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Distributed Online Optimization with Coupled Inequality Constraints over Unbalanced Directed Networks

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Abstract— **This paper studies a distributed online convex optimization problem, where agents in an unbalanced network cooperatively minimize the sum of their time-varying local cost functions subject to a coupled inequality constraint. To solve this problem, we propose a distributed dual subgradient tracking algorithm, called DUST, which attempts to optimize a dual objective by means of tracking the primal constraint violations and integrating dual subgradient and push-sum techniques. Different from most existing works, we allow the underlying network to be unbalanced with a column stochastic mixing matrix. We show that DUST achieves sublinear dynamic regret and constraint violation bounds, provided that the accumulated variation of the optimal sequence grows sublinearly. If the standard Slater's condition is additionally imposed, DUST acquires a smaller constraint violation bound than the alternative existing methods applicable to unbalanced networks. Simulations on a plug-in electric vehicle charging problem demonstrate the superior convergence of DUST.**

I. INTRODUCTION

Distributed online convex optimization (DOCO) has received considerable interest in recent years, motivated by its broad applications in dynamic networks with uncertainty, such as resource allocation for wireless network [1], target tracking [2], multi-robot surveillance [3], and medical diagnosis [4]. In these scenarios, each agent in a network holds a time-varying local cost function and only has access to its real-time local cost function after making a decision based on historical information. Compared with centralized online optimization, DOCO enjoys prominent advantages in privacy protection, alleviation of computation and communication burden, and robustness to channel failures [5].

There have been a great number of distributed algorithms for solving DOCO problems [2]–[4], [6]–[15]. Nevertheless, most of them are limited to unconstrained problems or simple set constraints, and do not allow for coupled inequality constraints that arise in many engineering applications. Coupled inequality constraints involve information from all agents, which poses a significant challenge to handle them in a distributed manner. To date, only a few distributed algorithms have been developed to address DOCO problems with coupled inequality constraints, including various variants of the saddle-point algorithm [10]–[13], a primal-dual dynamic mirror descent algorithm [14] that has been extended to bandit settings in [15], and a bandit distributed mirror descent push-sum algorithm [9]. However, among these works, [12]–[15] can only be applied to balanced networks with doubly stochastic mixing matrices. Although [9]–[11] consider unbalanced networks, their regret and constraint violation bounds are much greater than $\mathcal{O}(\sqrt{T})$ regret bound and $\mathcal{O}(T^{\frac{3}{4}})$ constraint violation bound in [13], not to mention $\mathcal{O}(\sqrt{T})$ constraint violation bound in [12], [14], [15].

To overcome the drawbacks of the aforementioned existing works, this paper focuses on the DOCO problem with a coupled inequality constraint over an unbalanced network with a column stochastic mixing matrix, and proposes a distributed dual subgradient tracking (DUST) algorithm to solve it. DUST attempts to address the dual problem of the constrained DOCO by emulating the subgradient method. In particular, it enables distributed implementation by introducing auxiliary variables to track the primal constraint violations, which can be viewed as estimated dual subgradients. It also harnesses the push-sum technique to tackle the network imbalance. The main contributions of this paper are elaborated as follows:

1) DUST is able to address DOCO with *coupled inequality* constraints over *unbalanced* networks with column stochastic mixing matrices, while the alternative methods in [12]–[15] require balanced interaction graphs.
2) We adopt *dynamic regret* as the performance measure of DUST, which is a more stringent metric than the static regret used in [9], [11]–[13].
3) We show that DUST achieves $\mathcal{O}(\sqrt{T}+V_T)$ dynamic regret and $\mathcal{O}(T^{\frac{3}{4}})$ constraint violation bounds, where T is a finite time horizon and V_T is the accumulated variation of the optimal sequence. Provided that V_T grows sublinearly, DUST is able to achieve sublinear dynamic regret and constraint violations. Moreover, the constraint violation bound is improved to $\mathcal{O}(\sqrt{T})$ if we additionally assume the Slater's condition. To the best of our knowledge, there are no existing distributed algorithms achieving comparable dynamic regret and constraint violation bounds for DOCO problems with coupled constraints over unbalanced networks.

The remainder of the paper is organized as follows. Section II formulates a DOCO with a coupled inequality constraint over unbalanced graphs with column stochastic mixing matrices. Section III develops the proposed DUST algorithm, and Section IV provides bounds of dynamic regret

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and constraint violations. Section V presents the numerical experiments, and Section VI concludes the paper.

Notations: let $\mathbb{R}^n$, $\mathbb{R}^n_+$ be the set of n-dimensional vectors, nonnegative vectors, respectively. For any set $X \subseteq \mathbb{R}^n$, $\mathrm{relint}(X)$ is its relative interior. $A \otimes B$ represents the Kronecker product of any two matrices A and B with arbitrary size. Let $[a]_+$ represents the component-wise projection of a vector $a \in \mathbb{R}^n$ onto $\mathbb{R}^n_+$. Denote I_d and $\mathbf{1}_p$ ($\mathbf{0}_p$) as the d-dimensional identity matrix and the all-one (all-zero) column vectors with p dimensions. Let $\|\cdot\|$ be the Euclidean norm. $\langle x, y\rangle$ represents the standard inner product of two vectors x and y. The notation $w_{ij,t}$ denotes the i,j-th component of matrix W_t at time t. Let $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ be the ceiling and floor functions, respectively. For a convex function $f : \mathbb{R}^n \to \mathbb{R}$, we denote $\partial f(x)$ as a subgradient of f at x, i.e., $f(y) \geq f(x) + \langle \partial f(x), y - x\rangle$, $\forall y \in \mathbb{R}^n$.

II. Problem Formulation

Consider a network over a finite time interval $\{1, \ldots, T\}$. The network at each time t is modeled as a directed graph $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$, where $\mathcal{V} = \{1, ..., N\}$ is the set of nodes and $\mathcal{E}_t \subseteq \{\{i, j\} : i, j \in \mathcal{V},\ i \neq j\}$ is the set of edges. A directed edge $(j, i) \in \mathcal{E}_t$ means that node i can receive a message from node j. Let $\mathcal{N}^{\text{in}}_{i,t} = \{j|(j, i) \in \mathcal{E}_t\} \cup \{i\}$ and $\mathcal{N}^{\text{out}}_{i,t} = \{j|(i, j) \in \mathcal{E}_t\} \cup \{i\}$ be the sets of in-neighbors and out-neighbors of node i, respectively. The mixing matrix W_t associated with $\mathcal{G}_t$ is defined as $w_{ij,t} > 0$ if $(j, i) \in \mathcal{E}_t$ or $i = j$, and $w_{ij,t} = 0$, otherwise. We assume each node $j \in \mathcal{V}$ only knows the weights related to its out-neighbors, i.e., $w_{ij,t}$, $\forall i \in \mathcal{N}^{\text{out}}_{j,t}$. We impose the following assumption on the interaction graph.

Assumption 1: $\{\mathcal{G}_t\}_{t=1}^T$ and $\{W_t\}_{t=1}^T$ satisfy the following:

1) There exists a constant $a \in (0, 1)$ such that for each $t \geq 1$, $w_{ij,t} > a$ if $w_{ij,t} > 0$.
2) For each $t \geq 1$, the mixing matrix W_t is column stochastic, i.e., $\sum_{i=1}^N w_{ij,t} = 1$ for all $j \in \mathcal{V}$.
3) There exists an integer $B > 0$ such that for any $k \geq 0$, the graph $(\mathcal{V}, \bigcup_{t=kB+1}^{(k+1)B} \mathcal{E}_t)$ is strongly connected.

An example of the mixing matrix that satisfies Assumption 1 is $w_{ij,t} = 1/d_{j,t}$, if $i \in \mathcal{N}^{\text{out}}_{j,t}$; otherwise, $w_{ij,t} = 0$, where $d_{j,t} = |\mathcal{N}^{\text{out}}_{j,t}|$ is the out-degree of node j at each time t. In this case, each node only needs to know its out-degree at each time t. Assumption 1 ensures that there exists a path from one node to every other nodes within the interval of length B. Assumption 1 is less restrictive than those in [12]–[15], which require W_t to be doubly stochastic.

