2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

ISS Lyapunov-Krasovskii theorem with point-wise dissipation: a V-stability approach

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Abstract— **We show that the existence of a Lyapunov-Krasovskii functional (LKF) with a point-wise dissipation suffices for ISS of time-delay systems, provided that uniform global stability can also be ensured using the same LKF. To prove this result, we develop a stability theory, in which the behavior of solutions is not assessed through the classical norm but rather through a specific LKF, which may provide significantly tighter estimates.**

Keywords: nonlinear control systems, input-to-state stability, time-delay systems, infinite-dimensional systems.

I. Introduction

Input-to-state stability (ISS), introduced by E.D. Sontag in the late 1980s [20], has become a central tool in the analysis and control of nonlinear dynamical systems [21], [12]. Originally defined in the context of ordinary differential equations (ODE), it has been extended more recently to infinite-dimensional systems [13], [7], including time-delay systems (TDS) [3].

For TDS, ISS can be established by means of Lyapunov-Krasovskii functionals (LKFs) [19]. As for ODEs, ISS holds if the LKF dissipates along the system's solutions, modulo a positive term involving the input norm. So far, the only general conditions to ensure ISS based on LKF impose that the dissipation can be expressed in terms of the LKF itself (which is also a necessary requirement for ISS [8]).

It has been conjectured in [4] that a point-wise dissipation, involving merely the norm of current value of the solution, is enough to guarantee ISS. While this conjecture has been proved for specific classes of systems [4], [2] and for the weaker notion of integral ISS [1], it has not yet been proved or disproved in its full generality. It is worth mentioning that, in [6], the authors employed an ISS LKF in the so-called "implication form" and showed that if a point-wise dissipation holds whenever the LKF dominates the input magnitude then ISS can be concluded. Still, this condition remains significantly more conservative than the existence of an ISS LKF with point-wise dissipation.

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This work has been supported by BayFrance, project FK-20-2022. A. Mironchenko has been supported by the German Research Foundation (DFG) (grant MI 1886/3-1).

Solving this conjecture would be interesting not only for the sake of mathematical curiosity, but also for more practical considerations, as a point-wise dissipation is usually easier to obtain than an LKF-wise one. It would also unify the theory with that of input-free systems, since it has been known for a long time that a point-wise dissipation is enough to conclude global asymptotic stability [9].

Despite significant efforts on this question, it is not even known whether the conjecture is true if we additionally assume that solutions are globally uniformly bounded or even if the origin is uniformly globally stable (UGS). In this paper, we partially solve this question by showing that if the LKF that dissipates point-wisely can also be used to establish UGS, then the ISS property holds.

Interestingly, our main result actually establishes a stronger property, which we call V-ISS. This property is similar to ISS, but measures the behavior of the system's solutions through a given LKF V rather than through the classical sup norm of the state. We believe this notion may be of some interest on its own as, depending on the considered LKF, it may provide a tighter estimate of the solutions' norm.

The paper is organized as follows. In Section II, we recall some basics about TDS, introduce the V-ISS concept, state our main result and highlight its added value with respect to [6]. In Section III, we adapt some other classical stability concepts, in the same spirit as V-ISS, and state a superposition principle for V-ISS. We also provide some LKF-based conditions to establish these properties, and give some technical observations that are needed in the proof of the main result, which is stated in Section IV. For input-free systems, our results recover and, in fact, strengthen the Krasovskii theorem for asymptotic stability [9].

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm and $|A|$ denotes the corresponding induced matrix norm of $A \in \mathbb{R}^{n\times n}$. Given intervals $\mathscr{I}, \mathscr{J} \subset \mathbb{R}$, $C(\mathscr{I}, \mathscr{J})$ denotes the set of continuous functions from $\mathscr{I}$ to $\mathscr{J}$. "For all $t \in \mathscr{I}$ *a.e.*" means for all $t \in \mathscr{I}$, with the possible of a set of measure zero. Given $\theta > 0$, $\mathscr{X} := C([-\theta,0],\mathbb{R})$. $\mathscr{U}$ denotes the set of all signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that are Lebesgue measurable and locally essentially bounded. Given an interval $\mathscr{I} \subset \mathbb{R}_{\geq 0}$ and a locally essentially bounded signal $u : \mathscr{I} \to \mathbb{R}^m$, $\|u\| := \operatorname{ess\,sup}_{t\in\mathscr{I}}|u(t)|$. Given $u \in \mathscr{U}^m$, $u_{\mathscr{I}} : \mathscr{I} \to \mathbb{R}^m$ denotes its restriction to the interval $\mathscr{I}$, in particular $\|u_{\mathscr{I}}\| = \operatorname{ess\,sup}_{t\in\mathscr{I}}|u(t)|$. Given $T \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, $\theta > 0$, $x \in C([-\theta,T),\mathbb{R}^n)$ and $t \in [0,T)$, $x_t \in \mathscr{X}^n$ is the history function defined as $x_t(\tau) := x(t+\tau)$ for all $\tau \in$

