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Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation

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***Abstract*— We propose relaxed Lyapunov-based conditions to ensure input-to-state stability (ISS) of nonlinear time-delay systems. Their strength lies in the fact that the dissipation rate of the Lyapunov-Krasovskii functional (LKF) involves only the current value of solution's norm rather than the LKF itself. The additional requirement takes the form of a growth condition between the dissipation rate and its maximal increase along the system's solutions. We show through examples that the obtained conditions are more general than existing techniques, including the strictification method through the addition of an exponential term in the integral kernel of the LKF, whose limitations are highlighted through a counter-example.**

I. Introduction

A fundamental tool to study stability of nonlinear time-delay systems is the Lyapunov-Krasovskii approach. For input-free systems, the existence of a Lyapunov-Krasovskii functional (LKF) that dissipates in a point-wise manner along the system solutions (namely, $\dot{V} \leq -\alpha(|x(t)|)$) is enough to conclude global asymptotic stability [11], [6]. Nevertheless, when addressing input-to-state stability (ISS, [19]), a dissipation involving the whole LKF itself (namely, $\dot{V} \leq -\alpha(V) + \gamma(|u|)$) is requested [18], [3].

This requirement often complicates the analysis, as an LKF-wise dissipation is often harder to obtain than a point-wise one. This problem is often circumvented by what can be called the "exponential trick", which consists in weighting the kernel of the integral term of the LKF by a convenient exponential term. This trick has been widely used in the literature of time-delay systems: see for instance [17], [15], [5], [7], [13] and it has been shown in [16] that it always provides a LKF-wise dissipation based on a point-wise one in the case when the bounds on the LKF and the dissipation rate are all quadratic functions.

A first contribution of this paper is to extend the class of systems for which this "exponential trick" works, by replacing the quadratic requirement by the assumption that the dissipation rate is of the same order (or dominates) the term under the integral. Nevertheless, despite the popularity of this method, we also show with a counter-example that the "exponential trick" cannot be systematically employed to derive a LKF-wise dissipation based on a point-wise one.

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It has been conjectured in [4] that a point-wise dissipation could be enough to conclude ISS. To date, this conjecture remains open, although several advances have been made on this question, as reviewed in [3]: it has been proved to hold for a weaker notion known as integral input-to-state stability [1] and some additional conditions have been proposed under which ISS can indeed be derived [4], [2], [14]. In particular, in [2] some growth restrictions have been proposed to establish an exponential version of ISS (exp-ISS) based on a point-wise dissipation. The extra condition essentially imposes that a quadratic function does not increase (or decrease) faster than exponentially along the system's solutions.

Until now, this approach was restricted to exp-ISS, which significantly limits its application. In this paper, we extend it to ISS and thus allow to cover a much wider class of time-delay systems, as illustrated through academic examples. Roughly speaking, our main result states that ISS holds under point-wise dissipation if the dissipation rate dominates at infinity its maximal increase along the system's solution.

II. Preliminaries and definitions

Notation. Given $\Delta \geq 0$, we denote by $\mathscr{X} := C([-\Delta,0],\mathbb{R})$, the set of continuous functions from $[-\Delta,0]$ to $\mathbb{R}$. The symbol $\mathscr{U}$ denotes the set of Lebesgue mesurable and locally essentially bounded functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$. Given $z \in \mathbb{R}^n$, $|z|$ denotes its Euclidean norm. Given an interval $I \subset \mathbb{R}_{\geq 0}$ and $u \in \mathscr{U}$, u_I denotes the restriction of u to I and $\|u_I\| := \operatorname{ess\,sup}_{\tau \in I} |u(\tau)|$. In particular, given $\phi \in \mathscr{X}^n$, $\|\phi\| = \max_{\tau \in [-\Delta,0]} |\phi(\tau)|$. We will also make use of comparison functions: $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ if it is zero at zero, continuous, and increasing; $\alpha \in \mathscr{K}_\infty$ if $\alpha \in \mathscr{K}$ and it is unbounded; $\beta \in \mathscr{KL}$ if, for each $t \geq 0$, $\beta(\cdot,t) \in \mathscr{K}$ and for each $s \geq 0$, $\beta(s,\cdot)$ is continuous, non-increasing, and tends to zero at infinity. Given a continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}$, $\nabla W : \mathbb{R}^n \to \mathbb{R}^n$ denotes its gradient. Given a functional $V : \mathscr{X}^n \to \mathbb{R}^n$, its Driver's derivative $D^+V : \mathscr{X}^n \times \mathbb{R}^n \to [-\infty,+\infty]$ is defined for all $(\phi,w) \in \mathscr{X}^n \times \mathbb{R}^n$ as (see [3] for more details): $D^+V(\phi,w) := \limsup_{h\to 0^+} \frac{V(\phi_{h,w})-V(\phi)}{h}$, where the function $\phi_{h,w} \in \mathscr{X}^n$ is defined by

