_{\mathsf{n}}}{\partial x}(F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)k_{\mathrm{FL}}(x)) \le -\frac{1}{2}\lambda_{\mathrm{m}}(Q)\|x\|^2 \tag{10b}$$

$$\left\| \frac{\partial V_{\mathsf{n}}}{\partial x} \right\| \le \lambda_{\mathrm{M}}(P)\|x\|. \tag{10c}$$

On the other hand, for a future use we make the following assumption on the CBF:

Assumption 2: There exists a CBF $B_{\mathsf{n}} : \mathbb{R}^2 \to \mathbb{R}$ for the nominal model (3) that is twice continuously differentiable and satisfies

$$B_{\mathsf{n}}(x) = -\phi, \quad \forall x \in \mathcal{X} \setminus \mathcal{D}_+ \tag{11}$$

for a compact subset $\mathcal{D}_+ \supset \mathcal{D}$ that does not contain 0 with a constant $\phi > 0$. □

It is remarked that, under mild conditions, one can always refine a pre-chosen CBF so that the resulting CBF satisfies Assumption 2. (For a constructive design, the readers are referred to [1], [3], [5], [6]).

Finally, combining the CLF $V_{\mathsf{n}}(x)$ and the CBF $B_{\mathsf{n}}(x)$ in Assumption 2 as in [5], we construct the CLBF for the nominal model (3) and the set $\mathcal{D}$ of unsafe states as follows:

$$W_{\mathsf{n}}(x) = V_{\mathsf{n}}(x) + \vartheta B_{\mathsf{n}}(x) + \kappa \tag{12}$$

where ϑ and κ are selected such that

$$\vartheta > \frac{c_2 c_3 - c_1 c_4}{\phi}, \quad \kappa = -c_1 c_4$$

with

$$\begin{aligned} c_1 &= \frac{1}{2}\lambda_{\mathrm{m}}(P), \quad c_2 = \frac{1}{2}\lambda_{\mathrm{M}}(P), \\ c_3 &= \max_{x \in \partial\mathcal{D}_+} \|x\|^2, \quad c_4 = \min_{x \in \partial\mathcal{D}} \|x\|^2. \end{aligned} \tag{13}$$

A direct consequence of selecting the parameters in W_{n} is [5, Eq. (24)]:

$$\mathcal{D} \subset \hat{\mathcal{D}} := \{x : W_{\mathsf{n}}(x) > 0\} \subset \mathcal{D}_+. \tag{14}$$

This implies that, if $W_{\mathsf{n}}(x(t)) \le 0$ is satisfied for all $t \ge 0$, then the system (1) is safe.

B. Modified Sontag's Universal Formula

Once a CLBF $W_{\mathsf{n}}(x)$ is selected as in (12), the well-known *Sontag's universal formula* suggests a stabilizing control law

$$u = k_{\mathrm{SU}}\big(L_{F_{\mathsf{n}}} W_{\mathsf{n}}, L_{G_{\mathsf{n}}} W_{\mathsf{n}}; \gamma\big) \tag{15}$$

where the function k_{SU} is defined as

$$k_{\mathrm{SU}}(\mathsf{a}, \mathsf{b}; \gamma) := \begin{cases} -\dfrac{\mathsf{a} + \sqrt{\mathsf{a}^2 + \gamma \mathsf{b}^4}}{\mathsf{b}}, & \text{if } \mathsf{b} \neq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{16}$$

where $\gamma > 0$ is a design parameter. It is well-known that, as long as $W_{\mathsf{n}}(x(0)) < 0$, the system (3) with $u =$

$k_{\mathrm{SU}}(L_{F_{\mathsf{n}}}W_{\mathsf{n}}, L_{G_{\mathsf{n}}}W_{\mathsf{n}};\gamma)$ is safe and its origin $x = 0$ is asymptotically stable, because

$$\frac{\partial W_{\mathsf{n}}}{\partial x}\left(F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)k_{\mathrm{SU}}\big(L_{F_{\mathsf{n}}}W_{\mathsf{n}}, L_{G_{\mathsf{n}}}W_{\mathsf{n}};\gamma\big)\right) < 0, \quad \forall x \in \mathcal{X} \setminus \{0\}.$$

When model uncertainty and external disturbance come into the picture as in (1), however, this attractive property of k_{SU} is not enough to deal with such a (even small) perturbation. For example, when the same control law is applied to the perturbed system $\dot{x} = F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)u + \Delta$ of (3), we have only

$$\frac{\partial W_{\mathsf{n}}}{\partial x}\left(F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)k_{\mathrm{SU}} + \Delta\right) < \frac{\partial W_{\mathsf{n}}}{\partial x}\Delta.$$

It means that, we cannot conclude that $W_{\mathsf{n}}(x(t))$ decreases near the unsafe region $\mathcal{D}$, and thus the safety may not be guaranteed in some cases.