We consider the distributed online problem with a globally coupled inequality constraint over the directed graph $\mathcal{G}_t$, where each node $i \in \mathcal{V}$ privately holds a time-varying local cost function $f_{i,t} : \mathbb{R}^{d_i} \to \mathbb{R}$, a constraint function $g_i : \mathbb{R}^{d_i} \to \mathbb{R}^p$, and a constraint set $X_i \subseteq \mathbb{R}^{d_i}$. Let $\mathbf{x} = [(x_1)^T, \ldots, (x_N)^T]^T \in \mathbb{R}^{\sum_{i=1}^N d_i}$ and $X = X_1 \times \cdots \times X_N$ be the Cartesian product of all the X_i's. At each time t, all nodes cooperate to minimize the sum of local cost functions while satisfying a coupled inequality constraint and set constraints, which can be written as

$$\begin{aligned} \underset{x_i, \forall i \in \mathcal{V}}{\text{minimize}} \quad & f_t(\mathbf{x}) := \textstyle\sum_{i=1}^N f_{i,t}(x_i) \\ \text{subject to} \quad & \textstyle\sum_{i=1}^N g_i(x_i) \leq \mathbf{0}_p, \\ & x_i \in X_i,\ \forall i \in \mathcal{V}, \end{aligned} \tag{1}$$

where the feasible set $\mathcal{X} := \{\mathbf{x} \in X | \sum_{i=1}^N g_i(x_i) \leq \mathbf{0}_p\}$ is assumed to be nonempty. Note that the local cost function $f_{i,t}$ is unrevealed to node i until it makes its decision $x_{i,t} \in X_i$ at time t. Since node i cannot access $f_{i,t}$ in advance, it is unlikely to obtain the exact optimal solution of problem (1). Thus, it is desirable to develop a distributed online algorithm that generates local decisions $x_{i,t}$, $i \in \mathcal{V}$ to track the optimal solution. We make the following assumption on problem (1).

Assumption 2: Problem (1) satisfies the following:

1) For each $i \in \mathcal{V}$, X_i is a compact convex set with diameter $R := \sup_{x_i, \tilde{x}_i \in X_i} \|x_i - \tilde{x}_i\|$.
2) For each $i \in \mathcal{V}$, $f_{i,t}$, $\forall t \geq 1$ and g_i are convex on X_i.

It is directly obtained from the compactness of X_i's and the convexity of f_i, g_i in Assumption 2 that there exist constants $F > 0$, $G > 0$ such that

$$\|g_i(x_i)\| \leq F,\ \forall x_i \in X_i, \forall i \in \mathcal{V}, \tag{2}$$

$$\|\partial f_{i,t}(x_i)\| \leq G,\ \|\partial g_i(x_i)\| \leq G,\ \forall x_i \in X_i, \forall i \in \mathcal{V}. \tag{3}$$

We adopt *dynamic regret* to measure the algorithm performance over the finite time horizon T [2], which is defined as the difference of the accumulated cost at the real-time decisions and at the optimal solution sequence, i.e.,

$$\mathrm{Reg}(T) := \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(x_{i,t}) - \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(x^*_{i,t}), \tag{4}$$

where $x^*_{i,t}$ is the i-th component of the optimal solution $\mathbf{x}^*_t = [(x^*_{1,t})^T, \ldots, (x^*_{N,t})^T]^T := \arg\min_{\mathbf{x} \in \mathcal{X}} \sum_{i=1}^N f_{i,t}(x_i)$ to problem (1). In contrast to the conventional metric *static regret* that is defined as the difference between the accumulated cost over time and the cost incurred by the best fixed decision when all functions are known in hindsight (i.e., $\sum_{t=1}^T \sum_{i=1}^N f_{i,t}(x^*_i)$, where $\mathbf{x}^* = [(x^*_1)^T, \ldots, (x^*_N)^T]^T := \arg\min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(x_i)$), the dynamic regret (4) allows the best decisions to vary with time and is a more stringent and suitable benchmark to capture the algorithm performance on a time-varying optimization problem [2], [3].

In addition, we define the cumulative constraint violation to measure whether the coupled inequality is satisfied in a longterm run as follows:

$$\mathrm{Reg}^c(T) := \left\| \left[\sum_{t=1}^T \sum_{i=1}^N g_i(x_{i,t}) \right]_+ \right\|. \tag{5}$$

Our goal is to design a distributed algorithm for solving the online problem (1) over $\mathcal{G}_t$ with superior dynamic regret and cumulative constraint violation bounds.

III. Algorithm Development

In this section, we propose a distributed dual subgradient tracking method to solve the distributed online problem with a coupled inequality constraint described in Section II.

First of all, let $L_t : \mathbb{R}^{\sum_{i=1}^N d_i} \times \mathbb{R}_+^p \to \mathbb{R}$ be the Lagrangian function associated with problem (1) at time t, given by

$$L_t(\mathbf{x}, \mu) = f_t(\mathbf{x}) + \mu^T \sum_{i=1}^N g_i(x_i), \tag{6}$$

where $\mu \geq \mathbf{0}_p$ is the Lagrange multiplier. We denote the dual function at time t as $D_t(\mu) := \min_{\mathbf{x}\in X}\{L_t(\mathbf{x}, \mu)\}$. The dual problem of problem (1) at time t is $\max_{\mu\geq \mathbf{0}_p} D_t(\mu)$. If we directly apply the dual subgradient method [25] to the online problem (1), we obtain the following updates: For arbitrarily given $\mu_1 \geq \mathbf{0}_p$ and each $t \geq 1$

$$\mathbf{x}_{t+1} = \arg\min\nolimits_{\mathbf{x}\in X}\{L_{t+1}(\mathbf{x}, \mu_t)\}, \tag{7}$$

$$\mu_{t+1} = [\mu_t + \sum_{i=1}^N g_i(x_{i,t+1})]_+, \tag{8}$$

where $\mathbf{x}_{t+1} = [(x_{1,t+1})^T, \ldots, (x_{N,t+1})^T]^T \in \mathbb{R}^{\sum_{i=1}^N d_i}$ can be viewed as an estimate of $\mathbf{x}_{t+1}^*$, i.e., the optimal solution of problem (1) at time $t+1$ and μ_{t+1} is an estimate of the optimal dual solution at time $t+1$. The updates (7)–(8) intend to use the subgradient method for solving the dual problem of problem (1) at time $t+1$, i.e, $\max_{\mu\geq\mathbf{0}_p} D_{t+1}(\mu)$. The update of μ_{t+1} in (8) involves the subgradient of the dual function $D_{t+1}(\mu)$ at μ_t, which is equal to $\sum_{i=1}^N g_i(x_{i,t+1})$ according to the Danskin's theorem [25].

However, (7) and (8) suffer from two issues. First, we have no prior knowledge of f_{t+1} when making decision $\mathbf{x}_{t+1}$. Second, the above updates require the global quantities μ_t and $\sum_{i=1}^N g_i(x_{i,t+1})$ at each time t so that they are not implementable in the distributed scenario.