$$\phi_{h,w}(\tau) := \begin{cases} \phi(\tau+h) & \text{if } \tau \in [-\Delta,-h] \\ \phi(0)+(\tau+h)w & \text{if } \tau \in (-h,0]. \end{cases}$$

We consider time-delay systems (TDS) of the form

$$\dot{x}(t) = f(x_t,u(t)). \tag{1}$$

The history function $x_t \in \mathscr{X}^n$ is defined as $x_t(\tau) := x(t+\tau)$ for all $\tau \in [-\Delta,0]$, where $\Delta \geq 0$ is the maximum delay of the system. The input u belongs to $\mathscr{U}^m$ and the vector field

$f : \mathscr{X}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Lipschitz on bounded sets with $f(0,0) = 0$. A central property for the stability and robustness analysis of TDS is the input-to-state stability, originally introduced in [19] for finite-dimensional systems and more recently extended to TDS as reviewed in [3].

Definition 1 (ISS): The TDS (1) is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathscr{KL}$ and $\mu \in \mathscr{K}_\infty$ such that, for all $x_0 \in \mathscr{X}^n$ and all $u \in \mathscr{U}^m$, its solution satisfies

$$|x(t,x_0,u)| \leq \beta(\|x_0\|,t) + \mu(\|u_{[0,t]}\|), \quad \forall t \geq 0.$$

A powerful tool to study input-to-state stability of (1) is the Lyapunov-Krasovskii approach. We first recall the definition of a Lyapunov-Krasovskii functional candidate.

Definition 2 (LKF): A functional $V : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$ is called a *Lyapunov-Krasovskii functional candidate (LKF)* if it is Lipschitz on bounded sets and, for some $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}_\infty$,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathscr{X}^n. \tag{2}$$

It is called *coercive* if, in addition,

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathscr{X}^n. \tag{3}$$

We may consider different types of dissipation of such a LKF along the system's solutions.

Definition 3 (Point-wise/LKF-wise ISS LKF): A LKF $V : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$ is said to be:

- an *ISS LKF with LKF-wise dissipation* if there exist $\alpha, \gamma \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,
$$D^+V(\phi, f(\phi,v)) \leq -\alpha(V(\phi)) + \gamma(|v|). \tag{4}$$
- an *ISS LKF with point-wise dissipation* if there exist $\alpha, \gamma \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,
$$D^+V(\phi, f(\phi,v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{5}$$

It is known since [9] that ISS is equivalent to the existence of an ISS LKF with LKF-wise dissipation. Due to (2), it can easily be seen that a point-wise dissipation is less restrictive than a LKF-wise one, as it requires negativity only in the current value of the solution's norm. This feature is appealing in practice, as it is often easier to get such a negativity rather than imposing negativity in terms of the whole LKF as in (4). It has been conjectured in [4] that the existence of an ISS LKF with point-wise dissipation is enough to conclude ISS. However, this conjecture has not yet been proved or disproved in its full generality.

III. The Exponential Trick

A. When it works

A classical way to obtain a LKF-wise dissipation based on a point-wise one can be referred to as the "exponential trick", which is particularly useful for LKFs of the form

$$W(\phi) := w_1(\phi(0)) + \int_{-\Delta}^{0} w_2(\phi(\tau))d\tau, \quad \forall \phi \in \mathscr{X}^n, \tag{6}$$

where $w_1, w_2 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ are positive definite and radially unbounded functions. This class of LKFs is widely used in the TDS literature, but often fails at guaranteeing a LKF-wise dissipation: only a point-wise dissipation is usually obtained. In order to obtain a LKF-wise dissipation, the "exponential trick" consists in adding an exponential term within the integral, namely to consider the alternative LKF

$$\tilde{W}(\phi) := w_1(\phi(0)) + k\int_{-\Delta}^{0} e^{c\tau} w_2(\phi(\tau))d\tau, \quad \forall \phi \in \mathscr{X}^n, \tag{7}$$

for some $k, c > 0$. In [16, Lemma 1], it was shown that, in the case where w_1, w_2 are quadratic functions and the dissipation rate is itself quadratic, the fact that W is an ISS LKF with point-wise dissipation ensures that $\tilde{W}$ is an ISS LKF with LKF-wise dissipation for suitably chosen constants k and c. The following proposition, proved in Section VI-A, shows that this method actually works for a wider class of LKFs.