A remedy to overcome this difficulty is to guarantee that

$$\frac{\partial W_{\mathsf{n}}}{\partial x}\left(F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)u\right) < -\rho\|x\|^2, \quad \forall x \in \mathcal{X} \setminus \{0\} \tag{17}$$

for some $\rho > 0$, With this kept in mind, we propose a modified version of the Sontag's universal formula (16), by adding $-\eta\mathsf{b}$ to the conventional structure (16) as follows: that is,

$$u = k_{\mathrm{MSU}}\big(L_{F_{\mathsf{n}}}W_{\mathsf{n}}, L_{G_{\mathsf{n}}}W_{\mathsf{n}};\gamma,\eta\big) \tag{18}$$

where $\gamma > 0$ and $\eta > 0$ are design parameters, and

$$k_{\mathrm{MSU}}(\mathsf{a},\mathsf{b};\gamma,\eta) := \begin{cases} -\dfrac{\mathsf{a} + \sqrt{\mathsf{a}^2 + \gamma\mathsf{b}^4} + \eta\mathsf{b}^2}{\mathsf{b}}, & \text{if } \mathsf{b} \neq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{19}$$

The following theorem says that such a modification helps to obtain the desired property.

Theorem 1: For a system (3) and a CLBF W_{n} in (12), the modified Sontag's universal formula (19) with $\sqrt{\gamma} < \eta$ satisfies

$$\frac{\partial W_{\mathsf{n}}}{\partial x}\left(F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x)k_{\mathrm{MSU}}\big(L_{F_{\mathsf{n}}}W_{\mathsf{n}}, L_{G_{\mathsf{n}}}W_{\mathsf{n}};\gamma,\eta\big)\right) \le -\rho\|x\|^2, \quad \forall x \in \mathcal{X} \setminus \{0\}$$

where

$$\rho := \min\left(\frac{\lambda_{\mathrm{m}}(Q)}{2}, \eta\left(\frac{\lambda_{\mathrm{m}}(Q)}{2\beta}\right)^2\right).$$

with a constant $\beta > 0$ such that $|u_{\mathrm{FL}}(x)| \le \beta\|x\|$. □

For clarity of explanation, we provide the proof of Theorem 1 in Subsection IV-A. We instead remark that, as a byproduct of adding $-\eta\mathsf{b}$ to k_{SU}, one can readily conclude that the origin of the nominal model (3) controlled by (18) turns out to be exponentially stable, not just asymptotically stable.

C. Disturbance Observer and Composite Control Law

As a tool for robustifying the modified Sontag's universal formula k_{MSU} in (18), we employ the DOB for (1). The main role of the DOB is to compensate the mismatch between the actual plant (1) and its nominal model (3) (usually called the *lumped disturbance* in the literature) approximately but with arbitrarily small size, so that the modified Sontag's universal formula in (18) can easily handle the remaining (possibly small) effect of the mismatch in the end. It is also pointed out that, since $x(t)$ is assumed to be measurable in our work, the proposed DOB here will be constructed in an easier-to-implement manner than the conventional ones.

As similar to the conventional approach, we construct the DOB with two low-pass filters called the Q-filters, and the inverse of a nominal model. (It is noted in advance that the configuration of the DOB can be found in Fig. 1(a).) For the former, it is enough to utilize the simplest low-pass filter (that has the DC gain as 1 and has the relative degree 1)

$$Q_{\mathsf{a}}(s) = Q_{\mathsf{b}}(s) = \frac{1}{\tau s + 1} \tag{20}$$

as the Q-filter in the DOB structure, in which $\tau > 0$ is a design parameter and remains undetermined yet. In the state space, each of Q_{a} and Q_{b} are represented as

$$\dot{q}_{\mathsf{a}} = -\frac{1}{\tau}q_{\mathsf{a}} + \frac{1}{\tau}u, \qquad y_{\mathsf{q,a}} = q_{\mathsf{a}} \tag{21a}$$

$$\dot{q}_{\mathsf{b}} = -\frac{1}{\tau}q_{\mathsf{b}} + \frac{1}{\tau}x_2, \qquad y_{\mathsf{q,b}} = q_{\mathsf{b}} \tag{21b}$$

where $(q_{\mathsf{a}}, q_{\mathsf{b}})$ is the state of the Q-filters. Without loss of generality, it is assumed that the initial condition $(q_{\mathsf{a}}(0), q_{\mathsf{b}}(0))$ of (21) is located in a compact set $\mathcal{Q}_0$.

Next, we present the inverse of a nominal model $\overline{P}_{\mathsf{n}}$ used in Fig. 1(a). Note first that $g_{\mathsf{n}}(x)$ in (3) is allowed to be dependent of x. Thus, direct use of the inverse of (3) possibly makes both design and analysis complicated. As an alternative with no restriction on $g_{\mathsf{n}}(x)$, we here will pre- and post-multiply $g_{\mathsf{n}}(x)/\overline{g}$ in the nominal model (3) with a constant $\overline{g} > 0$. (See Fig. 1(b) for the conceptual description.) The key idea is to rewrite (3) with a new (auxiliary) input variable $\overline{v} := (g_{\mathsf{n}}(x)/\overline{g})\overline{u}$ as

$$\dot{\overline{x}}_1 = \overline{x}_2, \quad \dot{\overline{x}}_2 = f_{\mathsf{n}}(\overline{x}) + \overline{g}\left(\frac{g_{\mathsf{n}}(x)}{\overline{g}}\overline{u}\right) = f_{\mathsf{n}}(\overline{x}) + \overline{g}\overline{v}$$

where $(\overline{x}_1, \overline{x}_2)$ and $\overline{u}$ are states and inputs of $\overline{P}_{\mathsf{n}}$, respectively. By regarding $\overline{v}$ as output, its inverse model $\overline{P}_{\mathsf{n}}$ can be expressed as

$$\overline{v} = \frac{1}{\overline{g}}\big(\dot{\overline{x}}_2 - f_{\mathsf{n}}(\overline{x})\big). \tag{22}$$

