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Higher-order retraction maps and construction of numerical methods for optimal control of mechanical systems

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***Abstract*— Retractions maps are used to define a discretization of the tangent bundle of the configuration manifold as two copies of the configuration manifold where the dynamics take place. Such discretization maps can be conveniently lifted to a higher-order tangent bundle to construct geometric integrators for the higher-order Euler-Lagrange equations. Given a cost function, an optimal control problem for fully actuated mechanical systems can be understood as a higher-order variational problem. In this paper we introduce the notion of a higher-order discretization map associated with a retraction map to construct geometric integrators for the optimal control of mechanical systems. In particular, we study applications to path planning for obstacle avoidance of a planar rigid body.**

I. INTRODUCTION

In this paper, we consider fully-actuated optimal control problems as higher-order variational problems (see [6] and [9]). Such problems are defined on the k^{th}-order tangent bundle $T^{(k)}Q$ of a differentiable manifold Q (see [17]). For a higher-order Lagrangian function $L : T^{(k)}Q \to \mathbb{R}$ and local coordinates $(q, \dot{q}, \dots, q^{(k)})$ on $T^{(k)}Q$ the higher-order variational problems are given by

$$\min_{q(\cdot)} \int_0^T L(q(t), \dot{q}(t), \dots, q^{(k)}(t))dt,$$

subject to the boundary conditions $q^{(j)}(0) = q_0^j$, $q^{(j)}(T) = q_T^j$ for $0 \leq j \leq k-1$, where $q^{(j)}(t) = \frac{d^j}{dt^j}q(t)$.

The relationship between higher-order variational problems and optimal control problems of fully-actuated mechanical systems comes from the fact that Euler-Lagrange equations are represented by a second-order Newtonian system and fully-actuated mechanical control systems have the form $F(q, \dot{q}, \ddot{q}) = u$, where u are the control inputs, as many as the dimension of the configuration manifold Q. If C is a cost function of an optimal control problem given by

$$\min_{(q(\cdot),u(\cdot))} \int_0^T C(q, \dot{q}, u)dt,$$

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it can be rewritten as a second-order variational problem replacing u by the above expression.

The notion of retraction map is an essential tool in different research areas like optimization theory, numerical analysis and interpolation (see [1] and references therein). A retraction map plays the role of generalizing the linear-search methods in Euclidean spaces to general manifolds. On a manifold with nonzero curvature to move along the tangent line does not guarantee that the motion stays on the manifold. The retraction map provides the tool to define the notion of moving in a direction of a tangent vector while staying on the manifold. That is why retraction maps have been widely used to construct numerical integrators of ordinary differential equations, since it allows us to move from a point and a velocity to one nearby point so that the differential equation can be discretized.

In [4] the classical notion of retraction map used to approximate geodesics is extended to the new notion of discretization maps, that is rigorously defined to become a powerful tool to construct geometric integrators. Using the geometry of the tangent and cotangent bundles, the authors were able to tangently and cotangent lift the map so that these lifts inherit the same properties as the original one and they continue to be discretization maps. In particular, the cotangent lift of a discretization map is a natural symplectomorphism, which plays a key role for constructing symplectic integrators. It was further applied in [5] to the construction of numerical methods for optimal control problems from a Hamiltonian perspective.

Geometric integrators for optimal control problems seen as second-order variational problems were studied in [13] (see also [14], [15]). The goal of this paper is to extend the notion of discretization map given in [4] to higher-order tangent bundles and, at the same time, to construct symplectic integrators for optimal control problems of fully-actuated mechanical systems. The results in this paper are a demonstration of the use of higher-order retraction maps in optimal control problems. In future research, we will show how retraction maps may be leveraged to deal with optimal control problems on Lie groups and homogeneous spaces, two of the most common situations in applications to robotics. Thus, the present paper lays the foundations of a more general program to construct geometric integrators for optimal control problems in non-linear manifolds.

The paper is structured as follows. Section II introduces the necessary tools on differential geometry and the geometric formalism for the dynamics of mechanical systems. Section III describes optimal control problems as higher-order variational problems and the Lagrangian and Hamiltonian characterization of necessary conditions for optimality. In Section IV we introduce retraction maps and discretization maps as well as the cotangent lift of discretization maps which allows the construction of symplectic integrators. In Section V we define higher-order discretization maps and describe the construction of symplectic integrators for higher-order mechanical systems. We employ this construction in Section VI to construct geometric integrators for optimal control of mechanical systems. In particular, we study applications to path planning for obstacle avoidance of planar rigid bodies.

II. Background on Geometric Mechanics

Let Q be a n-dimensional differentiable configuration manifold of a mechanical system with local coordinates (q^A), $1 \leq A \leq n$. Denote by TQ the tangent bundle (see, for instance, [27] for an introduction to the tangent bundle and mechanics on it). If T_qQ denotes the tangent space of Q at the point q, then $TQ := \cup_{q\in Q} T_qQ$, with induced local coordinates $(q^A, \dot{q}^A)$. There is a canonical projection $\tau_Q : TQ \to Q$, sending each vector v_q to the corresponding base point q. Note that in coordinates $\tau_Q(q^A, \dot{q}^A) = q^A$.

The vector space structure of T_qQ makes possible to consider its dual space, T_q^*Q, to define the cotangent bundle as $T^*Q := \cup_{q\in Q} T_q^*Q$, with local coordinates (q^A, p_A). There is a canonical projection $\pi_Q : T^*Q \to Q$, sending each momenta p_q to the corresponding base point q. Note that in coordinates $\pi_Q(q^A, p_A) = q^A$.

Given a Lagrangian function $L : TQ \to \mathbb{R}$, the corresponding Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0, \quad 1 \leq A \leq n. \tag{1}$$

Equations (1) determine a system of n second-order differential equations. If we assume that the Lagrangian is regular, i.e., the $(n \times n)$-matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$, $1 \leq A, B \leq n$, is non-singular, the local existence and uniqueness of solutions are guaranteed for any given initial condition by employing the Implicit Function Theorem.