$[-\theta,0]$. We also use the standard classes of comparison functions:

$$\begin{aligned}
\mathcal{K} &:= \{\gamma \in C(\mathbb{R}_+,\mathbb{R}_+) \,|\, \gamma(0)=0,\ \gamma \text{ is strictly increasing}\},\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\},\\
\mathcal{L} &:= \{\gamma \in C(\mathbb{R}_+,\mathbb{R}_+) \,|\, \gamma \text{ is decreasing, } \lim_{t\to\infty}\gamma(t)=0\},\\
\mathcal{KL} &:= \{\beta \in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+ \,|\, \beta(\cdot,t)\in\mathcal{K}\ \forall t\geq 0,\\
&\qquad\qquad \beta(r,\cdot)\in\mathcal{L}\ \forall r>0\}.
\end{aligned}$$

II. Preliminaries and main result

A. Time-delay systems

We consider retarded differential equations of the form

$$\dot{x}(t) = f(x_t, u(t)), \tag{1}$$

where $x_t \in \mathcal{X}^n$, $n\in\mathbb{N}$, denotes the history function defined as $x_t(s) := x(t+s)$ for all $s\in[-\theta,0]$, and $\theta>0$ is the fixed maximal time-delay involved in the dynamics. The input u is assumed to be in $\mathcal{U}^m$, $m\in\mathbb{N}$. The vector field f is assumed to satisfy the following.

Assumption 1: The vector field $f:\mathcal{X}^n\times\mathbb{R}^m\to\mathbb{R}^n$

(i) is Lipschitz continuous in its first argument on bounded subsets of $\mathcal{X}^n\times\mathbb{R}^m$, i.e., for all $C>0$, there exists $L_f(C)>0$, such that for all $\phi,\varphi\in\mathcal{X}^n$ with $\|\phi\|\leq C$ and $\|\varphi\|\leq C$ and all $v\in\mathbb{R}^m$ with $|v|\leq C$,

$$|f(\phi,v)-f(\varphi,v)| \leq L_f(C)\|\phi-\varphi\|; \tag{2}$$

(ii) is continuous jointly in both arguments;

(iii) satisfies $f(0,0)=0$.

By [3, Theorem 2], Assumption 1 guarantees that, for any initial condition $x_0\in\mathcal{X}^n$ and any input $u\in\mathcal{U}^m$, there is a unique maximal solution (in Caratheodory sense) of (1), which we denote by $x(\cdot,x_0,u)$. Given t in the domain of existence of this solution, the corresponding history function is denoted by $x_t(x_0,u)\in\mathcal{X}^n$. The triple $(\mathcal{X}^n,\mathcal{U}^m,\varphi)$, where φ is the flow mapping (x_0,u) and t in the maximal interval of existence to $\varphi(t,x_0,u):=x_t(x_0,u)$ defines an abstract control system in the sense of [15]. In view of Assumption 1, the system (1) satisfies the boundedness-implies-continuation (BIC) property, i.e., every maximal solution that is bounded on its whole domain of existence is defined on $\mathbb{R}_+$, see [3, Theorem 2].

B. Definitions

In this section, we introduce a variant of ISS, specifically tailored to the analysis of delay systems using Lyapunov-Krasovskii functionals. To this aim, we first recall the definition of Lyapunov-Krasovskii functional candidates [3].

Definition 2.1 (LKF candidate): A map $V\in C(\mathcal{X}^n,\mathbb{R}_+)$ is called a *Lyapunov-Krasovskii functional candidate (LKF candidate)*, if there are $\psi_1,\psi_2\in\mathcal{K}_\infty$, so that

$$\psi_1(|\phi(0)|)\leq V(\phi)\leq\psi_2(\|\phi\|)\quad\forall\phi\in\mathcal{X}^n. \tag{3}$$

It is said to be *coercive* if, in addition,

$$\psi_1(\|\phi\|)\leq V(\phi)\leq\psi_2(\|\phi\|)\quad\forall\phi\in\mathcal{X}^n.$$

Remark 2.2: We note that in [3], it is required that LKF candidates are also Lipschitz continuous. For our results, we, however, do not need this extra assumption.

Next, we revisit the notion of ISS by estimating the system's properties through an associated LKF candidate, rather than through the standard norms.