Finally, replacing $\overline{x} = (\overline{x}_1, \overline{x}_2)$ and $\dot{\overline{x}}_2$ with the measurement $x = (x_1, x_2)$ and the estimate $\dot{q}_{\mathsf{b}}$, respectively, we have

$$\hat{\overline{v}}_{\mathsf{n}} = \frac{1}{\overline{g}}\big(\dot{q}_{\mathsf{b}} - f_{\mathsf{n}}(x)\big) = \frac{1}{\overline{g}}\left(-\frac{1}{\tau}q_{\mathsf{b}} + \frac{1}{\tau}x_2 - f_{\mathsf{n}}(x)\right) \tag{23}$$

which represents the very $\overline{P}_{\mathsf{n}}^{-1}$ in Fig. 1(a).

The last ingredient for the DOB design is a saturation function $\overline{s}$ that is sufficiently smooth and satisfies

$$\begin{aligned} &\overline{s}(w) = w, \quad \forall w \in [w_{\rm m}^\star, w_{\rm M}^\star], \\ &\left|\frac{\partial \overline{s}}{\partial w}\right| \le 1, \quad \overline{s}(w) \text{ is bounded for all } w \in \mathbb{R} \end{aligned} \tag{24}$$

where $w_{\rm m}^\star$ and $w_{\rm M}^\star$ are some constants such that

$$\{w^\star \text{ in (25)} : x \in \mathcal{X},\ |d| \le l_d\} \subset [w_{\rm m}^\star, w_{\rm M}^\star]$$

with

$$w^\star := d + \frac{1}{g(x)}(f(x) - f_{\sf n}(x)) + \left(\frac{1}{\overline{g}} - 1\right)\frac{g_{\sf n}(x)}{g(x)} k_{\rm MSU}. \tag{25}$$

Summarizing so far, the composite control law consists of the modified Sontag's universal formula (19) and the q-dynamics (21) in the DOB structure such as

$$u = \frac{g_{\sf n}(x)}{\overline{g}} k_{\rm MSU} - \overline{s}\left(\frac{1}{\overline{g}}\left(-\frac{1}{\tau}q_{\sf b} + \frac{1}{\tau}x_2 - f_{\sf n}(x)\right) - q_{\sf a}\right). \tag{26}$$

IV. Mathematical Analysis

In this section, we prove that the proposed robust safety-critical control law (26) guarantees that the uncertain system (1) under (18) is robustly safe and stable against disturbance and uncertainty in Assumption 1. The following assumption is required to make the origin $x = 0$ reachable.

Assumption 3: For the CLBF $W_{\sf n}(x)$ in (12), the set

$$\hat{\mathcal{C}} := \mathcal{X} \setminus \hat{\mathcal{D}} = \{x \in \mathcal{X} : W_{\sf n}(x) \le 0\}$$

is connected. □

Our main result is summarized in the following theorem, while its proof will be presented in Subsection IV-C.

Theorem 2: Suppose that Assumptions 1–3 hold, and $W_{\sf n}(x(0)) < 0$. Then for any $\epsilon > 0$, there exist $\tau^\star > 0$ and $t^\star > 0$ such that for all $0 < \tau < \tau^\star$, the solution $(x(t), q(t))$ of the closed-loop system with (1), (19), (21), and (26) satisfies the following:

(a) $x(t) \in \hat{\mathcal{C}}$ for all $t \ge 0$;
(b) $\|x(t)\| < \epsilon$ for all $t \ge t^\star$. □

A. Proof of Theorem 1

In this subsection we prove Theorem 1 first. The proof of the theorem is divided into three parts, in terms of the value of $L_{G_{\sf n}} W_{\sf n}(x)$. For brevity, let for now

$$\mathsf{a}(x) := L_{F_{\sf n}} W_{\sf n}(x), \quad \mathsf{b}(x) := L_{G_{\sf n}} W_{\sf n}(x)$$

(**Case 1:** $\mathsf{b}(x) = 0$) From (10b) we have

$$\mathsf{a}(x) + \mathsf{b}(x) k_{\rm MSU} = \mathsf{a}(x) \le -\lambda_{\rm m}(Q)\|x\|^2 \le -\rho\|x\|^2.$$

(**Case 2:** $|\mathsf{b}(x)| \le (\lambda_{\rm m}(Q)/2\beta)\|x\|$) Note that, by (10b) and the definition of β, one has

$$\begin{aligned} \mathsf{a}(x) &\le -\lambda_{\rm m}(Q)\|x\|^2 + |\mathsf{b}(x)|\|k_{\rm FL}(x)\| \\ &\le -\lambda_{\rm m}(Q)\|x\|^2 + \left(\frac{\lambda_{\rm m}(Q)}{2\beta}\|x\|\right)(\beta\|x\|) \\ &\le -\frac{\lambda_{\rm m}(Q)}{2}\|x\|^2. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \mathsf{a}(x) + \mathsf{b}(x) k_{\rm MSU} &= -\sqrt{\mathsf{a}(x)^2 + \gamma \mathsf{b}(x)^4} - \eta \mathsf{b}(x)^2 \\ &\le -|\mathsf{a}(x)| + \sqrt{\gamma}\mathsf{b}(x)^2 - \eta \mathsf{b}(x)^2 \\ &\le \mathsf{a}(x) \le -\frac{\lambda_{\rm m}(Q)}{2}\|x\|^2 \\ &\le -\rho\|x\|^2 \end{aligned}$$

where the last three inequalities come from $\eta > \sqrt{\gamma}$ and $\mathsf{a}(x) < 0$ as above.