A Hamiltonian function $H : T^*Q \to \mathbb{R}$ is described by the total energy of a mechanical system and leads to Hamilton's equations on T^*Q, whose solutions are integral curves of the Hamiltonian vector field X_H taking values in $T(T^*Q)$ associated with H. Locally, $X_H(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$, that is,

$$\dot{q}^A = \frac{\partial H}{\partial p_A}, \quad \dot{p}_A = -\frac{\partial H}{\partial q^A}, \quad 1 \leq A \leq n. \tag{2}$$

Equations (2) determine a set of $2n$ first order ordinary differential equations (see [6], for instance, for more details).

A one-form α on Q is a map assigning to each point q a cotangent vector on q, that is, $\alpha(q) \in T_q^*Q$. Cotangent vectors acts linearly on vector fields according to $\alpha(X) = \alpha_i X^i \in \mathbb{R}$ if $\alpha = \alpha_i dq^i$ and $X = X^i \frac{\partial}{\partial q^i}$. Analogously, a two-form or a $(0,2)$-tensor field is a bilinear map that acts on a pair of vector fields to produce a number.

A symplectic form ω on a manifold Q is a $(0,2)$-type tensor field that is skew-symmetric and non-degenerate, i.e., $\omega(X,Y) = -\omega(Y,X)$ for all vector fields X and Y and if $\omega(X,Y) = 0$ for all vector fields X, then $Y \equiv 0$.

The set of vector fields and the set of 1-forms on Q are denoted by $\mathfrak{X}(Q)$ and $\Omega^1(Q)$, respectively. The symplectic form induces a linear isomorphism $\flat_\omega : \mathfrak{X}(Q) \to \Omega^1(Q)$, given by $\langle \flat_\omega(X), Y\rangle = \omega(X,Y)$ for any vector fields X, Y. The inverse of $\flat_\omega$ will be denoted by $\sharp_\omega$.

As described in [24], the cotangent bundle T^*Q of a differentiable manifold Q is equipped with a canonical exact symplectic structure $\omega_Q = -d\theta_Q$, where θ_Q is the canonical 1-form on T^*Q. In canonical bundle coordinates (q^A, p_A) on T^*Q, $\theta_Q = p_A\, \mathrm{d}q^A$ and $\omega_Q = \mathrm{d}q^A \wedge \mathrm{d}p_A$. Hamilton's equations can be intrinsically rewritten as $\imath_{X_H}\omega_Q : = \flat_\omega(X_H) = \mathrm{d}H$. Hamiltonian dynamics are characterized by the following two essential properties [20]:

- Preservation of energy by the Hamiltonian function:
$$0 = \omega_Q(X_H, X_H) = dH(X_H) = X_H(H)\,.$$
- Preservation of the symplectic form: If $\{\phi^t_{X_H}\}$ is the flow of X_H, then the pull-back of the differential form by the flow is preserved, $(\phi^t_{X_H})^*\omega_Q = \omega_Q$.

Recall that a pair (Q, ω_Q) is called a symplectic manifold if Q is a differentiable manifold and ω_Q is a symplectic 2-form. As a consequence, the restrictions of ω_Q to each $q \in Q$ makes the tangent space T_qQ into a symplectic vector space.

Definition 1: Let (Q_1, ω_1) and (Q_2, ω_2) be two symplectic manifolds, let $\phi : Q_1 \to Q_2$ be a smooth map. The map ϕ is called symplectic if the symplectic forms are preserved: $\phi^*\omega_2 = \omega_1$. Moreover, it is a *symplectomorphism* if ϕ is a diffeomorphism and ϕ^{-1} is also symplectic.

Let Q_1 and Q_2 be n-dimensional manifolds and $F : Q_1 \to Q_2$ be a smooth map. The *tangent lift* $TF : TQ_1 \to TQ_2$ of F is defined by $TF(v_q) = T_qF(v_q) \in T_{F(q)}Q_2$ where $v_q \in T_qQ_1$, and T_qF is the tangent map of F whose matrix is the Jacobian matrix of F at $q \in Q_1$.

As the tangent map T_qF is linear, the dual map $T_q^*F\colon T^*_{F(q)}Q_2 \to T_q^*Q_1$ is defined as follows:

$$\langle (T_q^*F)(\alpha_2), v_q\rangle = \langle \alpha_2, T_qF(v_q)\rangle \text{ for every } v_q \in T_qQ_1.$$

Note that $(T_q^*F)(\alpha_2) \in T_q^*Q_1$.

Definition 2: Let $F : Q_1 \to Q_2$ be a diffeomorphism. The vector bundle morphism $\widehat{F} : T^*Q_1 \to T^*Q_2$ defined by $\widehat{F} = T^*F^{-1}$ is called the cotangent lift of F^{-1}.
In other words, $\widehat{F}(\alpha_q) = T^*_{F(q)}F^{-1}(\alpha_q)$ where $\alpha_q \in T_q^*Q_1$. Obviously, $(T^*F^{-1}) \circ (T^*F) = \mathrm{Id}_{T^*Q_2}$.

A. Higher-order tangent bundles

The higher-order tangent bundle is essentially a generalization of the tangent space of the manifold Q to higher-order derivatives, when one interprets tangent vectors as the velocity vector of some curve in Q. Analogously, an element of the k-th order tangent bundle can be defined as an equivalence relation identifying all curves that match up to k-th order derivative.

Let $c_1, c_2 : \mathbb{R} \to Q$ be two curves on Q. Consider the equivalence relation $\sim_k$ at $0 \in \mathbb{R}$ determined by the following two conditions:

1) $c_1(0) = c_2(0)$;
2) $c_1^{(i)}(0) = c_2^{(i)}(0)$ for all $1 \leqslant i \leqslant k$, where the notation $c^{(i)}$ represents the i-th derivative of c.