Definition 2.3 (V-ISS / ISS): Given an LKF candidate $V:\mathcal{X}^n\to\mathbb{R}_+$, the system (1) is called *V-input-to-state stable (V-ISS)* if there exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$ such that, for all $x_0\in\mathcal{X}^n$ and all $u\in\mathcal{U}^m$,

$$V(x_t(x_0,u))\leq\beta(V(x_0),t)+\gamma(\|u\|),\quad\forall t\geq 0. \tag{4}$$

It is called *input-to-state stable (ISS)*, if it is V-ISS with $V(\phi)=\|\phi\|$.

It is worth noting that V-ISS may provide a tighter estimate on the solutions' norm. For instance, consider the following widely-used class of quadratic LKF candidates

$$V(\phi) := \phi(0)^\top P\phi(0)+\int_{-\theta}^0\phi(\tau)^\top Q\phi(\tau)d\tau,$$

where $P,Q\in\mathbb{R}^{n\times n}$ denote symmetric positive definite matrices. For such LKF candidates, V-ISS ensures an upper bound on the solution's norm in terms of $|x_0(0)|+\int_{-\theta}^0|x_0(\tau)|^2d\tau$, whereas the classical ISS would upper-bound them in terms of $\|x_0\|$, which may be significantly larger for some particular initial states.

The following statement clarifies the properties induced by V-ISS and relate it to the classical definition of ISS.

Proposition 2.4 (V-ISS $\Rightarrow$ ISS): Given a LKF candidate $V:\mathcal{X}^n\to\mathbb{R}_+$, consider the following statements:

i) System (1) is V-ISS.

ii) There exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$ such that, for all $x_0\in\mathcal{X}^n$ and all $u\in\mathcal{U}^m$, the flow of (1) satisfies

$$V(x_t(x_0,u))\leq\beta(\|x_0\|,t)+\gamma(\|u\|),\quad\forall t\geq 0. \tag{5}$$

iii) There exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$ such that, for all $x_0\in\mathcal{X}^n$ and all $u\in\mathcal{U}^m$, the flow of (1) satisfies

$$|x(t,x_0,u)|\leq\beta(\|x_0\|,t)+\gamma(\|u\|),\quad\forall t\geq 0. \tag{6}$$

iv) System (1) is ISS.

Then the following relations hold:

$$\text{i)}\Rightarrow\text{ii)}\ \Leftrightarrow\ \text{iii)}\ \Leftrightarrow\ \text{iv)}.$$

If V is coercive, then all four statements are equivalent.

Proof: i) $\Rightarrow$ ii) $\Rightarrow$ iii) follows easily from the fact that by definition V satisfies a sandwich condition as in (3). The fact that iv) $\Rightarrow$ ii) is also straightforward by noticing that $V(\phi)\leq\psi_2(\|\phi\|)$ for certain $\psi_2\in\mathcal{K}_\infty$ and for all $\phi\in\mathcal{X}^n$.

The equivalence between the items iii), iv) can be found, e.g., in [11, Proposition 1.4.2]. ■

To analyse the V-ISS property using ISS Lyapunov-Krasovskii functionals, we use the following notions of ISS LKF. They all rely on the *upper right-hand Dini derivative* of a map V along the solutions of system (1), defined for all $\phi \in \mathscr{X}^n$ and $u \in \mathscr{U}^m$ as

$$\dot{V}_u(\phi) := \limsup_{h \to 0^+} \frac{V(x_h(\phi,u)) - V(\phi)}{h}.$$

Definition 2.5 (Point-wise/LKF-wise ISS LKF): A LKF $V : \mathscr{X}^n \to \mathbb{R}_+$ is called:

- an *ISS LKF with LKF-wise dissipation in implication form* for the system (1) if there exist $\alpha, \chi \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,
$$V(\phi) \geq \chi(\|u\|) \quad \Rightarrow \quad \dot{V}_u(\phi) \leq -\alpha(V(\phi)). \quad (7)$$
- an *ISS LKF with point-wise dissipation in implication form* for the system (1) if there exist $\alpha, \chi \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,
$$|\phi(0)| \geq \chi(\|u\|) \quad \Rightarrow \quad \dot{V}_u(\phi) \leq -\alpha(|\phi(0)|). \quad (8)$$
- an *ISS LKF with point-wise dissipation in sum form* for the system (1) if there exist $\alpha, \chi \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,
$$\dot{V}_u(\phi) \leq -\alpha(|\phi(0)|) + \chi(\|u\|). \quad (9)$$

The following result states that any ISS LKF with point-wise dissipation sum form is also an ISS LKF with point-wise dissipation in implication form.