(**Case 3**: $|\mathsf{b}(x)| > (\lambda_{\rm m}(Q)/2\beta)\|x\|$) For sufficiently large $|\mathsf{b}(x)|$, we have

$$\begin{aligned} \mathsf{a}(x) + \mathsf{b}(x) k_{\rm MSU} &= -\sqrt{\mathsf{a}(x)^2 + \gamma \mathsf{b}(x)^4} - \eta \mathsf{b}(x)^2 \\ &\le -\eta \mathsf{b}(x)^2 \le -\eta\left(\frac{\lambda_{\rm m}(Q)}{2\beta}\right)^2\|x\|^2 \\ &\le -\rho\|x\|^2. \end{aligned}$$

Summing up all the cases concludes the theorem.

B. System Description in Singularly Perturbed Form

As the first step to prove the main result, we rewrite the closed-loop system (1), (19), (21) and (26) into a singularly perturbed form. To this end, define the following coordinate change for the state $q = (q_{\sf a}, q_{\sf b})$ of the DOB that is well-defined for any $\tau > 0$:

$$\xi := \begin{bmatrix} \xi_{\sf a} \\ \xi_{\sf b} \end{bmatrix} = \Phi_\xi(x, q) := \begin{bmatrix} q_{\sf a} + \dfrac{1}{\tau\overline{g}}(q_{\sf b} - x_2) \\ q_{\sf b} \end{bmatrix} \in \mathbb{R}^2.$$

Now, differentiating $\xi_{\sf a}$ in t along with (1) and (21), we have

$$\begin{aligned} \dot{\xi}_{\sf a} &= \dot{q}_{\sf a} + \frac{1}{\tau\overline{g}}(\dot{q}_{\sf b} - \dot{x}_2) \\ &= -\frac{1}{\tau}q_{\sf a} + \frac{1}{\tau}u \\ &\quad + \frac{1}{\tau\overline{g}}\left(-\frac{1}{\tau}q_{\sf b} + \frac{1}{\tau}x_2 - \big(f(x) + g(x)(u + d)\big)\right) \\ &= -\frac{1}{\tau}\xi_{\sf a} + \frac{1}{\tau}\left(1 - \frac{g(x)}{\overline{g}}\right)u - \frac{1}{\tau\overline{g}}(f(x) + g(x)d) \end{aligned}$$

On the other hand, the composite control input u in (26) can be represented in the (x, ξ)-coordinate as follows:

$$u = \frac{g_{\sf n}(x)}{\overline{g}} k_{\rm MSU} - \overline{s}\left(-\xi_{\sf a} - \frac{1}{\overline{g}}f_{\sf n}(x)\right). \tag{27}$$

Applying (27) to $\dot{\xi}_{\sf a}$ above gives

$$\begin{aligned} \tau\dot{\xi}_{\sf a} =& -\xi_{\sf a} - \left(1 - \frac{g(x)}{\overline{g}}\right)\overline{s}\left(-\xi_{\sf a} - \frac{1}{\overline{g}}f_{\sf n}(x)\right) && \text{(28a)} \\ &+ \left(1 - \frac{g(x)}{\overline{g}}\right)\frac{g_{\sf n}(x)}{\overline{g}}k_{\rm MSU} - \frac{1}{\overline{g}}(f(x) + g(x)d), \\ \tau\dot{\xi}_{\sf b} =& -\xi_{\sf b} + x_2. && \text{(28b)} \end{aligned}$$

which is the fast subsystem with sufficiently small τ in the overall system.

According to the singular perturbation theory, the fast variable $\xi := (\xi_{\mathsf{a}}, \xi_{\mathsf{b}})$ approaches to the *boundary layer* $(\xi_{\mathsf{a}}, \xi_{\mathsf{b}}) = (\xi_{\mathsf{a}}^\star, \xi_{\mathsf{b}}^\star)$ quickly, where $(\xi_{\mathsf{a}}^\star, \xi_{\mathsf{b}}^\star)$ is the solution of (28) with $\tau = 0$ while (t, x) regarded as being fixed or frozen: in other words, $(\xi_{\mathsf{a}}^\star, \xi_{\mathsf{b}}^\star)$ is the solution of

$$0 = -\xi_{\mathsf{a}}^\star - \left(1 - \frac{g(x)}{\overline{g}}\right) \overline{s} \left(-\xi_{\mathsf{a}}^\star - \frac{1}{\overline{g}} f_{\mathsf{n}}(x)\right) + \left(1 - \frac{g(x)}{\overline{g}}\right) \frac{g_{\mathsf{n}}(x)}{\overline{g}} k_{\mathrm{MSU}} - \frac{1}{\overline{g}}(f(x) + g(x)d), \tag{29a}$$