In this case, we say that c_1 and c_2 are $\sim_k$-related at 0. Moreover, the equivalence class of c determined by $\sim_k$ is called the k-jet of c and is represented by $j_0^{(k)}c$. The set of all k-jets at 0 is denoted by $J_0^{(k)}(\mathbb{R}, Q)$ in some general contexts. But, from now on, it will be denoted by $T^{(k)}Q$, the k-th order bundle of Q. The k-th order bundle of Q is a smooth manifold (see [17]) and admits several fibrations: $\pi_r^k : T^{(k)}Q \to T^{(r)}Q$ mapping $j_0^{(k)}c \mapsto j_0^{(r)}c$ for $0 \leqslant r < k$. Observe that for $r = 1$, $T^{(1)}Q = TQ$ and for $r = 0$, $T^{(0)}Q = Q$.

If (q^A) are local coordinates on the manifold Q, then the k-jet $j_0^{(k)}c$ is uniquely determined by the coordinates $(q^A, q^{(0)^A}, \ldots, q^{(k)^A})$, where

$$q^A = c^A(0), \quad q^{(r)^A} = \frac{1}{r!}c^{(r)^A}(0), \quad 1 \leqslant A \leqslant \dim Q.$$

In a sense, the local coordinates for k-jets are provided by the Taylor polynomial of c at 0.

Given a smooth map $F : Q_1 \to Q_2$, we define $T^{(k)}F : T^{(k)}Q_1 \to T^{(k)}Q_2$ by $T^{(k)}F(j_0^{(k)}c) = j_0^{(k)}(F \circ c)$, for some curve $c : \mathbb{R} \to Q_1$.

III. Variational Formulation of Optimal Control Problems for Mechanical Systems

There are some problems in which the functional to be minimized depends on higher-order derivatives of a curve. This is the case in interpolating problems [21], [31]; in generation of trajectories for quadrotors [28], [26], or in a generalization of least square problem on Riemannian manifolds [25].

The goal of this paper is to use discretization maps obtained from the retraction maps to produce numerical algorithms for the solutions of optimal control problems for fully actuated mechanical systems. The prototype problem in this paper is the optimization of the cost functional

$$\mathcal{J} = \int_0^T ||u||^2 \, dt$$

subjected to controlled Euler-Lagrange equations describing the dynamics of standard mechanical systems, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = u.$$

The cost functional may then be recast as the second-order functional

$$\mathcal{J} = \int_0^T \left|\left| \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \right|\right|^2 \, dt = \int_0^T \mathcal{L}(q, \dot{q}, \ddot{q}) \, dt.$$

Using the variational principle, necessary equations for a trajectory to be optimal are the second order Euler-Lagrange equations:

$$\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial q} = 0\,.$$

We want to use the results in [4] (see also [5]) to produce geometric numerical methods for the optimal control problem under study. But first, we need to get these results generalized for the higher-order tangent bundles. as well as to study the Hamiltonian version of optimality conditions.

A second-order Lagrangian $\mathcal{L}$ can be associated with a Lagrangian energy $E_{\mathcal{L}} : T^{(3)}Q \to \mathbb{R}$ defined by

$$E_{\mathcal{L}}(q, \dot{q}, \ddot{q}, q^{(3)}) = \dot{q}\hat{p}_{(0)} + \ddot{q}\hat{p}_{(1)} - L(q, \dot{q}, \ddot{q}),$$

where $\hat{p}_{(0)}$ and $\hat{p}_{(1)}$ are the generalized momenta given by

$$\hat{p}_{(0)} = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \;\; \hat{p}_{(1)} = \frac{\partial L}{\partial \ddot{q}}\,.$$

These momenta are conserved along solutions of the second-order Euler-Lagrange equations (see [17] for instance).

As usual the link between Lagrangian and Hamiltonian formalism is the corresponding Legendre transformation $\mathrm{Leg}_{\mathcal{L}} : T^{(3)}Q \to T^*(TQ)$ given by

$$\mathrm{Leg}_{\mathcal{L}}(q, \dot{q}, \ddot{q}, q^{(3)}) = (q, \dot{q}, \hat{p}_{(0)}, \hat{p}_{(1)})\,.$$

The associated Hamiltonian function $H : T^*(TQ) \to \mathbb{R}$ is given by

$$H(q, \dot{q}, \hat{p}_{(0)}, \hat{p}_{(1)}) = E_{\mathcal{L}} \circ \mathrm{Leg}_{\mathcal{L}}^{-1}(q, \dot{q}, \hat{p}_{(0)}, \hat{p}_{(1)}),$$

and the second-order Hamitlon equations are given by

$$\dot{q} = \frac{\partial H}{\partial \hat{p}_{(0)}}, \quad \ddot{q} = \frac{\partial H}{\partial \hat{p}_{(1)}}, \quad \dot{\hat{p}}_{(0)} = -\frac{\partial H}{\partial q}, \quad \dot{\hat{p}}_{(1)} = -\frac{\partial H}{\partial \dot{q}}.$$

IV. Discretization Maps

The first notion of retraction appearing in the literature can be found in [10] from a topological viewpoint. Later on, the notion of retraction map as defined below is used to obtain Newton's method on Riemannian manifolds [30], [3].

Definition 3: A *retraction map* on a manifold Q is a smooth mapping R from the tangent bundle TQ onto Q. Let R_q denote the restriction of R to T_qQ, the following properties are satisfied:

1) $R_q(0_q) = q$, where 0_q denotes the zero element of the vector space T_qQ.
2) With the canonical identification $T_{0_q}T_qQ \simeq T_qQ$, R_q satisfies
$$\mathrm{D}R_q(0_q) = T_{0_q}R_q = \mathrm{Id}_{T_qQ}, \qquad (3)$$
where Id_{T_qQ} denotes the identity mapping on T_qQ.