To overcome the two issues, we let $g(x) = [(g_1(x_1))^T, \ldots, (g_N(x_N))^T]^T \in \mathbb{R}^{Np}$, and construct the following algorithm: Given $\mathbf{x}_1 \in X$, $\mathbf{y}_1 = g(\mathbf{x}_1)$, $\boldsymbol{\mu}_1 = \mathbf{0}_{Np}$, for any $t \geq 1$,

$$\begin{aligned}\mathbf{x}_{t+1} = \arg\min\nolimits_{x\in X} \{&\alpha_t \partial f_t(\mathbf{x}_t)^T(\mathbf{x}-\mathbf{x}_t) \\ &+\langle (W_t\otimes I_p)\boldsymbol{\mu}_t,\ g(\mathbf{x})\rangle + \eta_t\|\mathbf{x}-\mathbf{x}_t\|^2\},\end{aligned} \tag{9}$$

$$\mathbf{y}_{t+1} = (W_t \otimes I_p)\mathbf{y}_t + g(\mathbf{x}_{t+1}) - g(\mathbf{x}_t), \tag{10}$$

$$\boldsymbol{\mu}_{t+1} = [(W_t\otimes I_p)\boldsymbol{\mu}_t + \mathbf{y}_{t+1}]_+, \tag{11}$$

where $\mathbf{y}_t = [(y_{1,t})^T, \ldots, (y_{N,t})^T]^T \in \mathbb{R}^{Np}$, $\boldsymbol{\mu}_t = [(\mu_{1,t})^T, \ldots, (\mu_{N,t})^T]^T \in \mathbb{R}^{Np}$, and W_t is the mixing matrix at time t described in Section II. Here, the parameters α_t is used to balance the objective optimization and the constraint violations at each time t and η_t is the stepsize.

The above updates (9)–(11) are capable of addressing the issues caused by (7)–(8). Specifically, we estimate the unknown f_{t+1} with the first-order approximation of f_t at $\mathbf{x}_t$, i.e., $f_t(\mathbf{x}_t) + \partial f_t(\mathbf{x}_t)^T(\mathbf{x}-\mathbf{x}_t)$, which is accessible. The proximal term $\eta_t\|\mathbf{x}-\mathbf{x}_t\|^2$ guarantees that (9) is well-posed and contributes to convergence. To enable distributed implementation of (9)–(11), let each node $i \in \mathcal{V}$ maintain local variables $x_{i,t}$, $y_{i,t}$, and $\mu_{i,t}$ at each time t, which are the i-th blocks of $\mathbf{x}_t$, $\mathbf{y}_t$, and $\boldsymbol{\mu}_t$. Here, $x_{i,t}$ is node i's estimate of $x_{i,t}^*$ and $\mu_{i,t}$ is node i's estimate of the dual optimal solution at time t, playing a similar role to μ_t in (8). Different from (7)–(8), we employ the terms $(W_t\otimes I_p)\boldsymbol{\mu}_t$ and $(W_t \otimes I_p)\boldsymbol{y}_t$ such that the updates of $x_{i,t+1}$, $y_{i,t+1}$, and $\mu_{i,t+1}$ in (9)–(11) only depend on local information and the neighbors' information, thus enabling distributed computation. If W_t satisfies row stochasticity and each $\mu_{i,t}$ reaches the same value μ_t, $\langle (W_t\otimes I_p)\boldsymbol{\mu}_t,\ g(\mathbf{x})\rangle$ in (9) is equivalent to $\mu_t^T\sum_{i=1}^N g_i(x_i)$ in (7) and $(W_t\otimes I_p)\boldsymbol{\mu}_t = \mu_t$ in (11). The local variable $y_{i,t}$ is capable of tracking the primal constraint violation $\sum_{i=1}^N g_i(x_{i,t})$ at time t since the column stochasticity of W_t and the initial condition $\mathbf{y}_1 = g(\mathbf{x}_1)$ prompt (10) to give $\sum_{i=1}^N y_{i,t} = \sum_{i=1}^N g_i(x_{i,t})$, $\forall t \geq 1$, which is shown by Lemma 1 in Section IV. Thus, at time $t+1$, each $y_{i,t+1}$ tracks $\sum_{i=1}^N g_i(x_{i,t+1})$ that can be regarded as the estimated subgradient of the dual function $D_{t+1}(\mu)$ at μ_t in (8) as we state before. Clearly, in (11), each node $i\in\mathcal{V}$ computes its estimate $\mu_{i,t+1}$ of the dual optimal solution at time $t+1$ by means of the weighted $\mu_{j,t}$ received from all its in-neighbors and the local variable $y_{i,t+1}$ that tracks the estimated dual subgradient of the dual function $D_{t+1}(\mu)$ at μ_t, i.e., $\sum_{i=1}^N g_i(x_{i,t+1})$. Consequently, (9)–(11) lead to a distributed dual subgradient tracking (DUST) algorithm.

Nevertheless, (9)–(11) are not applicable to unbalanced networks because the column stochastic matrix W_t causes that $\mu_{i,t}$, $\forall i \in \mathcal{V}$ cannot reach an identical value as they should. To cope with unbalanced graphs, we integrate the push-sum technique into (9)–(11) to eliminate the imbalance of interaction networks by constructing row-stochastic property of W_t. For convenience, we still refer to the resulting algorithm as DUST. Let each node $i \in \mathcal{V}$ maintain variables $c_{i,t} \in \mathbb{R}$ besides $x_{i,t}$, $y_{i,t}$, and $\mu_{i,t}$. The DUST algorithm is described as follows: Given $x_{i,1} \in X_i$, $y_{i,1} = g_i(x_{i,1})$, $c_{i,1} = 1$, $\mu_{i,1} = \mathbf{0}_p$, $\forall i \in \mathcal{V}$, for any $t \geq 1$, each node $i \in \mathcal{V}$ updates as follows:

$$c_{i,t+1} = \sum_{j\in\mathcal{N}_{i,t}^{\text{in}}} w_{ij,t}c_{j,t}, \tag{12}$$

$$\lambda_{i,t+1} = \frac{\sum_{j\in\mathcal{N}_{i,t}^{\text{in}}} w_{ij,t}\mu_{j,t}}{c_{i,t+1}}, \tag{13}$$

$$\begin{aligned}x_{i,t+1} = \arg\min\nolimits_{x_i\in X_i}\{&\alpha_t\partial f_{i,t}(x_{i,t})^T(x_i-x_{i,t}) \\ &+\langle \lambda_{i,t+1},\ g_i(x_i)\rangle + \eta_t\|x_i-x_{i,t}\|^2\},\end{aligned} \tag{14}$$

$$y_{i,t+1} = \sum_{j\in\mathcal{N}_{i,t}^{\text{in}}} w_{ij,t}y_{j,t} + g_i(x_{i,t+1}) - g_i(x_{i,t}), \tag{15}$$

$$\mu_{i,t+1} = \Big[\sum_{j\in\mathcal{N}_{i,t}^{\text{in}}} w_{ij,t}\mu_{j,t} + y_{i,t+1}\Big]_+, \tag{16}$$

where the values of $x_{i,1}$, $y_{i,1}$, and $\mu_{i,1}$ follow from the initialization of the algorithm (9)–(11). The updates (14)–(16) are obtained by simply rewriting (9)–(11). The updates (12), (13), (15), and (16) require each node i to collect $w_{ij,t}c_{j,t}$, $w_{ij,t}\mu_{j,t}$, and $w_{ij,t}y_{j,t}$ from its every in-neighbor $j\in\mathcal{N}_{i,t}^{\text{in}}$, which suggests that DUST only needs communi-

Algorithm 1 DUST

1: **Initialization**:
2: Each node $i \in \mathcal{V}$ sets $x_{i,1} \in X_i$, $c_{i,1} = 1$, $\mu_{i,1} = \mathbf{0}_p$, and $y_{i,1} = g_i(x_{i,1})$.
3: **for** $t = 1, 2, \ldots, T$ **do**
4: Each node $j \in \mathcal{V}$ sends its local information $w_{ij,t}c_{j,t}$, $w_{ij,t}\mu_{j,t}$, and $w_{ij,t}y_{j,t}$ to every out-neighbor $i \in \mathcal{N}_{j,t}^{\text{out}}$. After receiving the information from its in-neighbor $j \in \mathcal{N}_{i,t}^{\text{in}}$, each node $i \in \mathcal{V}$ updates $c_{i,t+1}$ according to (12) and then computes $\lambda_{i,t+1}$ according to (13).
5: Each node $i \in \mathcal{V}$ updates $x_{i,t+1}$ according to (14).
6: Each node $i \in \mathcal{V}$ updates $y_{i,t+1}$ according to (15).
7: Each node $i \in \mathcal{V}$ updates $\mu_{i,t+1}$ according to (16).
8: **end for**

cation between neighboring nodes. Algorithm 1 details all these actions taken by the nodes.