$$0 = -\xi_{\mathsf{b}}^\star + x_2. \tag{29b}$$

Following the same procedure of [12] with the properties of the saturation function $\overline{s}$, one can conclude that the solution of (29) is uniquely determined as

$$\xi_{\mathsf{a}}^\star = \left(\frac{1}{g(x)} - \frac{1}{\overline{g}}\right)(f_{\mathsf{n}}(x) + g_{\mathsf{n}}(x) k_{\mathrm{MSU}}) - \frac{1}{g(x)}(f(x) + g(x)d), \tag{30a}$$

$$\xi_{\mathsf{b}}^\star = x_2. \tag{30b}$$

We now define the error variable for ξ as

$$\tilde{\xi} = \begin{bmatrix} \tilde{\xi}_{\mathsf{a}} \\ \tilde{\xi}_{\mathsf{b}} \end{bmatrix} = \begin{bmatrix} \xi_{\mathsf{a}} - \xi_{\mathsf{a}}^\star \\ \xi_{\mathsf{b}} - \xi_{\mathsf{b}}^\star \end{bmatrix}, \tag{31}$$

and compute the overall error dynamics (which is expressed in a singularly perturbed form) as

$$\dot{x} = F_{\mathsf{n}}(x) + G_{\mathsf{n}}(x) k_{\mathrm{MSU}} + \begin{bmatrix} 0 \\ g(x)\tilde{\delta}_w(t, x, \tilde{\xi}) \end{bmatrix} \tag{32a}$$

$$\tau \dot{\tilde{\xi}}_{\mathsf{a}} = -\tilde{\xi}_{\mathsf{a}} + \left(1 - \frac{g(x)}{\overline{g}}\right) \tilde{\delta}_w(t, x, \tilde{\xi}) - \tau \dot{\xi}_{\mathsf{a}}^\star \tag{32b}$$

$$\tau \dot{\tilde{\xi}}_{\mathsf{b}} = -\tilde{\xi}_{\mathsf{b}} - \tau \dot{\xi}_{\mathsf{b}}^\star \tag{32c}$$

where

$$\tilde{\delta}_w(t, x, \tilde{\xi}) := -\overline{s}(-\tilde{\xi}_{\mathsf{a}} - w^\star) - w^\star$$

and $w^\star$ is defined in (25) and satisfies

$$w^\star = -\xi_{\mathsf{a}}^\star - \frac{1}{\overline{g}} f_{\mathsf{n}}(x).$$

Note that $w^\star \in \mathcal{W}^\star$ so that $w^\star = \overline{s}(w^\star)$. This implies that

$$0 \le \tilde{\xi}_{\mathsf{a}} \tilde{\delta}_w(t, x, \tilde{\xi}) \le \|\tilde{\xi}_{\mathsf{a}}\|^2 \tag{33}$$

for any $\tilde{\xi}_{\mathsf{a}} \in \mathbb{R}$, which will be used in the analysis to come.

C. Proof of Theorem 2

For the analysis of the overall error dynamics (32), we define a CLBF-like function

$$W(x, \tilde{\xi}) = W_{\mathsf{n}}(x) + V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}) + V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{a}}) \tag{34}$$

where $W_{\mathsf{n}}(x)$ is the CLBF in (12), and V_{a} and V_{b} are defined as

$$V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}) := \frac{1}{2}\tilde{\xi}_{\mathsf{a}}^2, \quad V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}) := \frac{1}{2}\tilde{\xi}_{\mathsf{b}}^2.$$

Conceptually, we prove Theorem 2 by showing that

- $W_{\mathsf{n}}(x(t)) \le 0$ for all $t \ge 0$, and
- $W(x(t), \tilde{\xi}(t))$ converges (approximately) to $W(0, 0) = W_{\mathsf{n}}(0) = \vartheta B_{\mathsf{n}}(0) + \kappa$ as time goes on, by which one can say that $V_{\mathsf{n}}(x(t))$ converges to 0.

For both items, we will observe that $\dot{W} < 0$ in the entire time for sufficiently small τ, while additional attention needs to be paid for the first item. Indeed, even though $W_{\mathsf{n}}(x(0))$ is negative, $W(x(0), \tilde{\xi}(0))$ can be positive, and even become larger as τ gets smaller. This means that $\dot{W} < 0$ does not imply the first item directly.

We tackle this issue by dividing the entire time interval $[0, \infty)$ into two parts: the transient period $[0, t_1)$ and the steady-state period $[t_1, \infty)$. Suppose $W_{\mathsf{n}}(x(t)) = -\varsigma$, and here $t_1 > 0$ is selected such that

$$W_{\mathsf{n}}(x(t)) \le -\frac{1}{2}\varsigma, \quad \forall t \in [0, t_1] \tag{35}$$

holds for any value of τ. Such a t_1 always exists because

$$\dot{W}_{\mathsf{n}} = \frac{\partial W_{\mathsf{n}}}{\partial x}\dot{x} = \frac{\partial W_{\mathsf{n}}}{\partial x}(f(x) + g(x)(u + d))$$

must be bounded with τ-independent bounds, due to Assumption 1 and the saturation function $\overline{s}$.

The following lemma says that, $\tilde{\xi}(t)$ converges to $\tilde{\xi} = 0$ quickly in the transient period $[0, t_1)$, so that W turns out to be negative at the end of the transient.