The condition (3) is known as *local rigidity condition* since, given $\xi \in T_qQ$, the curve $\gamma_\xi(t) = R_q(t\xi)$ has ξ as tangent vector at q, i.e. $\dot{\gamma}_\xi(t) = \langle DR_q(t\xi), \xi\rangle$ and, in consequence, $\dot{\gamma}_\xi(0) = \mathrm{Id}_{T_qQ}(\xi) = \xi$.

A typical example of a retraction map is the exponential map, exp, on Riemannian manifolds given in [18, Chapter 3.2]. Therefore, the image of ξ through the exponential map is a point on the Riemannian manifold (Q, g) obtained by moving along a geodesic a length equal to the norm of ξ starting with the velocity $\xi/\|\xi\|$, that is,

$$\exp_q(\xi) = \sigma(\|\xi\|) \ ,$$

where σ is the unit speed geodesic such that $\sigma(0) = q$ and $\dot{\sigma}(0) = \xi/\|\xi\|$.

Next, we define a generalization of the retraction map in Definition 3 that allows a discretization of the tangent bundle of the configuration manifold leading to the construction of numerical integrators as described in [4]. Given a point and a velocity, we obtain two nearby points that are not necessarily equal to the initial base point.

Definition 4: A map $R_d \colon U \subset TQ \to Q \times Q$ given by

$$R_d(q, v) = (R^1(q, v), R^2(q, v)),$$

where U is an open neighborhood of the zero section 0_q of TQ, defines a *discretization map on* Q if it satisfies

1) $R_d(q, 0) = (q, q)$,
2) $T_{0_q} R^2_q - T_{0_q} R^1_q \colon T_{0_q} T_q Q \simeq T_q Q \to T_q Q$ is equal to the identity map on $T_q Q$ for any q in Q, where R^a_q denotes the restrictions of R^a, $a = 1, 2$, to $T_q Q$.

Thus, the discretization map R_d is a local diffeomorphism from some neighborhood of the zero section of TQ.

If $R^1(q, v) = q$, the two properties in Definition 4 guarantee that the both properties in Definition 3 are satisfied by R^2. Thus, Definition 4 generalizes Definition 3.

Example 1: The mid-point rule on an Euclidean vector space can be recovered from the following discretization map: $R_d(q, v) = \left(q - \dfrac{v}{2}, q + \dfrac{v}{2}\right)$.

A. Cotangent lift of discretization maps

As the Hamiltonian vector field takes value on TT^*Q, the discretization map must be on T^*Q, that is, $R_d^{T^*} : TT^*Q \to T^*Q \times T^*Q$. Such a map is obtained by cotangently lifting a discretization map $R_d \colon TQ \to Q \times Q$, so that the construction $R_d^{T^*}$ is a symplectomorphism. In order to do that, we need the following three symplectomorphisms (see [4] and [5] for more details):

- The cotangent lift of the diffeomorphism $R_d \colon TQ \to Q \times Q$ as described in Definition 2.
- The canonical symplectomorphism:

$$\alpha_Q \colon T^*TQ \longrightarrow TT^*Q$$

 such that $\alpha_Q(q, v, p_q, p_v) = (q, p_v, v, p_q)$.

- The symplectomorphism between $(T^*(Q \times Q), \omega_{Q\times Q})$ and $(T^*Q \times T^*Q, \Omega_{12} := pr_2^*\omega_Q - pr_1^*\omega_Q)$:

$$\Phi : T^*Q \times T^*Q \longrightarrow T^*(Q \times Q) \ ,$$

 given by $\Phi(q_0, p_0; q_1, p_1) = (q_0, q_1, -p_0, p_1)$.

Diagram in Fig. 1 summarizes the construction procress from R_d to $R_d^{T^*}$:

Proposition 1: [4] Let $R_d \colon TQ \to Q \times Q$ be a discretization map on Q. Then

$$R_d^{T^*} = \Phi^{-1} \circ \widehat{R_d} \circ \alpha_Q \colon TT^*Q \to T^*Q \times T^*Q$$

is a discretization map on T^*Q.

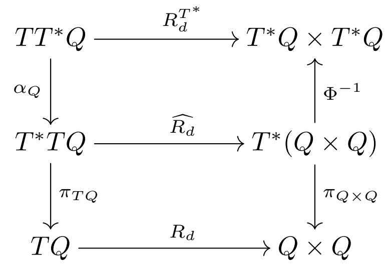

Fig. 1: Definition of the cotangent lift of a discretization.

Corollary 1: [4] The discretization map $R_d^{T^*} = \Phi^{-1} \circ (TR_d^{-1})^* \circ \alpha_Q \colon T(T^*Q) \to T^*Q \times T^*Q$ is a symplectomorphism between $(T(T^*Q), \mathrm{d}_T\omega_Q)$ and $(T^*Q \times T^*Q, \Omega_{12})$.

Example 2: On $Q = \mathbb{R}^n$ the discretization map $R_d(q, v) = \left(q - \frac{1}{2}v, q + \frac{1}{2}v\right)$ is cotangently lifted to

$$R_d^{T^*}(q, p, \dot{q}, \dot{p}) = \left(q - \frac{1}{2}\,\dot{q}, p - \frac{\dot{p}}{2};\ q + \frac{1}{2}\,\dot{q}, p + \frac{\dot{p}}{2}\right) .$$

V. Higher-order discretization maps

In [4], the authors show how to lift a discretization map to the tangent and cotangent bundles. Next, we are going to see how to lift a discretization map to a one on a higher-order tangent bundle.

Let $R_d : TQ \to Q \times Q$ be a discretization map on Q, then we can lift it to the map

$$T^{(k)}R_d : T^{(k)}(TQ) \to T^{(k)}Q \times T^{(k)}Q,$$

defined by $T^{(k)}R_d(j_0^{(k)}\gamma) = j_0^{(k)}(R_d \circ \gamma)$ for $\gamma : I \to TQ$.