Proposition 2.6 (Sum form $\Rightarrow$ implication form): For system (1), if V is an ISS LKF with point-wise dissipation in sum form then it is also an ISS LKF with point-wise dissipation in implication form.

Proof: If V is an ISS LKF with point-wise dissipation in sum form, then (9) holds with some $\alpha, \chi \in \mathscr{K}_\infty$. Thus,

$$|\phi(0)| \geq \alpha^{-1} \circ 2\chi(\|u\|) \quad \Rightarrow \quad \dot{V}_u(\phi) \leq -\frac{1}{2}\alpha(|\phi(0)|),$$

and the claim follows. ■

Our main result will also exploit the following relaxation of the concept of ISS LKF in implication form.

Definition 2.7 (UGS LKF): A LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$ is called a *UGS LKF* for (1) if, for all $\phi \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,

$$V(\phi) \geq \chi(\|u\|) \quad \Rightarrow \quad \dot{V}_u(\phi) \leq 0. \quad (10)$$

As will be formalized in Section III-C, the existence of a UGS LKF ensures uniform global stability of solutions.

C. Main result

In [4], it has been conjectured that the existence of an ISS LKF with point-wise dissipation in sum form is enough to ensure ISS. In light of Proposition 2.6, this conjecture would be solved if we managed to show that the existence of an ISS LKF V with point-wise dissipation in implication form is enough to ensure ISS. To date, this conjecture remains open, but our main result states that ISS (and, actually, V-ISS) indeed holds if V is also a UGS LKF.

Theorem 2.8 (ISS under point-wise dissipation): Let Assumption 1 hold. If there exists a LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$ which is simultaneously an ISS LKF with point-wise dissipation (in either implication or sum form) and a UGS LKF for (1), then (1) is V-ISS and, in particular, ISS.

The proof of this result requires the introduction of further notions related to V-stability and related LKF tools. It is therefore postponed to Section IV.

Let us briefly discuss the novelty of Theorem 2.8. In [6], the authors have considered a variant of ISS LKF, which imposes the following implication:

$$V(\phi) \geq \gamma(\|u\|) \quad \Rightarrow \quad \dot{V}_u(\phi) \leq -\alpha(|\phi(0)|). \quad (11)$$

This condition lies halfway between (7) and (8), in the sense that the dissipation is requested in a point-wise manner but it needs to hold whenever the LKF qualitatively dominates the input norm. It has been shown in [6, Theorem 2] that it is sufficient to ensure ISS. This result can be seen as a corollary of Theorem 2.8 as (11) implies that V is both an ISS LKF with point-wise dissipation in implication form, and a UGS LKF. Our result therefore strengthens [6, Theorem 2] in three different ways. First, Theorem 2.8 ensures not merely ISS but also V-ISS, which is a potentially stronger property. Second, our requirements on V are also weaker than those in [6, Theorem 2], as V is requested to decay only when $|\phi(0)| \geq \gamma(\|u\|)$. Finally, our requirements on the nonlinearity f (Assumption 1) are weaker than those in [6, Theorem 2]. Namely, we do not assume Lipschitz continuity of f with respect to its second argument (the input u), which was important in [6].

The proof indicates that if the gain χ in (8) is identically zero, the gain γ in the V-ISS estimate (4) can also be picked null. This observation yields the following.

Corollary 2.9 (V-ISS with zero gain): If (1) admits an ISS LKF with point-wise dissipation (in implication or sum form) with gain $\chi \equiv 0$ (in (8) or (9)), then it is V-ISS with gain $\gamma \equiv 0$ in (4).

Proof: Checking the proof and Theorem 2.8 (and, in particular, the proof of Proposition 3.4 that we use there), we see that the system (1) satisfies the V-UGS property with 0 gain and V-ULIM property with 0 gain. This implies by arguments similar to those in [16, Theorem 2] that (1) is V-ISS with zero gain. ■

The property of ISS with zero gain is sometimes referred to as uniform global asymptotic stability [16], which finds its roots in the ISS literature on finite-dimensional systems and was instrumental for the derivation of converse Lyapunov results [10]. Corollary 2.9 thus extends the classical global asymptotic result by Krasovskii [5, Chapter 5, Theorem 2.1, p. 132] from input-free systems to systems with inputs, and we even obtain V-ISS with zero gain, in contrast to ISS with zero gain as claimed in the original statement.

III. V-STABILITY THEORY

On our way to proving Theorem 2.8, we develop the theory of V-stability notions, in which the solutions' behavior is not evaluated through the classical $\|\cdot\|$-norm, but rather through a particular LKF. This section aims to introduce other useful V-stability notions, to provide LKF conditions to establish them in practice, and, more importantly, to give a superposition theorem for V-ISS in this new setup.