Lemma 1: Under the same hypothesis of Theorem 2 and (35), there exists τ_1 such that for all $0 < \tau < \tau_1$, the solution $(x(t), \tilde{\xi}(t))$ satisfies

$$W\big(x(t_1), \tilde{\xi}(t_1)\big) \le -\frac{1}{4}\varsigma. \tag{36}$$

□

Proof: Since W_{n} satisfies (35), it is enough to show that

$$V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}(t_1)) + V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}(t_1)) \le \frac{1}{4}\varsigma.$$

For this, we first compute $\dot{V}_{\mathsf{a}}$ as follows:

$$\begin{aligned} \dot{V}_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}) &= -\frac{1}{\tau}\tilde{\xi}_{\mathsf{a}}^2 + \frac{1}{\tau}\left(1 - \frac{g(x)}{\overline{g}}\right)\tilde{\xi}_{\mathsf{a}}\tilde{\delta}_w(t, x, \tilde{\xi}) - \tilde{\xi}_{\mathsf{a}}\dot{\xi}_{\mathsf{a}}^\star. \\ &\le -\frac{1}{\tau}\frac{g(x)}{\overline{g}}\tilde{\xi}_{\mathsf{a}}^2 - \tilde{\xi}_{\mathsf{a}}\dot{\xi}_{\mathsf{a}}^\star \\ &\le -\frac{1}{\tau}\frac{l_{g,\mathrm{m}}}{\overline{g}}\tilde{\xi}_{\mathsf{a}}^2 - \tilde{\xi}_{\mathsf{a}}\dot{\xi}_{\mathsf{a}}^\star \end{aligned} \tag{37}$$

where we use (33) for the first inequality. After some computations, one may have

$$\frac{d}{dt}\sqrt{V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}(t))} \le -\frac{1}{\tau}\frac{\overline{g}}{l_{g,\mathrm{m}}}\sqrt{V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}(t))} + \sigma_{\mathsf{a}}$$

$$\frac{d}{dt}\sqrt{V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}(t))} \le -\frac{1}{\tau}\sqrt{V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}(t))} + \sigma_{\mathsf{b}},$$

where

$$\sigma_{\mathsf{a}} = \frac{\sqrt{2}}{2}\|\dot{\xi}_{\mathsf{a}}^\star\|_\infty, \ \sigma_{\mathsf{b}} = \frac{\sqrt{2}}{2}\|\dot{\xi}_{\mathsf{b}}^\star\|_\infty$$

are bounded above. By applying the comparison lemma [11, Lemma B.2] to each $\sqrt{V_{\mathsf{a}}}$ and $\sqrt{V_{\mathsf{b}}}$ and by using the fact that

$$\|\tilde{\xi}_{\mathsf{a}}(0)\| \le \upsilon_0 + \frac{1}{\tau}\upsilon_1, \quad \|\tilde{\xi}_{\mathsf{b}}(0)\| \le \upsilon_2$$

for some τ-independent constants $\upsilon_i > 0$, it is derived that

$$\begin{aligned} &V_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}(t_1)) + V_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}(t_1)) \\ &\leq \left(\frac{\upsilon_0 + (1/\tau)\upsilon_1}{\sqrt{2}} e^{-\frac{1}{\tau}\frac{l_{g,\mathrm{m}}}{\overline{g}} t_1} + \tau \frac{\overline{g}}{l_{g,\mathrm{m}}} \sigma_{\mathsf{a}} \right)^2 \\ &\quad + \left(\frac{\upsilon_2}{\sqrt{2}} e^{-\frac{1}{\tau} t_1} + \tau \sigma_{\mathsf{b}} \right)^2 . \end{aligned}$$

Note that the right-hand side term of the inequality above decreases as τ gets smaller, which completes the proof of the lemma. ■

We now show that in the steady-state period $[t_1, \infty)$, $W(x(t), \tilde{\xi}(t))$ initiated at $W(x(t_1), \tilde{\xi}(t_1)) < 0$ will decrease as long as $\|x(t)\| \geq \epsilon$.

Lemma 2: Under the same hypothesis of Theorem 2, there exists τ_2 such that for all $0 < \tau < \tau_2$, the solution $(x(t), \tilde{\xi}(t))$ satisfies

$$\dot{W}(x(t), \tilde{\xi}(t)) \leq -2\rho \left(V_{\mathsf{n}}(x(t)) - \frac{1}{2}\lambda_{\mathrm{m}}(P)\epsilon^2 \right) \quad (38)$$

for all $t \geq t_1$.