Consider the natural equivalence $\Phi^{(k)} : T(T^{(k)}Q) \to T^{(k)}(TQ)$ defined using the following construction (see [23] or [12, Sec. V]): for each $X \in T(T^{(k)}Q)$ there exists a curve $c : \mathbb{R} \to Q$ such that $X = j_0^{(1)}(j_0^{(k)}c)$. Then, we have that

$$\Phi^{(k)}(X) = j_0^{(k)}(j_0^{(1)}c).$$

The identification between the higher-order tangent bundles $T^{(k)}(TQ) \cong T(T^{(k)}Q)$ allows to define the map $R_d^{(k)} : T(T^{(k)}Q) \to T^{(k)}Q \times T^{(k)}Q$ given by $R_d^{(k)} = T^{(k)}R_d \circ \Phi^{(k)}$.

The following lemma will be useful in the proof of the Theorem below.

Lemma 1: Let $F : M \to N$ be a smooth map and $\gamma_t : \mathbb{R} \to M$ a smooth family of maps, i.e., $\gamma : \mathbb{R}^2 \to M$ defined by $\gamma(t, s) = \gamma_t(s)$ is a smooth map. Then,

$$\left.\frac{d}{dt}\right|_{t=0} j_0^{(k)}(F \circ \gamma_t) = (\Phi_N^{(k)})^{-1} j_0^{(k)} \left(\left.\frac{d}{dt}\right|_{t=0} (F \circ \gamma_t) \right)$$

where $\Phi_N^{(k)} : T(T^{(k)}N) \to T^{(k)}(TN)$ is the canonical identification.

Proof: As

$$\left.\frac{d}{dt}\right|_{t=0} j_0^{(k)}(F \circ \gamma_t) = j_0^{(1)}(j_0^{(k)}(F \circ \gamma_t)) \, ,$$

using the natural equivalence $\Phi_N^{(k)}$

$$\left.\frac{d}{dt}\right|_{t=0} j_0^{(k)}(F \circ \gamma_t) = (\Phi_N^{(k)})^{-1}\left(j_0^{(k)}(j_0^{(1)}(F \circ \gamma_t))\right),$$

the result follows. ■

Now, we can prove that the map $R_d^{(k)}$ is a discretization map on the higher-order bundle $T^{(k)}Q$.

Theorem 1: Let R_d be a discretization map on Q, the lift to the higher-order tangent bundle $R_d^{(k)}: T(T^{(k)}Q) \to T^{(k)}Q \times T^{(k)}Q$ is a discretization map on $T^{(k)}Q$.

Proof: Let $\Phi^{(k)} : T(T^{(k)}Q) \to T^{(k)}TQ$ be the diffeomorphism identifying both manifolds.

First, we shall prove that given $z \in T^{(k)}Q$, we have that $R_d^{(k)}(0_z) = (z, z)$, where 0_z is the zero section of the bundle $T(T^{(k)}Q) \to T^{(k)}Q$.

The image of the zero section under $\Phi^{(k)}$ is the k-th jet lift of the zero section on Q, that is, $\Phi^{(k)}(0_z) = T^{(k)}\hat{0}(z)$, where $\hat{0} : Q \to TQ$, as it is easily checked choosing natural coordinates on the higher-order tangent bundle. Thus, $R_d^{(k)}(0_z) = T^{(k)}R_d(j^{(k)}\hat{0}(z))$.

Using the definition of the k-th jet lift

$$T^{(k)}R_d(j^{(k)}\hat{0}(z)) = j_0^{(k)}(R_d \circ \hat{0})(z).$$

In addition, since R_d is a discretization map, we have that $R_d \circ \hat{0} = \mathrm{Id}_Q \times \mathrm{Id}_Q$. Hence,

$$\begin{aligned} R_d^{(k)}(0_z) &= T^{(k)}(\mathrm{Id}_Q \times \mathrm{Id}_Q)(z) \\ &= (\mathrm{Id}_{T^{(k)}Q} \times \mathrm{Id}_{T^{(k)}Q})(z) = (z, z). \end{aligned}$$

Next, let $R_{d,z}^{(k)}$ be the restriction of $R_d^{(k)}$ to the space $T_z(T^{(k)}Q)$, where $z \in T^{(k)}Q$. We can write $R_{d,z}^{(k)} = T^{(k)}R_d \circ \Phi_z^{(k)}$, where $\Phi_z^{(k)}$ is the restriction of $\Phi^{(k)}$ to $T_z(T^{(k)}Q)$.

Moreover, if $R_{d,z}^{(k),a}$ denotes the composition of $R_{d,z}^{(k)}$ with the projection onto the ath-factor, $a = 1, 2$, then we will prove that $T_{0_z}R_{d,z}^{(k),2}(X_z) - T_{0_z}R_{d,z}^{(k),1}(X_z) = X_z$ for all $X_z \in T_z(T^{(k)}Q)$ and $z \in T^{(k)}Q$ under the identification $T_{0_z}T_z(T^{(k)}Q) \equiv T_z(T^{(k)}Q)$. We have that

$$\left(R_{d,z}^{(k),2} - R_{d,z}^{(k),1}\right)(X_z) = \left(T^{(k)}R_d^2 - T^{(k)}R_d^1\right) \circ \Phi_z^{(k)}(X_z).$$

Therefore,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} &\left(R_{d,z}^{(k),2} - R_{d,z}^{(k),1}\right)(tX_z) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(T^{(k)}R_d^2 - T^{(k)}R_d^1\right) \circ \Phi_z^{(k)}(tX_z) \\ &= \left.\frac{d}{dt}\right|_{t=0} T^{(k)}\left(R_d^2 - R_d^1\right) \circ \Phi_z^{(k)}(tX_z) \\ &= \left.\frac{d}{dt}\right|_{t=0} j_0^{(k)}\left((R_d^2 - R_d^1)(tY(q))\right), \end{aligned}$$

where $q = \pi_0^k(z)$, $Y(q) \in TQ$ is a curve such that $j_0^{(k)}Y = \Phi_z^{(k)}(X_z)$. Moreover, if $\tilde{\pi}_0^k : T^{(k)}TQ \to TQ$ and using $\tau_Q \circ T\pi_0^k = \pi_0^k \circ \tau_{T^{(k)}Q}$, then $Y(q) \in T_qQ$ and $Y = \tilde{\pi}_0^k(\Phi^{(k)}(X_z)) = T\pi_0^k(X_z)$ because the diagram in Fig. 2 is commutative.