A. Definitions

In the same way as the classical ISS notion can be extended to V-ISS, we can consider the following notions.

Definition 3.1 (V-stability notions): Given a LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$, the system (1) is called

- *V-uniformly globally stable (V-UGS)* if there exist $\sigma, \gamma \in \mathscr{K}_\infty$ such that, for all $x_0 \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,

$$V(x_t(x_0,u)) \leq \sigma(V(x_0)) + \gamma(\|u\|), \quad \forall t \geq 0. \tag{12}$$

- *V-uniform limit property (V-ULIM)* if there exists $\gamma \in \mathscr{K}_\infty \cup \{0\}$ so that, for every $\varepsilon, r > 0$, there exists a $\tau = \tau(\varepsilon, r) \geq 0$ such that, for all x_0 with $V(x_0) \leq r$ and all $u \in \mathscr{U}^m$ with $\|u\| \leq r$, there is a $t \in [0,\tau]$ such that

$$V(x_t(x_0,u)) \leq \varepsilon + \gamma(\|u\|). \tag{13}$$

The system (1) is called *UGS*, if it is V-UGS with $V(\phi) = \|\phi\|$, and similarly for *ULIM*.

The UGS property has already been used in the TDS literature [14], [18]. The ULIM notion shares some similarities with the more classical LIM property [22], [13], at the difference that the maximal time needed for (13) is here required to be uniform on bounded balls of both initial states and inputs. The following result can be shown analogously to Proposition 2.4.

Proposition 3.2 (V-UGS $\Rightarrow$ UGS): Given a LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$, if (1) is V-UGS, then it is UGS.

B. Superposition theorem for V-ISS

In the same way as their classical counterparts [14], [18], we can characterize V-ISS based on the combination of V-ULIM and V-UGS.

Theorem 3.3 (V-ISS superposition theorem): Given a LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$, the system (1) is V-ISS if and only if (1) is both V-UGS and V-ULIM.

If $V(x) = \|x\|$ for all $x \in \mathscr{X}^n$, Theorem 3.3 reduces to part of the ISS superposition theorem for general control systems proved in [15]. Although this result is instrumental for the proof of our main result, its proof is too long to be included in this paper and is thus omitted. The proof goes along the lines of the proof of the corresponding result in [15], and consists of several lemmas relating the ULIM and UGS notions with other central properties including the uniform asymptotic gain property.

C. Lyapunov-Krasovskii condition for V-UGS

The next result states that the existence of a UGS LKF V, as introduced in Definition 2.7, guarantees V-UGS.

Proposition 3.4 (LKF condition for V-UGS): If (1) admits a UGS LKF V then it is V-UGS (and thus UGS).

Proof: By assumption, there exists $\chi \in \mathscr{K}_\infty$ such that (10) holds for all $\phi \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$. Pick any $x_0 \in \mathscr{X}^n$ and any $u \in \mathscr{U}^m$. Then the maximal solution $x(\cdot, x_0, u)$ of (1) exists on some interval $[-\theta, t_m(x_0,u))$ with $t_m(x_0,u) \in (0,+\infty]$. We consider two cases, whether or not $V(x_0) \leq \chi(\|u\|)$. First let $V(x_0) \leq \chi(\|u\|)$. Seeking a contradiction, assume that there is a time $t_2 \in (0, t_m(x_0,u))$ such that $V(x_{t_2}(x_0,u)) > \chi(\|u\|)$. Let t_1 be maximal time $t \in [0,t_2)$ such that $V(x_t(x_0,u)) = \chi(\|u\|)$, which exists by continuity of solutions.

Due to the continuity of solutions, $V(x_t(x_0,u)) > \chi(\|u\|)$ for all $t \in (t_1,t_2)$, and hence it holds from (10) that

$$\dot{V}_{u(t+\cdot)}(x_t(x_0,u)) \leq 0, \quad \forall t \in (t_1,t_2),$$

and thus $V(x_t(x_0,u)) \leq V(x_{t_1}(x_0,u)) = \chi(\|u\|)$ for all $t \in (t_1,t_2)$, a contradiction. We conclude for the case $V(x_0) \leq \chi(\|u\|)$ that

$$V(x_t(x_0,u)) \leq \chi(\|u\|), \quad \forall t \in [0, t_m(x_0,u)). \tag{14}$$