Proof: The time derivative of W_{n} along with (1) whose input is $u = k_{\mathrm{MSU}}$ is calculated as

$$\begin{aligned} \dot{W}_{\mathsf{n}} &\leq -\rho\|x\|^2 + \left\| \frac{\partial W_{\mathsf{n}}}{\partial x} \right\| |g(x)||\tilde{\delta}_w| \\ &\leq -\rho\|x\|^2 + \mu\|\tilde{\xi}_{\mathsf{a}}\| \end{aligned}$$

where $\mu := l_{g,\mathrm{M}} \cdot \max_{x\in\mathcal{X}} \|\partial W_{\mathsf{n}}/\partial x\| < \infty$. From this and (37), we have with the Young's inequality that

$$\begin{aligned} \dot{W} &\leq -\rho\|x\|^2 + \mu\|\tilde{\xi}_{\mathsf{a}}\| + \dot{V}_{\mathsf{a}}(\tilde{\xi}_{\mathsf{a}}) + \dot{V}_{\mathsf{b}}(\tilde{\xi}_{\mathsf{b}}) \\ &\leq -\rho\|x\|^2 - \frac{1}{\tau}\frac{l_{g,\mathrm{m}}}{\overline{g}}\|\tilde{\xi}_{\mathsf{a}}\|^2 - \frac{1}{\tau}\|\tilde{\xi}_{\mathsf{b}}\|^2 \\ &\qquad + (\mu + \sqrt{2}\sigma_{\mathsf{a}})\|\tilde{\xi}_{\mathsf{a}}\| + \sqrt{2}\sigma_{\mathsf{b}}\|\tilde{\xi}_{\mathsf{b}}\| \\ &\leq -2\rho V_{\mathsf{n}} + \tau\Big(\frac{\overline{g}}{l_{g,\mathrm{m}}}(\mu + \sqrt{2}\sigma_{\mathsf{a}})^2 + 2\sigma_{\mathsf{b}}^2\Big). \end{aligned}$$

Thus if τ is sufficiently small, (38) is true. ■

It is obtained from (38) that $W(x(t), \tilde{\xi}(t))$ decreases unless $V_{\mathsf{n}}(x(t)) < (1/2)\lambda_{\mathrm{m}}(P)\epsilon^2$. We note that, with ϵ small enough (which does not lose the generality of the proof), $x(t)$ eventually enters $\mathcal{X} \setminus \mathcal{D}_+ \subset \hat{\mathcal{C}}$ and does not leave forever, by which Item (a) of Theorem 2 is obtained. On the other hand, for given ϵ and $\tau < \tau_2$ satisfying Lemma 2, there exists $t_2 \geq t_1$ such that

$$\frac{1}{2}\lambda_{\mathrm{m}}(P)\|x(t)\|^2 \leq V_{\mathsf{n}}(x(t)) < \frac{1}{2}\lambda_{\mathrm{m}}(P)\epsilon^2$$

for all $t \geq t_2$. This concludes the proof of the second part of the theorem.

V. Simulation: Inverted Pendulum on a Cart

In this section, we perform a simulation to verify the validity of the proposed control scheme. Consider an inverted pendulum on a cart in Fig. 2 governed by the (normalized) dynamics [11, Appendix A.11]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = a\big(\sin x_1 + \cos x_1(u + d)\big),$$

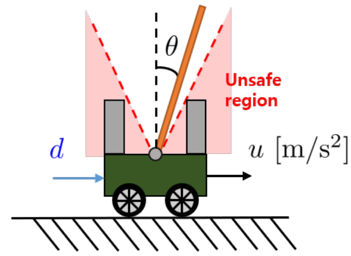


Fig. 2. Inverted pendulum on a cart

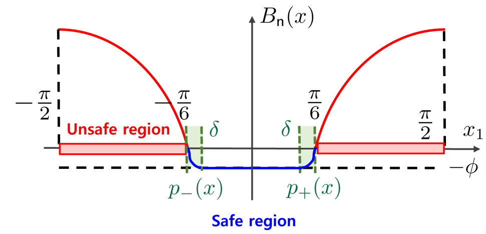


Fig. 3. Structure of CBF

$K_1 = 1$ $K_2 = 4$	$c_1 = 0.0660$ $c_2 = 1.1840$ $c_3 = \left(\frac{\pi}{2}\right)^2 + 1$ $c_4 = \left(\frac{\pi}{6}\right)^2$	$\phi = 1$ $\vartheta = 0.6854$ $\kappa = -0.045$	$\tau = 0.02$ $\eta = 10$ $\gamma = 2$ for k_{MSU} $\gamma = 2 + \eta^2$ for k_{SU}

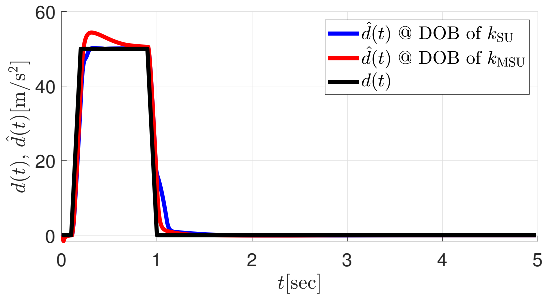
TABLE I
Design parameters used for simulation

where $x_1 = \theta$ [rad] and $x_2 = \dot{\theta}$ [rad/s] are state variables that are assumed to belong in a compact set $\mathcal{X} := \{(x_1, x_2) \in \mathbb{R}^2 : -\pi/2 \leq x_1 \leq \pi/2, -1 \leq x_2 \leq 1\}$ as a physical constraint. On the other hand, the acceleration u $[m/s^2]$ of the cart serves as the control input, and d $[m/s^2]$ represents an unknown disturbance. The parameter a is determined in the normalization process and depends on the inertia, length, and so on. In this simulation, we assume that a is uncertain and belongs to the set $[0.8, 1.2]$, and its nominal value a_{n} is set as 1.