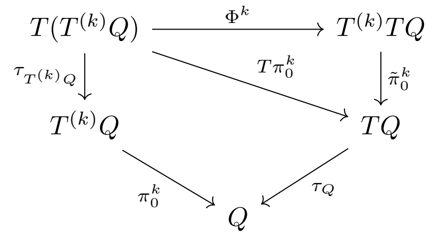

Fig. 2: Commutative diagram.

Using Lemma 1 we have that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} &\left(R_{d,z}^{(k),2} - R_{d,z}^{(k),1}\right)(tX_z) \\ &= (\Phi_z^k)^{-1} j_0^k \left(\left.\frac{d}{dt}\right|_{t=0} (R_{d,q}^2 - R_{d,q}^1)(tY(q)) \right). \end{aligned}$$

Using the second property from discretization maps, we obtain

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} &\left(R_{d,z}^{(k),2} - R_{d,z}^{(k),1}\right)(tX_z) \\ &= (\Phi_z^k)^{-1} j_0^k(Y) = X_z, \end{aligned}$$

where the last step follows from the definition of Y. ■

Example 3: Consider the midpoint discretization map

$$R_d(q, v) = \left(q - \frac{1}{2}v, q + \frac{1}{2}v\right).$$

The lift of the midpoint to $T(TQ)$ is

$$TR_d(q, v, \dot{q}, \dot{v}) = \left(q - \frac{1}{2}v, q + \frac{1}{2}v, \dot{q} - \frac{1}{2}\dot{v}, \dot{q} + \frac{1}{2}\dot{v}\right)$$

and the second lift to $T^{(2)}(TQ)$ is

$$\begin{gathered} T^{(2)}R_d(q, v; \dot{q}, \dot{v}; \ddot{q}, \ddot{v}) = \\ \left(q - \frac{1}{2}v, q + \frac{1}{2}v, \dot{q} - \frac{1}{2}\dot{v}, \dot{q} + \frac{1}{2}\dot{v}, \ddot{q} - \frac{1}{2}\ddot{v}, \ddot{q} + \frac{1}{2}\ddot{v}\right) \end{gathered}$$

Under the natural equivalence between higher-order tangent bundles, the map $R_d^{(2)} : T(T^{(2)}Q) \to T^{(2)}Q \times T^{(2)}Q$ is given by

$$\begin{gathered} R_d^{(2)}(q, \dot{q}, \ddot{q}; v, \dot{v}, \ddot{v}) = \\ \left(q - \frac{1}{2}v, \dot{q} - \frac{1}{2}\dot{v}, \ddot{q} - \frac{1}{2}\ddot{v}; q + \frac{1}{2}v, \dot{q} + \frac{1}{2}\dot{v}, \ddot{q} + \frac{1}{2}\ddot{v}\right). \end{gathered}$$

Then $R_d^{(2)}(q, \dot{q}, \ddot{q}; 0, 0, 0) = (q, \dot{q}, \ddot{q}; q, \dot{q}, \ddot{q})$,

$$T_{0_{(q,\dot{q},\ddot{q})}} R_{d,(q,\dot{q},\ddot{q})}^{(2),2} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

$$T_{0_{(q,\dot{q},\ddot{q})}} R_{d,(q,\dot{q},\ddot{q})}^{(2),1} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

Therefore, $T_{0_{(q,\dot{q},\ddot{q})}} R_{d,(q,\dot{q},\ddot{q})}^{(2),2} - T_{0_{(q,\dot{q},\ddot{q})}} R_{d,(q,\dot{q},\ddot{q})}^{(2),1} = Id$, and $R_d^{(2)}$ is a discretization map under the suitable identifications.

Example 4: Consider the initial point discretization map on the sphere $R_d : T\mathbb{S}^2 \to \mathbb{S}^2 \times \mathbb{S}^2$

$$R_d(q, \xi) = \left(q, \frac{q+\xi}{\|q+\xi\|}\right).$$

The lift to $T(T\mathbb{S}^2)$ is the map $TR_d \colon T(T\mathbb{S}^2) \to T\mathbb{S}^2 \times T\mathbb{S}^2$:

$$TR_d(q,\xi,\dot{q},\dot{\xi}) = \left(q, \dot{q}, \frac{q+\xi}{\|q+\xi\|}, \frac{\dot{q}+\dot{\xi}}{\|q+\xi\|} - \frac{\xi\cdot\dot{\xi}(q+\xi)}{\|q+\xi\|^3}\right)$$

and the second lift $T^{(2)}(T\mathbb{S}^2)$ is

$$\begin{aligned}T^{(2)}R_d(q,\xi,\dot{q},\dot{\xi},\ddot{q},\ddot{\xi}) = \Bigg(& TR_d(q,\xi,\dot{q},\dot{\xi}), \ddot{q}, \frac{\ddot{q}+\ddot{\xi}}{\|q+\xi\|} \\ & - \frac{2\xi\cdot\dot{\xi}(\dot{q}+\dot{\xi}) + (\dot{\xi}\cdot\dot{\xi} + \xi\cdot\ddot{\xi})(q+\xi)}{\|q+\xi\|^3} \\ & + \frac{3\xi\cdot\dot{\xi}(q+\xi)}{\|q+\xi\|^5}\Bigg).\end{aligned}$$

Composing with the natural identifications, we obtain a discretization map on $T^{(2)}\mathbb{S}^2$.