Proposition 1 (Exponential trick): Consider the LKF W defined in (6) for some continuously differentiable, positive definite and radially unbounded functions $w_1, w_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Assume that there exist $\alpha, \gamma \in \mathscr{K}_\infty$ such that W satisfies the point-wise dissipation estimate (5) for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$. If there exists $p > 0$ such that

$$\alpha(|x|) \geq p w_2(x), \quad \forall x \in \mathbb{R}^n, \tag{8}$$

then there exist $k, c > 0$ such that the functional $\tilde{W}$ in (7) is an ISS LKF with LKF-wise dissipation, and (1) is ISS.

Condition (8) imposes that the dissipation rate α somehow dominates the term under the integral sign in W. In the case when both these functions are quadratic, this additional condition is immediately fulfilled, meaning that Proposition 1 encompasses [16, Lemma 1] as a particular case.

B. When it does not work

In view of the success of this method, a natural question is whether the "exponential trick" constitutes a systematic way to contruct a LKF-wise dissipation based on a point-wise one. The following example gives a negative answer. It provides an LKF of the form (6) with point-wise dissipation for which, no matter how we select the constants k and c, the corresponding LKF (7) does not dissipate LKF-wise. As we will see, the considered one-dimensional time-delay system turns out to be ISS. The proof is given in Section VI-B.

Proposition 2 (Limitation on the exponential trick): Consider the scalar TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{9}$$

and the LKF W defined as

$$W(\phi) := \frac{\phi(0)^4}{4} + \int_{-1}^{0} \phi(\tau)^4 d\tau, \quad \forall \phi \in \mathscr{X}^n, \tag{10}$$

meaning (6) with $w_1(z) := z^4/4$ and $w_2(z) := z^4$ for all $z \in \mathbb{R}$. Then we have the following:

i) the TDS (9) is ISS
ii) the LKF W is an ISS LKF with point-wise dissipation
iii) given any $k, c > 0$, the corresponding LKF $\tilde{W}$ (as in (7)) is not an ISS LKF with LKF-wise dissipation.

IV. ISS FROM A POINT-WISE DISSIPATION

Motivated by the above limitation of the "exponential trick", our objective here is to provide alternative ways to ensure ISS based on a point-wise dissipation. Our main result, proved in Section VI-C, provides a growth rate condition, linking the point-wise dissipation rate to its maximal increase along the system's solutions, under which ISS indeed holds.

Theorem 1 (ISS under point-wise dissipation): Assume that there exist a functional $V : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets and satisfying $V(0) = 0$, $\alpha, \gamma \in \mathscr{K}_\infty$ and $Q \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ positive definite and radially unbounded such that, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|). \qquad (11)$$

Assume further that there exists $\sigma \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma\left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))\right) + \gamma(|v|). \qquad (12)$$

Then the TDS (1) is ISS provided that

$$\liminf_{s \to +\infty} \frac{\alpha(s)}{\sigma(se^\Delta)} > 0. \qquad (13)$$

Theorem 1 generalizes the exp-ISS result given in [2, Theorem 3], which focuses on the case when Q is quadractic and α and σ are both linear. Note that V is not required here to be a LKF, as no specific bounds on V are imposed. The left-hand side of (12) corresponds to the derivative of the function Q along the solutions of (1). The function σ thus provides an estimate on the maximal increase of Q along the system's solutions. *Per se*, it constitutes a mild requirement but, according to the growth condition (13), this function σ has to be dominated by the dissipation rate α at infinity.

Since Q is a function of $\phi(0)$ only, the dissipation inequality (11) indeed corresponds to a point-wise dissipation. But this appears more clearly in the following statement, which provides a growth condition that no longer depends on the maximal delay Δ: see [12] for the proof.