Remark 1: DUST allows the algorithm parameters α_t and η_t to be time-varying and they can be set as $\alpha_t = \sqrt{t}$ and $\eta_t = t$ according to the theoretical results in Section IV. Different from [13], [14] whose parameters are related to the time horizon T, we allow α_t and η_t to be time-varying without knowing T in advance, which provides flexibility in terminating the algorithm.

IV. Dynamic Regret and Constraint Violation Bounds

In this section, we provide the dynamic regret and constraint violation bounds of DUST.

A. Auxiliary Lemmas

In this subsection, we present the following key lemmas to elucidate the role of variable $y_{i,t}$ and establish connections between the constraint violation and the global objective value with dual variables.

Lemma 1: Suppose Assumptions 1 and 2 hold. Then, for any $t \geq 1$,

$$\sum_{i=1}^{N} y_{i,t} = \sum_{i=1}^{N} g_i(x_{i,t}), \tag{17}$$

$$\|y_{i,t}\| \leq B_y, \tag{18}$$

where $B_y = \frac{8N^2F\sqrt{p}}{r}(1 + \frac{2}{1-\sigma}) + (N+2)F$, $r := \inf_{t=1,2,\ldots}(\min_{i\in[N]}\{W_t \cdots W_1 \mathbf{1}_N\}_i)$, and $\sigma \in (0,1)$ satisfy $r \geq \frac{1}{N^{NB}}$, $\sigma \leq \left(1 - \frac{1}{N^{NB}}\right)^{\frac{1}{NB}}$.

Lemma 1 shows that the local estimator $y_{i,t}$ is capable of tracking the the sum of local constraint function values at each time t. The proof of Lemma 1 is similar to Lemma 1 in [12] and Lemma 4 in [10], and we omit it here.

Lemma 2: Suppose Assumptions 1 and 2 hold. Let $\bar{\mu}_{T+1} = \frac{1}{N}\sum_{i=1}^{N}\mu_{i,T+1}$. Then, for any $T \geq 1$,

$$\sum_{t=1}^{T}\sum_{i=1}^{N} g_i(x_{i,t}) \leq N\bar{\mu}_{T+1} + NGR\mathbf{1}_p. \tag{19}$$

Proof: See Appendix A. ■

Lemma 2 states that the cumulative constraint violation is bounded by the dual variable $\bar{\mu}_{T+1}$. Thus, by finding the upper bound of $\bar{\mu}_{T+1}$, we can obtain the upper bound of cumulative constraint violation. The following lemma provides a bound on the change of dual variable $\bar{\mu}_t$.

Lemma 3: Suppose Assumptions 1 and 2 hold. Then, for any $t \geq 1$ and arbitrary $\tilde{x}_{i,t} \in X_i$, $i \in \mathcal{V}$,

$$\begin{aligned}
&\frac{N}{2}\|\bar{\mu}_{t+1}\|^2 - \frac{N}{2}\|\bar{\mu}_t\|^2 \\
&\leq (\frac{N}{2} + \frac{N}{r})B_y^2 + \frac{NG^2\alpha_t^2}{4\eta_t} + (2B_y + 2F)\sum_{i=1}^{N}\|\bar{\mu}_t - \lambda_{i,t+1}\| \\
&\quad + \sum_{i=1}^{N}\alpha_t \partial f_{i,t}(x_{i,t})^T(\tilde{x}_{i,t} - x_{i,t}) + \sum_{i=1}^{N}\langle\bar{\mu}_t,\ g_i(\tilde{x}_{i,t})\rangle \\
&\quad + \sum_{i=1}^{N}\eta_t(\|x_{i,t} - \tilde{x}_{i,t}\|^2 - \|x_{i,t+1} - \tilde{x}_{i,t}\|^2).
\end{aligned} \tag{20}$$

Proof: See Appendix B. ■

Lemma 3 establishes the relationship between the change of dual variables and the first-order information of the global objective functions, where the former involves constraint violations according to Lemma 2 and the latter is related to the dynamic regret bound. By choosing $\tilde{x}_{i,t}$ appropriately and utilizing the convexity of local functions as well as Lemmas 1–2, we obtain the dynamic regret and constraint violation bounds based on Lemma 3.

B. Main Results

Theorem 1: Suppose Assumptions 1 and 2 hold. If we set

$$\alpha_t = \sqrt{t},\ \eta_t = t, \tag{21}$$

then for any $t \geq 1$,

$$\mathrm{Reg}(T) = \mathcal{O}(\sqrt{T}) + \mathcal{O}(V_T), \tag{22}$$

where $V_T := \sum_{t=1}^{T}\sqrt{t}\sum_{i=1}^{N}\left\|x_{i,t+1}^* - x_{i,t}^*\right\|$ and $x_{i,t}^*$ is the i-th component of the optimal solution $\mathbf{x}_t^* := \arg\min_{\mathbf{x}\in\mathcal{X}}\sum_{i=1}^{N} f_{i,t}(x_i)$ to problem (1).

Proof: See Appendix C. ■

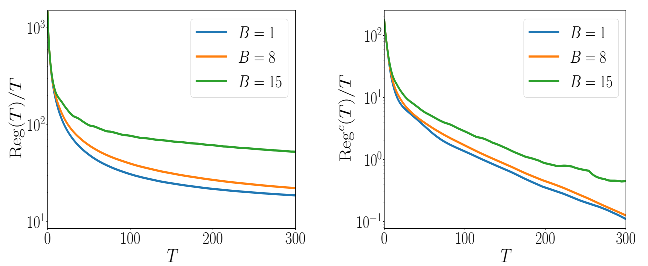
Theorem 1 shows that the dynamic regret grows sublinearly with T if V_T, the accumulated variation of the optimal sequence, is sublinear, which requires the online problem (1) does not change too drastically. Intuitively, the sublinearity guarantees that $\mathrm{Reg}(T)/T$ converges to 0 as T goes to infinity. It should be noted that if $V_T = 0$, the result reduces to an $\mathcal{O}(\sqrt{T})$ bound with respect to the static regret.

In addition, Theorem 1 indicates that DUST has stronger results than other existing algorithms applicable to coupled inequality constraints. As is shown in Table I, the static regret bounds in [9], [11] is strictly greater than $\mathcal{O}(\sqrt{T})$ and the dynamic regret bound in [10] is also worse than ours. Although [12] achieves the same static regret bound as DUST, it requires the boundedness of $\mu_{i,t}$, which is a rather restrictive assumption, while DUST does not. The works [13]–[15] are only applied to balanced networks with doubly stochastic mixing matrices, where [13] only focuses on the static regret. The dynamic regret bounds in [14],

TABLE I

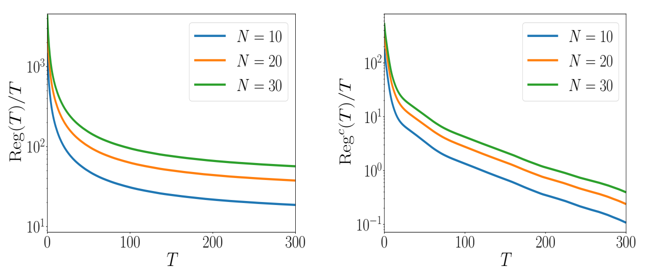
A COMPARISON OF DUST AND RELATED WORKS. LEGEND: $\surd$ MEANS THE CONDITION IS SATISFIED, SC STANDS FOR THE REQUIREMENT OF THE SLATER'S CONDITION, $\kappa \in (0, \frac{1}{4})$, $\theta \in (0, 1)$, $D_T = \sum_{t=1}^{T} \sum_{i=1}^{N} \left\| x_{i,t+1}^* - x_{i,t}^* \right\|$, AND $V_T = \sum_{t=1}^{T} \sqrt{t} \sum_{i=1}^{N} \left\| x_{i,t+1}^* - x_{i,t}^* \right\|$.