Corollary 2: Let $R_d^{(k)} : T(T^{(k)}Q) \to T^{(k)}Q \times T^{(k)}Q$ be a higher-order discretization map on $T^{(k)}Q$. The cotangent lift $\left(R_d^{(k)}\right)^{T^*} : T(T^*(T^{(k)}Q)) \to T^*(T^{(k)}Q) \times T^*(T^{(k)}Q)$ is discretization map on $T^*(T^{(k)}Q)$.

Proof: In Fig. 1 the discretization map at the bottom line can be replaced by the higher-order discretization $R_d^{(k)}$, whose existence has been proved in Theorem 1. Such a map can be cotangently lifted as in Proposition 1 to obtain the following discretization map $\left(R_d^{(k)}\right)^{T^*} : T\left(T^*(T^{(k)}Q)\right) \to T^*(T^{(k)}Q) \times T^*(T^{(k)}Q)$. ■

A. Geometric integrators on the higher-tangent bundle

The framework for the construction of geometric integrators is established by Proposition 5.1 in [4] wich reads:

Proposition 2: If R_d is a discretization map on Q and $H : T^*Q \to \mathbb{R}$ is a Hamiltonian function, then the equation

$$\begin{aligned}&(R_d^{T^*})^{-1}(q_0,p_0,q_1,p_1) = \\ &\sharp_\omega\left(hdH\left[\tau_{T^*Q}\circ(R_d^{T^*})^{-1}(q_0,p_0,q_1,p_1)\right]\right)\end{aligned}$$

written for the cotangent lift of R_d is a symplectic integrator.

The previous proposition adapts perfectly to our case since a higher-order Lagrangian on $T(T^{(k)}Q)$ has the corresponding Hamiltonian function on $T^*(T^{(k)}Q)$. The cotangent lift in Proposisition 1 can be replaced by the higher-order cotangent lift in Corollary 2. As a result, we have constructed a symplectic integrator for the Hamiltonian version of the higher-order dynamics.

VI. Application to Optimal Control problems

Suppose that on $Q = \mathbb{R}^n$ we have the optimal control problem with cost functional

$$\mathcal{J} = \int_0^T \frac{1}{2}||u||^2\ dt$$

subjected to the Euler-Lagrange controlled dynamics $\ddot{q} = u$.

This problem can be recasted as the second-order variational problem $\mathcal{J} = \int_0^T \frac{1}{2}||\ddot{q}||^2\ dt$ with the second-order Lagrangian $\mathcal{L} = \frac{1}{2}||\ddot{q}||^2$ on $T^{(2)}Q$. Necessary conditions for a trajectory to be optimal is to fulfill the second-order Euler-Lagrange equations, which in this case give the spline equations $q^{(4)} = 0$. However, as described in Section III, the Hamiltonian for this second-order Lagrangian system is defined on T^*TQ:

$$H(q,\dot{q},\hat{p}_{(0)},\hat{p}_{(1)}) = \frac{1}{2}\hat{p}_{(1)}^2 + \hat{p}_{(0)}\dot{q}. \tag{4}$$

A discretization map on $T^*TQ = T^*(T^{(1)}Q)$ is obtained by cotangently lifting a first-order discretization map on TQ, that corresponds with the tangent lift of a discretization map on Q as defined in [4].

As in Example 3, the midpoint discretization map $R_d(q,\dot{q}) = \left(q - \frac{1}{2}\dot{q}, q + \frac{1}{2}\dot{q}\right)$ is used to define $R_d^{(1)} : T(TQ) \to TQ \times TQ$.

The first-order cotangent lift of the midpoint on $T^*(TQ)$ is a discretization map as proved in Corollary 2:

$$\begin{aligned}&\left(R_d^{(1)}\right)^{T^*}(q,\dot{q},p_0,p_1;\dot{q},\ddot{q},\dot{p}_{(0)},\dot{p}_{(1)}) = \\ &\quad\Big(q - \frac{1}{2}\dot{q}, \dot{q} - \frac{1}{2}\ddot{q}, p_{(0)} - \frac{\dot{p}_{(0)}}{2}, p_{(1)} - \frac{\dot{p}_{(1)}}{2}; \\ &\quad q + \frac{1}{2}\dot{q}, \dot{q} + \frac{1}{2}\ddot{q}, p_{(0)} + \frac{\dot{p}_{(0)}}{2}, p_{(1)} + \frac{\dot{p}_{(1)}}{2}\Big).\end{aligned} \tag{5}$$

As the Hamiltonian vector field associated with the Hamiltonian function (4) takes values in $T(T^*TQ)$, by Proposition 2, $\left(R_d^{(1)}\right)^{T^*}$ generates the following symplectic numerical scheme on T^*TQ:

$$\begin{aligned}&\frac{q_1 - q_0}{h} = \frac{\dot{q}_1 + \dot{q}_0}{2}, \quad \frac{\dot{q}_1 - \dot{q}_0}{h} = \frac{\hat{p}_{(1)1} + \hat{p}_{(1)0}}{2}, \\ &\frac{\hat{p}_{(0)1} - \hat{p}_{(0)0}}{h} = 0, \quad \frac{\hat{p}_{(1)1} - \hat{p}_{(1)0}}{h} = -\frac{\hat{p}_{(0)1} + \hat{p}_{(0)0}}{2}.\end{aligned}$$

Working out the expressions we obtain:

$$\begin{aligned}q_1 &= q_0 + h\dot{q}_0 + \frac{h^2}{2}\hat{p}_{(1)0} + \frac{h^3}{4}\hat{p}_{(0)0}, \\ \dot{q}_1 &= \dot{q}_0 + h\hat{p}_{(1)0} + \frac{h^2}{2}\hat{p}_{(0)0}, \\ \hat{p}_{(0)1} &= \hat{p}_{(0)0}, \quad \hat{p}_{(1)1} = \hat{p}_{(1)0} - h\hat{p}_{(0)0}.\end{aligned}$$