We now proceed to the second case, namely when $V(x_0) > \chi(\|u\|)$. Then either $V(x_t(x_0,u)) > \chi(\|u\|)$ for all $t \in [0, t_m(x_0,u))$, or there is some minimal time $t_3 > 0$ so that $V(x_{t_3}(x_0,u)) = \chi(\|u\|) \leq V(x_0)$. Arguing as above, we see that in that case

$$V(x_t(x_0,u)) \leq V(x_0), \quad \forall t \in [0,t_3). \tag{15}$$

For $t > t_3$ we have by cocycle property and above arguments that for all $t \in [t_3, t_m(x_0,u))$

$$V(x_t(x_0,u)) = V(x_{t-t_3}(x_{t_3}(x_0,u), u(t_3+\cdot))) \leq \chi(\|u\|). \tag{16}$$

We conclude from (14) and (15) that, in all cases,

$$V(x_t(x_0,u)) \leq \max\{V(x_0), \chi(\|u\|)\}, \quad \forall t \in [0, t_m(x_0,u)). \tag{17}$$

Since (1) satisfies the BIC property (see, e.g., [3, Theorem 2]), (17) ensures that $t_m(x_0,u) = +\infty$, and thus this estimate holds for all $t \geq 0$, and V-UGS follows. UGS is then a consequence of Proposition 3.5. ■

D. Bounds on solutions' norm

We finally present some technical results providing bounds on the solutions (and on their derivative) of a V-UGS system.

While the definition of V-UGS provides an upper bound on $V(x_t)$, a bound on the whole history norm can be obtained after one full delay period.

Proposition 3.5 (Bound on history norm): Given a LKF candidate $V : \mathscr{X}^n \to \mathbb{R}_+$, if (1) is V-UGS then there are $\sigma, \gamma \in \mathscr{K}_\infty$ such that, for all $x_0 \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$,

$$\|x_t(x_0,u)\| \leq \sigma(V(x_0)) + \gamma(\|u\|), \quad \forall t \geq \theta. \tag{18}$$

Proof: Since V is a LKF candidate and (1) is V-UGS, there are $\psi_1, \tilde{\sigma}, \tilde{\gamma} \in \mathcal{K}_\infty$ so that, for all $x_0 \in \mathcal{X}^n$ and $u \in \mathcal{U}^m$,

$$\psi_1(|x(t,x_0,u)|) \le \tilde{\sigma}(V(x_0)) + \tilde{\gamma}(\|u\|), \quad \forall t \ge 0.$$

As $\psi_1^{-1}(a+b) \le \psi_1^{-1}(2a) + \psi_1^{-1}(2b)$ $\forall a,b \ge 0$, we have

$$|x(t,x_0,u)| \le \psi_1^{-1}\big(2\tilde{\sigma}(V(x_0))\big) + \psi_1^{-1}\big(2\tilde{\gamma}(\|u\|)\big).$$

Consequently, for all $t \ge \theta$,

$$\|x_t(x_0,u)\| = \max_{\tau\in[-\theta,0]} |x(t+\tau,x_0,u)| \le \sigma(V(x_0)) + \gamma(\|u\|),$$

with $\sigma := \psi_1^{-1} \circ 2\tilde{\sigma}$ and $\gamma := \psi_1^{-1} \circ 2\tilde{\gamma}$. ∎

V-UGS also provides a bound on the solutions' derivative after a full delay period. To establish this fact, we first make the following observation.

Proposition 3.6 (Bound on vector field): Under Assumption 1, there exist $\xi_1, \xi_2 \in \mathcal{K}_\infty$ such that

$$|f(\phi,v)| \le \xi_1(\|\phi\|) + \xi_2(|v|), \quad \forall \phi \in \mathcal{X}^n, v \in \mathbb{R}^m. \tag{19}$$

Proof: Pick any $\phi \in \mathcal{X}^n$ and any $v \in \mathbb{R}^m$. Due to Lipschitz continuity of f on bounded balls w.r.t. the first argument, there is a strictly increasing continuous function L (characterizing a Lipschitz constant of f), so that

$$\begin{aligned}|f(\phi,v)| &\le |f(0,v)| + |f(\phi,v) - f(0,v)| \\ &\le \xi(|v|) + L(\max\{\|\phi\|,|v|\})\|\phi\|,\end{aligned}$$

where $\xi(s) := \max_{|v|\le s} |f(0,v)|$ for all $s \ge 0$. As $f(0,0) = 0$ it holds that $\xi(0) = 0$. Since f is continuous in its second argument, it also holds that ξ is a continuous nondecreasing function: see [12, Lemma A.27], and hence it can be upperbounded by $\mathcal{K}_\infty$-function. Furthermore, we have

$$\begin{aligned}|f(\phi,v)| &\le \xi(|v|) + L(\max\{\|\phi\|,|v|\}) \max\{\|\phi\|,|v|\} \\ &\le \xi(|v|) + L(\|\phi\|)\|\phi\| + L(|v|)|v|.\end{aligned}$$

The function $L(\cdot)$ can be majorized by a continuous increasing function, and thus $r \mapsto L(r)r$ can be majorized by a $\mathcal{K}_\infty$ function. These considerations establish Proposition 3.6. ∎

Based on this, we have the following.