	[9]	[10]	[11]	[12]	[13]	[14]	DUST
unbalanced network	$\surd$	$\surd$	$\surd$				$\surd$
dynamic regret		$\surd$				$\surd$	$\surd$
regret bound	$\mathcal{O}(T^{\frac{1}{2}+\kappa})$	$\mathcal{O}(T^{\frac{1}{2}+2\kappa}) + \mathcal{O}(V_T)$	$\mathcal{O}(T^{\frac{1}{2}+2\kappa})$	$\mathcal{O}(\sqrt{T})$	$\mathcal{O}(T^{\max\{\theta,1-\theta\}})$	$\mathcal{O}(\max\{T^{\kappa} D_T, T^{\max\{\theta,1-\theta\}}\})$	$\mathcal{O}(\sqrt{T}) + \mathcal{O}(V_T)$
constraint violation bound	$\mathcal{O}(T^{1-\kappa})$ $(No\ SC)$	$\mathcal{O}(T^{1-\frac{\kappa}{2}})$ (SC)	$\mathcal{O}(T^{1-\frac{\kappa}{2}})$ (SC)	$\mathcal{O}(\sqrt{T})$ (SC)	$\mathcal{O}(T^{\max\{\frac{1}{2}+\frac{\theta}{2},1-\frac{\theta}{2}\}})$ $(No\ SC)$	$\mathcal{O}(T^{1-\frac{\theta}{2}})$ $(No\ SC)$ $\mathcal{O}(\max\{\theta, 1-\theta\})$ (SC)	$\mathcal{O}(T^{\frac{3}{4}})$ $(No\ SC)$ $\mathcal{O}(\sqrt{T})$ (SC)



(a) (b)

Fig. 1. Effects of network connectivity factor B on (a) $\text{Reg}(T)/T$ and (b) $\text{Reg}^c(T)/T$ when $N = 10$.



(a) (b)

Fig. 2. Effects of node number N on (a) $\text{Reg}(T)/T$ and (b) $\text{Reg}^c(T)/T$ when $B = 2$.

[15] depend on the accumulated error of optimal sequence $\sqrt{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \left\| x_{i,t+1}^* - x_{i,t}^* \right\|$, which is larger than V_T in (22), leading to a larger bound than DUST.

Next, we present a bound on constraint violation.

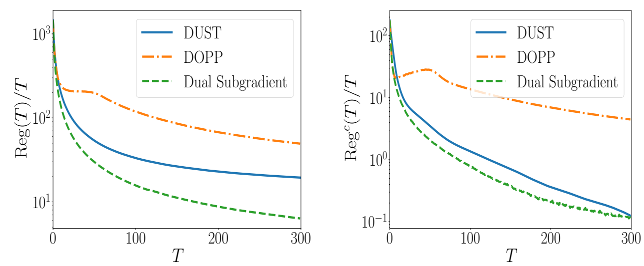
Theorem 2: Suppose all the conditions in Theorem 1 hold. Then for any $t \geq 1$,

$$\text{Reg}^c(T) = \mathcal{O}(T^{\frac{3}{4}}). \tag{23}$$

Proof: See Appendix D. ■

Theorem 2 shows that DUST achieves $\mathcal{O}(T^{\frac{3}{4}})$ constraint violation bound. Table I manifests that the result in (23) is superior than [9]–[11] whose constraint violation bound is strictly greater than $\mathcal{O}(T^{\frac{3}{4}})$, and is competitive compared with [13]–[15]. Theorem 2 holds without assuming the Slater's condition that allows us to handle equality constraints by converting an equality into two inequalities.

The following theorem shows that $Reg^c(T)$ is improved to $\mathcal{O}(\sqrt{T})$ if all local constraint functions $g_i, \forall i \in \mathcal{V}$ satisfy the Slater's condition, which is commonly assumed in [10], [11], [14].



(a) (b)

Fig. 3. Comparison of DUST with DOPP [10] and the centralized dual subgradient (7)–(8) on (a) $\text{Reg}(T)/T$ and (b) $\text{Reg}^c(T)/T$.

Assumption 3 (Slater's condition): There exists a constant $\epsilon > 0$ and a point $\hat{x}_i \in \text{relint}(X_i)$, $\forall i \in \mathcal{V}$ such that $\sum_{i=1}^{N} g_i(\hat{x}_i) \leq -\epsilon \mathbf{1}_p$.

Theorem 3: Suppose Assumptions 1–3 hold. If we set η_t and V_t as these in Theorem 1. Then, for any $t \geq 1$,

$$\text{Reg}^c(T) = \mathcal{O}(\sqrt{T}). \tag{24}$$

Proof: See Appendix E. ■

To the best of our knowledge, DUST is the first distributed algorithm achieving $\mathcal{O}(\sqrt{T})$ dynamic regret bound and $\mathcal{O}(T^{\frac{3}{4}})$ constraint violation bound for DOCO problems with coupled inequality constraints over unbalanced networks, let alone achieving $\mathcal{O}(\sqrt{T})$ constraint violation bound, which is also confirmed by Table I. Unlike [20]–[22] whose constraint violation bounds are affected by the dynamic optimal decisions x_t^*, $\forall t \geq 1$, our results are independent of them.

Remark 2: As the number of nodes N and the network connectivity factor B grow, the bounds of $\text{Reg}(T)$ and $\text{Reg}^c(T)$ in (22)–(24) increase accordingly, which can be observed from Appendixes C–E. This statement is verified via a numerical example in the following section.

V. NUMERICAL EXAMPLE

We apply DUST to solve the plug-in electric vehicles (PEVs) charging problem, where the charging cost of each PEV varied with time due to fluctuations in charging losses and energy prices at different time instances. The goal is to find an optimal charging schedule over a time period such that the sum of the local charging cost of all PEVs is minimized at each time instance and the network power resource constraints are satisfied [10], [13]. We formulate the

PEVs charging problem at each time t as:

$$\begin{aligned} &\underset{x_i\in X_i,\forall i\in\mathcal{V}}{\text{minimize}} && \textstyle\sum_{i=1}^{N} c_{i,t}(x_i) \\ &\text{subject to} && \textstyle\sum_{i=1}^{N} A_i x_i - D/N \le \mathbf{0}_p, \end{aligned} \tag{25}$$

where x_i represents the charging rate of PEV i, $c_{i,t}(x_i) := a_{i,t}/2\|x_i\|^2 + b_{i,t}^T x_i$ is the charging cost function of PEV i at time t [23], and X_i is the local constraint set involving maximum charging power and desired final state of charge of PEV i. The coupled constraint $\sum_{i=1}^{N} A_i x_i - D/N \le \mathbf{0}_p$ guarantees that the aggregate charging power of all PEVs is less than the maximum power D that the network can deliver. In our simulation, each $a_{i,t}$ and $b_{i,t}$ are randomly generated from uniform distributions $[0.5, 1]$ and $(0, 1]^{d_i}$, respectively, where $d_i = 24$ is the dimension of x_i. According to the set-up in [24], there are 48 coupled inequalities, i.e., the rate aggregation matrix $A_i \in \mathbb{R}^{48\times 24}$ and each local set X_i is determined by 197 inequalities. The values of A_i, D, and X_i are obtained by referring to [24].

To investigate the convergence performance of DUST and the effects of network connectivity factor B and node number N on the convergence performance of DUST, we run DUST with different B and different N, where the step (5) in Algorithm 1 is solved by CYXPY. Fig. 1 and Fig. 2 plot the evolution of $\mathrm{Reg}(T)/T$ and $\mathrm{Reg}^c(T)/T$ with $B = 1, 8, 15$ when N is fixed as 10 and $N = 10, 20, 30$ when B is fixed as 2, respectively. From the two figures, we observe that DUST is able to achieve sublinear convergence in terms of regret and constraint violations. In addition, it can be seen that the convergence speed becomes slower when B or N increases, which is consistent with our analysis in Remark 2.