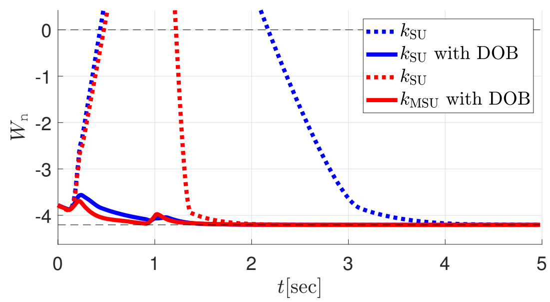
We now construct the proposed robust safety-critical controller, with the unsafe region set as $\mathcal{D} := \{x \in \mathcal{X} : \pi/6 < |x_1| < \pi/2\}$ to avoid collision between the pole and an object on the cart. First, take $K_1 = 1$ and $K_2 = 4$ from which P, c_1, and c_2 are determined as the solution of (8) and (13). Next, define $\mathcal{D}_+ := \{x \in \mathcal{X} : \pi/6 - \delta < |x_1| < \pi/2\}$ with $\delta = \pi/36$, and then c_3, c_4 and κ can be selected as in (13). We design a CBF $B_{\mathsf{n}}(x)$ as in the following, in order to satisfy (5a)–(5c) and Assumption 2 for all $x \in \mathcal{X}$:

$$B_{\mathsf{n}}(x) = \begin{cases} -b(x_1 + \frac{\pi}{2})^2 + b(-\frac{\pi}{6} + \frac{\pi}{2})^2, & \text{if } -\frac{\pi}{2} < x_1 < -\frac{\pi}{6} \\ p_-(x_1), & \text{if } -\frac{\pi}{6} \leq x_1 \leq -\frac{\pi}{6} + \delta \\ p_+(x_1), & \text{if } \frac{\pi}{6} - \delta \leq x_1 \leq \frac{\pi}{6} \\ -b(x_1 - \frac{\pi}{2})^2 + b(\frac{\pi}{6} - \frac{\pi}{2})^2, & \text{if } \frac{\pi}{6} < x_1 < \frac{\pi}{2} \\ -\phi, & \text{otherwise} \end{cases}$$

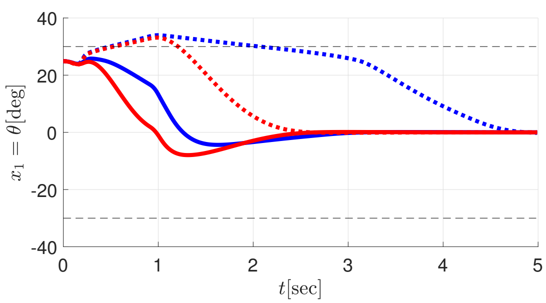
where $b = 10$ and $p_-(x)$ and $p_+(x)$ are cubic polynomials



(a) Disturbance $d(t)$ (black) and its estimates $\hat{d}(t)$ (red and blue)



(b) CLBF $W_{\mathsf{n}}(x(t))$



(c) $x_1(t) = \theta(t)$

Fig. 4. Simulation results controlled by the conventional Sontag's universal formula $k_{\rm SU}$ (blue) and with its modified version $k_{\rm MSU}$ (red) with and without the DOB (dashed and solid, respectively)

to connect the quadratic function and the constant function $x = -\phi$ smoothly, as depicted in Fig. 3. Choose $\phi = 1$, and then ϑ can be taken to satisfy (13). The remaining parameters τ, η and γ used in the DOB and other parts of the controller are selected as in Table I, satisfying $\sqrt{\gamma} < \eta$.

In addition to the proposed controller (26), three types of control laws are designed for comparison: the conventional Sontag's universal formula with and without the DOB, and the modified Sontag's universal formula without the DOB. Simulation results on these four controllers are presented in Fig. 4. For the simulation, set the initial condition of (1) as $x(0) = (\pi/6 - \delta - \pi/1800, 0)$, which is close to but not in $\hat{\mathcal{D}}$. The disturbance $d(t)$ is assumed to be a pulse signal such that $d(t) = 50$ for $0.1 \le t \le 1$, and $d(t) = 0$ otherwise. It is seen in Fig. 4(a) that the DOBs in both cases capture the disturbance $d(t)$, which enhances robustness of the Sontag's universal formula-based controllers as in Fig. 4(b), (c) as expected in the analysis. On the other hand, without the DOB the system turns out to be unsafe as $W_{\mathsf{n}}(x(t))$ gets larger. It is also needed to notice that the state $x(t)$ with $u = k_{\rm MSU}$ converges to the origin faster than that with $u = k_{\rm SU}$, due to the modification of the control law.

VI. Conclusion

In this paper, we proposed a controller for safe and robust stabilization of uncertain nonlinear systems, in the presence of not only model uncertainty and disturbance but also unsafe regions. The proposed controller consists of two components: the first is a modified version of the Sontag's universal formula, while the other part is the DOB that compensates disturbances and uncertainty. It can be observed in the mathematical analysis that the proposed controller enforces $W_{\mathsf{n}}(x(t))$ to remain negative and to converge to $W_{\mathsf{n}}(0)$ as time goes on, by which robust safety and stability of the closed-loop system is simultaneously guaranteed. Future works will include extension to a class of n-order nonlinear systems and application to robotics system.

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