Lemma 3.7 (Bound on solutions' derivative): Given a LKF candidate $V : \mathcal{X}^n \to \mathbb{R}_+$, assume that (1) is V-UGS, and let Assumption 1 hold. Then there exist $\mu_1, \mu_2 \in \mathcal{K}_\infty$ so that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$,

$$|\dot{x}(t,x_0,u)| \le \mu_1(V(x_0)) + \mu_2(\|u\|), \quad \forall t \ge \theta \ a.e. \tag{20}$$

Proof: By Proposition 3.5, there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$,

$$\|x_t(x_0,u)\| \le \sigma_1(V(x_0)) + \sigma_2(\|u\|), \quad \forall t \ge \theta.$$

It follows from Proposition 3.6 that there exist $\xi_1, \xi_2 \in \mathcal{K}_\infty$ such that, for almost all $t \ge \theta$,

$$\begin{aligned}|\dot{x}(t,x_0,u)| &= \big|f(x_t(x_0,u),u(t))\big| \\ &\le \xi_1(\|x_t(x_0,u)\|) + \xi_2(|u(t)|) \\ &\le \xi_1\big(\sigma_1(V(x_0)) + \sigma_2(\|u\|)\big) + \xi_2(\|u\|) \\ &\le \xi_1\big(2\sigma_1(V(x_0))\big) + \xi_1\big(2\sigma_2(\|u\|)\big) + \xi_2(\|u\|)\end{aligned}$$

and Lemma 3.7 follows with $\mu_1 := \xi_1 \circ 2\sigma_1$ and $\mu_2 := \xi_1 \circ 2\sigma_2 + \xi_2$. ∎

IV. Proof of Theorem 2.8

Now we can establish our main result. In view of Proposition 2.6, it is enough to assume that V is both an ISS LKF with point-wise dissipation in implication form and a UGS LKF. The proof consists in exploiting the V-ISS superposition theorem (Theorem 3.3) and thus to show that (1) is both V-UGS and V-ULIM. The former is a direct consequence of Proposition 3.4. For the latter, seeking a contradiction, assume that the system (1) does not have V-ULIM property with some function τ to be defined later and with the gain $\gamma := \psi_2 \circ 2\chi$, where $\chi \in \mathcal{K}_\infty$ is a Lyapunov gain as in Definitions 2.7 and 2.5 (if the Lyapunov gains are different, we can define χ as the maximum of both) and $\psi_2 \in \mathcal{K}_\infty$ is an upper bound on V as in the sandwich condition (3). Hence, there are some $r, \varepsilon > 0$, some $x_0 \in \mathcal{X}^n$ with $V(x_0) \le r$, and some $u \in \mathcal{U}^m$ with $\|u\| \le r$ such that

$$V(x_t(x_0,u)) \ge \varepsilon + \gamma(\|u\|), \quad \forall t \in [0,\tau(r,\varepsilon)]. \tag{21}$$

By (3), it follows that

$$\psi_2(\|x_t(x_0,u)\|) \ge \varepsilon + \psi_2 \circ 2\chi(\|u\|), \quad \forall t \in [0,\tau(r,\varepsilon)], \tag{22}$$

which in turn implies that

$$\|x_t(x_0,u)\| \ge \max\big\{\psi_2^{-1}(\varepsilon), 2\chi(\|u\|)\big\}, \quad \forall t \in [0,\tau(r,\varepsilon)].$$

Hence, there exists an increasing finite sequence of time instants $t_k \in [0,\tau(r,\varepsilon)]$, $k \in \{0,1,\dots,K\}$, $K \in \mathbb{N}$, satisfying $t_k - t_{k-1} \le \theta$ for all such k, such that

$$|x(t_k,x_0,u)| \ge \max\big\{\psi_2^{-1}(\varepsilon), 2\chi(\|u\|)\big\}. \tag{23}$$

Note that

$$K \ge \frac{\tau(\varepsilon,r)}{\theta} - 1. \tag{24}$$

By Lemma 3.7, there exist $\mu_1, \mu_2 \in \mathcal{K}_\infty$ such that

$$|\dot{x}(t,x_0,u)| \le \mu_1(V(x_0)) + \mu_2(\|u\|) \le \mu(r), \quad \forall t \ge \theta \ a.e., \tag{25}$$

where $\mu := \mu_1 + \mu_2$. For each k, consider the interval

$$I_k := \left[t_k - \frac{\psi_2^{-1}(\varepsilon)}{2\mu(r)}, t_k + \frac{\psi_2^{-1}(\varepsilon)}{2\mu(r)}\right].$$