We compare DUST with the distributed online primal-dual push-sum (DOPP) in [10] that is also developed based on column stochastic mixing matrices and the centralized dual subgradient method (7)–(8). For a fair comparison, we set $\kappa = 0.2$ for DOPP so that it achieves possibly best convergence performance. Fig. 3 presents the evolution of $\mathrm{Reg}(T)/T$ and $\mathrm{Reg}^c(T)/T$ of DUST, DOPP, and the dual subgradient when $N = 10$, $B = 4$. It is evident that DUST outperforms DOPP in terms of both the regret and constraint violations and achieves competitive performance compared with the centralized dual subgradient method, which demonstrates the distinguished performance of DUST.

VI. Conclusion

We have constructed a distributed dual subgradient tracking (DUST) algorithm to solve the DOCO problem with a globally coupled inequality constraint over unbalanced networks. To develop it, we integrate the push-sum technique into the dual subgradient method. The subgradients with respect to dual variables can be estimated by primal constraint violations, which are tracked by local auxiliary variables, enabling distributed implementation. We show that DUST achieves sublinear dynamic regret and constraint violations if the accumulated variation of the optimal sequence is also sublinear. Our theoretical results are stronger than those of existing distributed algorithms applicable to unbalanced networks, which is verified via numerical experiments.

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APPENDIX

A. Proof of Lemma 2

Proof: Let $\hat{\mu}_{i,t} = \sum_{j\in\mathcal{N}_{i,t}^{\text{in}}} w_{ij,t}\mu_{j,t}$ and $\bar{\mu}_t = \frac{1}{N}\sum_{i=1}^N \mu_{i,t}$. In light of (16), $\mu_{i,t+1} \geq \hat{\mu}_{i,t} + y_{i,t+1}$. Summing this inequality from $i=1$ to N gives $\bar{\mu}_{t+1} \geq \bar{\mu}_t + \frac{1}{N}\sum_{i=1}^N g_i(x_{i,t+1})$, which leads to $\sum_{t=1}^T\sum_{i=1}^N g_i(x_{i,t+1}) \leq N\sum_{t=1}^T(\bar{\mu}_{t+1} - \bar{\mu}_t) \leq N\bar{\mu}_{T+1} \leq N\|\bar{\mu}_{T+1}\|$. Invoking to the convexity of g_i gives $\sum_{t=1}^T\sum_{i=1}^N g_i(x_{i,t}) \leq \sum_{t=1}^T\sum_{i=1}^N g_i(x_{i,t+1}) + NGR\mathbf{1}_p \leq N\bar{\mu}_{T+1} + NGR\mathbf{1}_p$. ■

B. Proof of Lemma 3

Let and $\epsilon_{i,t+1} = [\hat{\mu}_{i,t} + y_{i,t+1}]_+ - \hat{\mu}_{i,t}$. Similar to [10, eq.(104)] and [10, Lemma 3], we have

$$\|\epsilon_{i,t+1}\| \leq B_y,\ r \leq c_{i,t} \leq N,\ \forall t \geq 1, \tag{26}$$

where B_y and r are given in Lemma 1. Based on the definition of $\epsilon_{i,t+1}$, we rewrite (16) as $\mu_{i,t+1} = \hat{\mu}_{i,t} + \epsilon_{i,t+1}$. Summing it from $i=1$ to N yields

$$\bar{\mu}_{t+1} = \bar{\mu}_t + \frac{1}{N}\sum_{i=1}^N \epsilon_{i,t+1}, \tag{27}$$

which gives for all $\lambda \in \mathbb{R}_+^p$,

$$\|\bar{\mu}_{t+1}-\lambda\|^2 \leq \|\bar{\mu}_t-\lambda\|^2 + \frac{2}{N}\sum_{i=1}^N \epsilon_{i,t+1}^T(\bar{\mu}_t-\lambda) + B_y^2. \tag{28}$$

The last inequality in (28) follows from (26). By following the line of proof in [10, eq.(113)] and using (18) and (26),

$$\epsilon_{i,t+1}^T(\bar{\mu}_t-\lambda) \leq \frac{B_y^2}{r} + \sum_{i=1}^N y_{i,t+1}^T(\lambda_{i,t+1}-\lambda) + B_y\|\bar{\mu}_t-\lambda_{i,t+1}\|,$$

The term $y_{i,t+1}^T(\lambda_{i,t+1}-\lambda)$ can be obtained

$$\sum_{i=1}^N y_{i,t+1}^T(\lambda_{i,t+1}-\lambda) = \sum_{i=1}^N y_{i,t+1}^T(\lambda_{i,t+1}-\bar{\mu}_t+\bar{\mu}_t-\lambda)$$

$$\leq (B_y+F)\sum_{i=1}^N\|\bar{\mu}_t-\lambda_{i,t+1}\| + \sum_{i=1}^N g_i(x_{i,t+1})^T(\lambda_{i,t+1}-\lambda), \tag{29}$$

where the last inequality utilizes Lemma 1 and (2). Let $S_{i,t}(x_i,\lambda_i) = \alpha_t\partial f_{i,t}(x_{i,t})^T(x_i - x_{i,t}) + \langle\lambda_i,\ g_i(x_i)\rangle + \eta_t\|x_i-x_{i,t}\|^2$. Obviously, we have $\sum_{i=1}^N g_i(x_{i,t+1})^T(\lambda_{i,t+1}-\lambda) = \sum_{i=1}^N S_{i,t}(x_{i,t+1},\lambda_{i,t+1}) - S_{i,t}(x_{i,t+1},\lambda) \leq \sum_{i=1}^N S_{i,t}(\tilde{x}_{i,t},\lambda_{i,t+1}) - S_{i,t}(x_{i,t+1},\lambda) - \eta_t\|x_{i,t+1}-\tilde{x}_{i,t}\|^2$, $\forall\tilde{x}_{i,t} \in X_i$, which follows from the $2\eta_t$-strong convexity of $S_{i,t}(x_i,\lambda_{i,t+1})$. Combing it with $\sum_{i=1}^N S_{i,t}(\tilde{x}_{i,t},\lambda_{i,t+1}) - S_{i,t}(\tilde{x}_{i,t},\bar{\mu}_t) \leq F\sum_{i=1}^N\|\bar{\mu}_t-\lambda_{i,t+1}\|$ yields

$$\sum_{i=1}^N g_i(x_{i,t+1})^T(\lambda_{i,t+1}-\lambda) \leq F\sum_{i=1}^N\|\bar{\mu}_t-\lambda_{i,t+1}\|$$

$$+\sum_{i=1}^N S_{i,t}(\tilde{x}_{i,t},\bar{\mu}_t) - S_{i,t}(x_{i,t+1},\lambda) - \eta_t\|x_{i,t+1}-\tilde{x}_{i,t}\|^2. \tag{30}$$

Let $\lambda = \mathbf{0}_p$. Imitating the Lemma 4 in [17] leads to $-S_{i,t}(x_{i,t+1},\lambda) \leq \frac{NG^2\alpha_t^2}{4\eta_t}$. By combing this inequality with (28)–(30), dividing both sides by $\frac{2}{N}$, and substituting the expressions of $S_{i,t}(\tilde{x}_{i,t},\bar{\mu}_t)$ give (20). Thus, Lemma 3 holds.