A. Obstacle avoidance problem

The following application is an optimal control problem with obstacle avoidance, which is usually cast as a second-order variational problem of the form

$$\int_0^T \left(\frac{1}{2}||\ddot{q}||^2 + V(q)\right)\ dt \tag{6}$$

(see [7], [8], [19]). The second order Lagrangian is in this case $\mathcal{L} = \frac{1}{2}||\ddot{q}||^2 + V(q)$ and the necessary equations for a trajectory to be optimal is the fulfilment of the Euler-Lagrange equations which in this case are the fouth-order system: $q^{(4)}+\nabla V(q)=0$. The Hamiltonian for this second-order Lagrangian system is

$$H(q,\dot{q},\hat{p}_{(0)},\hat{p}_{(1)})=\frac{1}{2}\hat{p}_{(1)}^2+\hat{p}_{(0)}\dot{q}-V(q).$$

The symplectic method given by Proposition 2 associated with the Hamiltonian function above and with the discretization in (5) is

$$\begin{aligned} &\frac{q_1-q_0}{h}=\frac{\dot{q}_1+\dot{q}_0}{2}, \quad \frac{\dot{q}_1-\dot{q}_0}{h}=\frac{\hat{p}_{(1)1}+\hat{p}_{(1)0}}{2},\\ &\frac{\hat{p}_{(0)1}-\hat{p}_{(0)0}}{h}=\nabla V\left(\frac{q_1+q_0}{2}\right),\\ &\frac{\hat{p}_{(1)1}-\hat{p}_{(1)0}}{h}=-\frac{\hat{p}_{(0)1}+\hat{p}_{(0)0}}{2}. \end{aligned} \tag{7}$$

B. *Obstacle avoidance for a planar rigid body*

Let us examine a particular example of the situation in the previous subsection. Suppose $Q=SE(2)$, that all maps are considered in a local coordinate chart with coordinates $q=(x,y,\theta)$ and that the artificial potential appearing in (6) has the form

$$V(x,y,\theta)=\frac{\tau}{x^2+y^2-r^2}.$$

We simulate the optimal trajectory of the previous problem using the integrator in (7) with $\tau=1\times10^{-20}$, $r=1$ and taking$N=400$ steps with a step-size $h=0.01$. The initial position and velocity of the particle are $(-5,0,0)$, $(1,0,0)$, respectively, and the final position and velocities are $(6,0,0)$ and $(8,0,0)$, respectively. To measure the norm we use the euclidean metric. To enforce the boundary conditions a shooting method was ran together with the integrator.

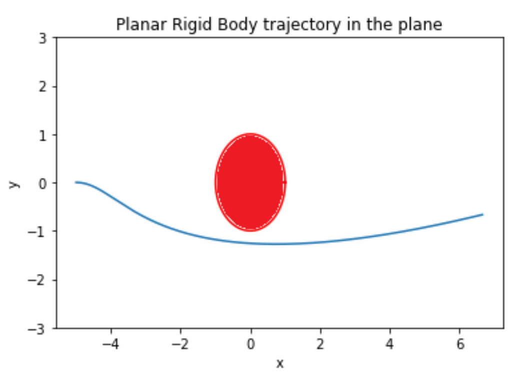

Fig. 3: In blue, the trajectory of the optimal solution in the xy plane. In red, a circular obstacle.

VII. Conclusions & Further applications

In this paper we have shown how to obtain discretization maps in higher-order tangent bundles by lifting discretization maps on the base manifold. Furthermore, we have shown some simple examples of higher-order discretization maps and simple applications to the construction of numerical integrators for optimal control problems. However, as we will describe below, the range of applications has still much to explore.

A. *Numerical methods for splines on the sphere*

Given a Riemannian manifold (Q,g) and the associated exponential map $\exp_q : T_qQ\to Q$, the following map

$$R_d(q,\xi)=\left(\exp_q(-\xi/2),\exp_q(\xi/2)\right)$$

is a discretization map because it satisfies the properties in Definition 4. An example of discretization maps that can be associated with the exponential map is, for instance, on the sphere S^2 with the Riemannian metric induced by the restriction of the standard metric on $\mathbb{R}^3$. The exponential map is given by

$$\exp_q(\xi)=\cos(\|\xi\|)\,q+\sin(\|\xi\|)\,\frac{\xi}{\|\xi\|}, \qquad \xi\in T_qS^2\,. \tag{8}$$

Higher-order discretization maps can be used in the problem of finding higher-order Riemannian polynomials, defined in [29], [16], [25], [31] as the critical curves of the higher-order functional

$$\int_0^T \frac{1}{2}\langle\frac{D^k\gamma}{dt^k},\frac{D^k\gamma}{dt^k}\rangle\,dt,$$

where $\frac{D^k\gamma}{dt^k}$ denotes k-th covariant derivative.

In future work, we will apply the previous construction to obtain higher-order geometric integrators to numerically obtain Riemannian polynomials.

B. *Discretization maps for systems on Lie groups*

Optimal control problems in Lie groups are extremely important because of the applications in robotics. Using the left-trivialized tangent bundle to have the identification $TG\approx G\times\mathfrak{g}$, the exponential map can be, for instance, used for defining a discretization map on the Lie group:

$$R_d(g,\xi)=\left(g\cdot\exp(-\xi/2),g\cdot\exp(\xi/2)\right).$$

In this scenario, higher-order lifts of $R_{d,g}$ are associated with higher-order derivatives of the map $R_{d,g}:\mathfrak{g}\to G\times G$. Then, we might generate geometric integrators for the problem (VII-A) on Lie Groups.

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