As ψ_2 is an upper bound for V, it can be chosen arbitrarily large. Thus, we can assume that $\psi_2(s) \ge s$ for all $s \ge 0$ and that ψ_2^{-1} is picked such that $\frac{\psi_2^{-1}(\varepsilon)}{\mu(r)} \le \theta$, so that the above intervals do not overlap. In view of (25), for all $k \in \{0,\dots,K\}$, we have for all $t \in I_k$ that

$$|x(t,x_0,u)| \ge \max\left\{\frac{\psi_2^{-1}(\varepsilon)}{2}, 2\chi(\|u\|) - \frac{\psi_2^{-1}(\varepsilon)}{2}\right\}.$$

Note that if $c > \max\{a, 2b-a\}$ for some $a,b,c \ge 0$, then $c > \max\{a,b\}$ (consider $a > b$ and $a \le b$). It follows that

$$|x(t,x_0,u)| \ge \max\left\{\frac{1}{2}\psi_2^{-1}(\varepsilon), \chi(\|u\|)\right\}, \quad \forall t \in I_k. \tag{26}$$

Since $\psi_2(s) \geq s$ for all $s \geq 0$, (21) ensures that

$$V(x_t(x_0,u)) \geq \chi(\|u\|), \quad \forall t \in [0, \tau(r,\varepsilon)].$$

Consequently, we get from (10) that

$$\dot{V}_{u(t+\cdot)}(x_t(x_0,u)) \leq 0, \quad \forall t \in [0, \tau(r,\varepsilon)]. \tag{27}$$

Using first [17, Lemma 3.4], and then (27), it follows that

$$\begin{aligned} V(x_{\tau(r,\varepsilon)}(x_0,u)) - V(x_0) &\leq \int_0^{\tau(r,\varepsilon)} \dot{V}_{u(t+\cdot)}(x_t(x_0,u))dt \\ &\leq \sum_{k=0}^{K} \int_{I_k} \dot{V}_{u(t+\cdot)}(x_t(x_0,u))dt. \end{aligned}$$

Using (8) and (26) on the intervals I_k, we get that

$$\begin{aligned} V(x_{\tau(r,\varepsilon)}(x_0,u)) - V(x_0) &\leq -\sum_{k=0}^{K} \int_{I_k} \alpha(|x(t,x_0,u)|)dt \\ &\leq -\sum_{k=0}^{K} \int_{I_k} \alpha \circ \frac{1}{2}\psi_2^{-1}(\varepsilon)dt \\ &\leq -(K+1)\frac{\psi_2^{-1}(\varepsilon)}{\mu(r)} \alpha \circ \frac{1}{2}\psi_2^{-1}(\varepsilon) \\ &\leq -\frac{\tau(r,\varepsilon)\psi_2^{-1}(\varepsilon)}{\theta\mu(r)} \alpha \circ \frac{1}{2}\psi_2^{-1}(\varepsilon), \end{aligned}$$

where the last inequality results from (24). This implies that

$$r \geq V(x_0) \geq \frac{\tau(r,\varepsilon)\psi_2^{-1}(\varepsilon)}{\theta\mu(r)} \alpha \circ \frac{1}{2}\psi_2^{-1}(\varepsilon). \tag{28}$$

For the particular choice

$$\tau(r,\varepsilon) := \frac{4r\theta\mu(r)}{\psi_2^{-1}(\varepsilon)\alpha \circ \frac{1}{2}\psi_2^{-1}(\varepsilon)}, \tag{29}$$

(28) yields a contradiction. Thus, (1) satisfies the V-ULIM estimate (13) with the function τ given in (29) and the gain $\gamma = \psi_2 \circ 2\chi$, which concludes the proof.

V. Conclusion and perspectives

We have demonstrated that ISS can be derived from an ISS LKF with point-wise dissipation, provided that the same LKF can be used to establish UGS. For this, we have relied on a superposition principle for a variant of ISS, in which solutions are estimated through the LKF rather than through the classical $\|\cdot\|$-norm of the state.

While our result relaxes the ISS conditions imposed in [6], it is still far from solving the original question posed in [4], namely whether a point-wise dissipation is enough to guarantee ISS. A potential next step in that direction would be to show that ISS indeed holds under a point-wise dissipation if the system is assumed to be UGS, thus without assuming a common LKF for both ISS and UGS.

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