C. Proof of Theorem 1

For any $t \geq 1$, let $\tilde{x}_{i,t} = x_{i,t}^*$, $\forall i \in \mathcal{V}$. With $\bar{\mu}_t \geq \mathbf{0}_p$, $\langle\bar{\mu}_t, \sum_{i=1}^N g_i(x_{i,t}^*)\rangle \leq 0$. By virtual of the convexity of $f_{i,t}$, we have $\sum_{i=1}^N \alpha_t\partial f_{i,t}(x_{i,t})^T(x_{i,t}^*-x_{i,t}) \leq \alpha_t\sum_{i=1}^N f_{i,t}(x_{i,t}^*) - f_{i,t}(x_{i,t})$. Equipped with these, we divide (20) both sides by α_t and then sum it from $t=1$ to T,

$$\sum_{t=1}^T\sum_{i=1}^N f_{i,t}(x_{i,t}) - \sum_{t=1}^T\sum_{i=1}^N f_{i,t}(x_{i,t}^\star) \leq \underbrace{(\frac{N}{2}+\frac{N}{r})\sum_{t=1}^T\frac{B_y^2}{\alpha_t}}_{S_1}$$

$$+\underbrace{\sum_{t=1}^T\frac{NG^2\alpha_t}{4\eta_t}}_{S_2} + \frac{N}{2}\underbrace{\sum_{t=1}^T\frac{1}{\alpha_t}(\|\bar{\mu}_t\|^2 - \|\bar{\mu}_{t+1}\|^2)}_{S_3}$$

$$+(2B_y+2F)\underbrace{\sum_{t=1}^T\frac{1}{\alpha_t}\sum_{i=1}^N\|\bar{\mu}_t-\lambda_{i,t+1}\|}_{S4}$$

$$+\underbrace{\sum_{t=1}^T\frac{\eta_t}{\alpha_t}\sum_{i=1}^N\left(\|x_{i,t}^\star - x_{i,t}\|^2 - \|x_{i,t+1}-x_{i,t}^\star\|^2\right)}_{S_5}. \tag{31}$$

Below, we analyze the upper bounds of each S_i, $i=1,\ldots,5$. With $\alpha_t = \sqrt{t}$ and $\eta_t = t$, it is easy to obtain

$$S_1 \leq (NB_y^2 + \frac{2NB_y^2}{r})\sqrt{T},\ S_2 \leq \frac{NG^2\sqrt{T}}{2}, \tag{32}$$

$$S_3 = \|\bar{\mu}_1\|^2 + \sum_{t=2}^T(\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}})\|\bar{\mu}_t\|^2 - \frac{1}{\alpha_t}\|\bar{\mu}_{T+1}\|^2 \leq 0, \tag{33}$$

where (32) follows from $\sum_{t=1}^T\frac{1}{\sqrt{t}} \leq 1 + \int_{t=1}^T t^{-1/2}dt \leq 2\sqrt{T}$. By referring to [16, Lemma 1], we obtain

$$\sum_{i=1}^N\|\bar{\mu}_t - \lambda_{i,t+1}\| \leq \frac{8N^2B_y\sqrt{p}}{r}\sum_{k=1}^t\sigma^{t-k}, \tag{34}$$

which leads to

$$S_4 \leq \frac{8N^2B_y\sqrt{p}}{r}\sum_{t=1}^T\frac{1}{\alpha_t}\sum_{k=1}^t\sigma^{t-k} \leq \frac{16N^2B_y\sqrt{p}\sqrt{T}}{r(1-\sigma)}, \tag{35}$$

which comes from $\sum_{t=1}^T\frac{1}{\alpha_t}\sum_{k=1}^t\sigma^{t-k} \leq \sum_{t=0}^{T-1}\sigma^t\sum_{k=1}^T\frac{1}{\alpha_k}$. Similar to [19, Theorem 2], the term S_5 is bounded by

$$S_5 \leq 2NR^2\sqrt{T} + 2NRV_T, \tag{36}$$

where $V_T := \sum_{t=1}^T\sqrt{t}\sum_{i=1}^N\left\|x_{i,t+1}^* - x_{i,t}^*\right\|$. Combing (31) with (32)–(36) gives Theorem 1.

D. Proof of Theorem 2

By $\tilde{x}_{i,t} = \tilde{x}_i$, $\forall i \in \mathcal{V}$ that satisfies $\sum_{i=1}^N g_i(\tilde{x}_i) \le \mathbf{0}_p$, we have $\langle \bar{\mu}_t, \sum_{i=1}^N g_i(\tilde{x}_i)\rangle \le 0$. Based on this and $\alpha_t = \sqrt{t}$, $\eta_t = t$, summing (20) from $t = 1$ to T yields

$$\begin{aligned}\frac{N}{2}\sum_{t=1}^{T}(\|\bar{\mu}_{t+1}\|^2 - \|\bar{\mu}_t\|^2) &= \frac{N}{2}\|\bar{\mu}_{T+1}\|^2 \le (\frac{N}{2}+\frac{N}{r})TB_y^2\\ &+ \sum_{t=1}^{T}\frac{NG^2\alpha_t^2}{4\eta_t} + (2B_y+2F)\sum_{t=1}^{T}\sum_{i=1}^{N}\|\bar{\mu}_t - \lambda_{i,t+1}\|\\ &+ \sum_{t=1}^{T}\sum_{i=1}^{N}\alpha_t\partial f_{i,t}(x_{i,t})^T(\tilde{x}_{i,t}-x_{i,t})\\ &+ \sum_{t=1}^{T}\sum_{i=1}^{N}\eta_t(\|x_{i,t}-\tilde{x}_{i,t}\|^2 - \|x_{i,t+1}-\tilde{x}_{i,t}\|^2)\\ &\le (\frac{N}{2}+\frac{N}{r})TB_y^2 + \frac{NG^2T}{4} + (2B_y+2F)\frac{8N^2B_y\sqrt{p}T}{r(1-\sigma)}\\ &+ NGRT^{\frac{3}{2}} + 2TNR^2, \end{aligned} \tag{37}$$

where (34), Cauchy–Schwarz inequality, Assumption 2, (3), (36), and $\sum_{t=1}^T \alpha_t \le 1 + \int_{t=1}^T t^{1/2}dt \le T^{\frac{3}{2}}$ are used to infer the last inequality. The inequality (37) implies $\|\bar{\mu}_{T+1}\| = \mathcal{O}(T^{\frac{3}{4}})$. By substituting it into the following inequality that is derived from Lemma 2

$$\mathrm{Reg}^c(T) \le N\|\bar{\mu}_{T+1}\| + NGR\sqrt{p}, \tag{38}$$

Theorem 2 holds.

E. Proof of Theorem 3

Lemma 4: Let $\tau = \lceil\sqrt{t}\rceil$, $\delta = B_y + \epsilon$. For any $t \ge 1$,

$$\|\bar{\mu}_t\| \le 4\delta\sqrt{t} + \theta_t(\tau) + \frac{16\sqrt{t}\delta^2}{\epsilon}\log\frac{32\delta^2}{\epsilon^2} + 6B_y. \tag{39}$$

where $\theta_t(\tau) = (1+\frac{2}{r})\frac{B_y^2}{\epsilon} + \frac{G^2}{2\epsilon} + \frac{(2B_y+2F)16NB_y\sqrt{p}}{r\epsilon(1-\sigma)} + \frac{4GR\alpha_t}{\epsilon} + \frac{4R^2\eta_t}{\epsilon\tau} + (2B_y^2+\epsilon)\tau$.

Proof: We first bound the difference between $\|\bar{\mu}_{t+1}\|$ and $\|\bar{\mu}_t\|$, $\forall t \ge 1$, i.e.,

$$-B_y \le \|\bar{\mu}_{t+1}\| - \|\bar{\mu}_t\| \le B_y, \tag{40}$$

where (26) and (27) give rise to the right-hand inequality. The left-hand inequality can be obtained from $\|\bar{\mu}_t\| - \|\bar{\mu}_{t+1}\| \le \|\bar{\mu}_{t+1} - \bar{\mu}_t\| = \|\frac{1}{N}\sum_{i=1}^N \epsilon_{i,t+1}\| \le B_y$.

Let $\tilde{x}_{i,t} = \hat{x}_i$ and $\triangle_s = \frac{1}{2}\|\bar{\mu}_{s+1}\|^2 - \frac{1}{2}\|\bar{\mu}_s\|^2$. Summing (20) from $s = t, t+1, \ldots, t+\tau-1$, we have

$$\begin{aligned}\sum_{s=t}^{t+\tau-1}\triangle_s \le& (\frac{1}{2}+\frac{1}{r})B_y^2\tau + \frac{G^2\tau}{4} + \eta_{t+\tau-1}R^2 - \epsilon\sum_{s=t}^{t+\tau-1}\|\bar{\mu}_s\|\\ &+ \frac{2B_y+2F}{N}\sum_{s=t}^{t+\tau-1}\sum_{i=1}^{N}\|\bar{\mu}_s - \lambda_{i,s+1}\| + GR\sum_{s=t}^{t+\tau-1}\alpha_s, \end{aligned} \tag{41}$$

where $\eta_{t+\tau-1}R^2$ is obtained by referring to (36) and the term $-\epsilon\sum_{s=t}^{t+\tau-1}\|\bar{\mu}_s\|$ is derived based on Assumption 3. Since $1 \le \tau \le t+1$ and $V_s = \sqrt{s}$, $\sum_{s=t}^{t+\tau-1}\alpha_s \le 2\tau\alpha_t$ and $\eta_{t+\tau-1} \le 2\eta_t$. By resorting to (34) and (40),

$$\sum_{s=t}^{t+\tau-1}\sum_{i=1}^{N}\|\bar{\mu}_s - \lambda_{i,s+1}\| \le \frac{8N^2B_y\sqrt{p}}{r(1-\sigma)}\tau, \tag{42}$$

$$\sum_{s=t}^{t+\tau-1}\|\bar{\mu}_s\| \ge \sum_{s=t}^{t+\tau-1}(\|\bar{\mu}_t\| - (s-t)B_y) \ge \tau\|\bar{\mu}_t\| - \tau^2B_y, \tag{43}$$

which together with (41) results in

$$\begin{aligned}\sum_{s=t}^{t+\tau-1}\triangle_s \le& (\frac{1}{2}+\frac{1}{r})B_y^2\tau + \frac{G^2\tau}{4} + 2R^2\eta_t + 2GR\tau\alpha_t\\ &+ \frac{(2B_y+2F)8NB_y\sqrt{p}}{r(1-\sigma)}\tau + \epsilon\tau^2B_y - \epsilon\tau\|\bar{\mu}_t\|.\end{aligned}$$

This inequality implies $\|\bar{\mu}_{t+\tau}\|^2 = \|\bar{\mu}_t\|^2 + 2\sum_{s=t}^{t+\tau-1}\triangle_s \le \|\bar{\mu}_t\|^2 - 2\epsilon\tau\|\bar{\mu}_t\| + \epsilon\tau\theta_t(\tau)$ according to the definition of $\theta_t(\tau)$. Thus, if $\|\bar{\mu}_t\| \ge \theta_t(\tau)$, we have

$$\|\bar{\mu}_{t+\tau}\| - \|\bar{\mu}_t\| \le -\frac{\epsilon\tau}{2}, \forall t \ge 1. \tag{44}$$

Next we utilize (44) to bound $\|\bar{\mu}_t\|$. Consider the case $t \ge 6$. Let $\delta = B_y + \epsilon$, $\xi = \frac{\epsilon}{2}$, $\tilde{r} = \frac{\xi}{4\lceil\sqrt{t}\rceil\delta^2}$, and $\rho = 1 - \frac{\tilde{r}\xi\tau}{2}$, which implies $0 < \rho < 1$. Denote $w_t = \|\bar{\mu}_t\| - \|\bar{\mu}_{t-\tau}\|$. According to (40), $w_t = \sum_{s=t-\tau}^{t-1}\|\bar{\mu}_{s+1}\| - \|\bar{\mu}_s\| \le \tau B_y \le \tau\delta$. Like Lemma 6 in [18], $e^{\tilde{r}\|\bar{\mu}_t\|} = e^{\tilde{r}(w_t + \|\bar{\mu}_{t-\tau}\|)} \le e^{\tilde{r}\|\bar{\mu}_{t-\tau}\|}(1 + \tilde{r}w_t + \frac{1}{2}\tilde{r}\tau\xi)$. Note that $t - \tau \ge 1$, $\forall t \ge 6$. If $\|\bar{\mu}_{t-\tau}\| \ge \theta_{t-\tau}(\tau)$, we have $w_t = \|\bar{\mu}_t\| - \|\bar{\mu}_{t-\tau}\| \le -\frac{\epsilon\tau}{2} = -\xi\tau$ by (44), which implies

$$e^{\tilde{r}\|\bar{\mu}_t\|} \le \rho e^{\tilde{r}\|\bar{\mu}_{t-\tau}\|} + e^{\tilde{r}\tau\delta}e^{\tilde{r}\theta_{t-\tau}(\tau)}. \tag{45}$$

It is easy to verify that (45) holds if $\|\bar{\mu}_{t-\tau}\| < \theta_{t-\tau}(\tau)$. Moreover, $\forall t \ge 6$, $\lfloor\frac{t}{\tau}\rfloor = k$ for some $k \ge 2$. Consequently, $t - (k-2)\tau \ge 1$. Thus, we can apply (45) for $s = t, t-\tau, \ldots, t-(k-2)\tau$ to obtain

$$\begin{aligned}e^{\tilde{r}\|\bar{\mu}_t\|} &\le \rho e^{\tilde{r}\|\bar{\mu}_{t-\tau}\|} + e^{\tilde{r}\tau\delta}e^{\tilde{r}\theta_{t-\tau}(\tau)}\\ &\le \rho^{k-1}e^{\tilde{r}\|\bar{\mu}_{t-(k-1)\tau}\|} + e^{\tilde{r}\delta\tau}\sum_{i=1}^{k-1}\rho^{i-1}e^{\tilde{r}\theta_{t-i\tau}}\\ &\le \rho^{k-1}e^{2\tilde{r}\tau\delta} + e^{\tilde{r}\delta\tau}e^{\tilde{r}\theta_t}\sum_{i=1}^{k-1}\rho^{i-1} \le \frac{e^{2\tilde{r}\tau\delta}e^{\tilde{r}\theta_t}}{1-\rho},\end{aligned} \tag{46}$$

where the third inequality was resulted from: (1) $\|\bar{\mu}_{t-(k-1)\tau}\| \le (t-(k-1)\tau)B_y \le 2\tilde{r}\tau\delta$ according to (40) and $t-(k-1)\tau \le 2\tau$; (2) $0 < \theta_{t-i\tau} \le \theta_t$ because θ_t increases with t. From (46) and $\tau = \lceil\sqrt{t}\rceil \le 2\sqrt{t}$, we have

$$\begin{aligned}\|\bar{\mu}_t\| &\le 2\tau\delta + \theta_t(\lceil\sqrt{t}\rceil) + \frac{1}{\tilde{r}}\log\frac{1}{1-\rho}\\ &\le 4\delta\sqrt{t} + \theta_t(\lceil\sqrt{t}\rceil) + \frac{16\sqrt{t}\delta^2}{\epsilon}\log\frac{32\delta^2}{\epsilon^2} + 6B_y.\end{aligned} \tag{47}$$

Consider the case $t < 6$. It is straightforward to obtain $\|\bar{\mu}_t\| \le tB_y \le 6B_y$. Thus, Lemma 4 holds. ■

Since $\theta_t(\lceil\sqrt{t}\rceil) = \mathcal{O}(\sqrt{t})$ according to the definition of θ_t in Lemma 4, from (47), $\|\bar{\mu}_t\| = \mathcal{O}(\sqrt{t})$. Combing it with (38) gives $\mathrm{Reg}^c(T) = \mathcal{O}(\sqrt{T})$. Thus, Theorem 3 holds.