Corollary 1 (Delay-free condition): Assume that there exist a functional $V : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets with $V(0) = 0$, $\gamma \in \mathscr{K}_\infty$ and a continuously differentiable $\alpha \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \qquad (14)$$

Assume further that there exists a function $\sigma \in \mathscr{K}_\infty$ such that, for all $\phi \in \mathscr{X}^n$ with $\phi(0) \neq 0$ and all $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma(\|\phi\|) + \gamma(|v|), \qquad (15)$$

with $Q(\cdot) := \alpha(|\cdot|)$. Then the TDS (1) is ISS provided that

$$\liminf_{s \to +\infty} \frac{\alpha(s)}{\sigma(s)} > 0. \qquad (16)$$

The proofs of Theorem 1 and Corollary 1 are constructive, in the sense that we explicitly build an ISS LKF with LKF-wise dissipation. In addition, the constructed LKF turns out to be coercive, which might constitute an additional interesting feature. The proof relies on the following result, established in [8, Lemma 6.7] and recalled in [3, Lemma 2], which was already the cornerstone of [2].

Lemma 1: Given any $Q \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ and any $c > 0$, the functional $V : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$ defined as

$$V(\phi) := \max_{\tau \in [-\Delta, 0]} Q(\phi(\tau)) e^{c\tau}, \quad \forall \phi \in \mathscr{X}^n,$$

is Lipschitz on bounded sets and, given any $\phi \in \mathscr{X}^n$ and any $w \in \mathbb{R}^n$, $\dot{V} := D^+V(\phi, w)$ satisfies

$$Q(\phi(0)) < V(\phi) \;\Rightarrow\; \dot{V} \leq -cV(\phi)$$
$$Q(\phi(0)) = V(\phi) \;\Rightarrow\; \dot{V} \leq \max\{-cV(\phi), \nabla Q(\phi(0)) w\}.$$

V. ILLUSTRATIVE EXAMPLES

The following example illustrates the applicability of Theorem 1 and Proposition 1, and underlines their novelty with respect to [2, Theorem 3] and [16, Lemma 1].

Example 1: Consider the scalar TDS

$$\dot{x}(t) = -2x(t)^3 + x(t)x(t-\Delta)^2 + x(t)u(t). \qquad (17)$$

By letting $f(\phi, v) := -2\phi(0)^3 + \phi(0)\phi(-\Delta)^2 + \phi(0)v$ for all $\phi \in \mathscr{X}$ and all $v \in \mathbb{R}$, and considering the LKF W defined in (6) with $w_1(z) := z^2$ and $w_2(z) := z^4$ for all $z \in \mathbb{R}$, we have:

$$\begin{aligned}
&D^+W(\phi, f(\phi, v)) \\
&= -3\phi(0)^4 + 2\phi(0)^2\phi(-\Delta)^2 + 2\phi(0)^2 v - \phi(-\Delta)^4 \\
&\leq -3\phi(0)^4 + \phi(0)^4 + \phi(-\Delta)^4 + \phi(0)^4 + v^2 - \phi(-\Delta)^4 \\
&\leq -\phi(0)^4 + v^2 \\
&\leq -\alpha(Q(\phi(0))) + \gamma(|v|),
\end{aligned}$$

where $\alpha(s) = \gamma(s) := s^2$ for all $s \geq 0$ and $Q(z) := z^2$ for all $z \in \mathbb{R}$. Clearly, W is not an ISS LKF with LKF-wise dissipation, yet it is indeed an ISS LKF with point-wise dissipation. We can apply Theorem 1 to establish ISS by noticing that

$$\begin{aligned}
\nabla Q(\phi(0)) f(\phi, v) &= -4\phi(0)^4 + 2\phi(0)^2\phi(-\Delta)^2 + 2\phi(0)^2 v \\
&\leq -4\phi(0)^4 + \phi(0)^4 + \phi(-\Delta)^4 + \phi(0)^4 + v^2 \\
&\leq \sigma\left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))\right) + \gamma(|v|),
\end{aligned}$$

with $\sigma(s) := s^2$. With these functions, it holds that $\lim_{s \to +\infty} \frac{\alpha(s)}{\sigma(se^\Delta)} = e^{-2\Delta} > 0$, making the growth condition (13) fulfilled. Thus, by Theorem 1, the system (17) is ISS. Notice that this system is not exponentially ISS, meaning that [2, Theorem 3] cannot be applied. To see this more clearly, exponential ISS would imply that the input-free system $\dot{x}(t) = -2x(t)^3 + x(t)x(t-\Delta)$ is globally exponentially stable. However, considering the initial state defined as $x_0(t) = a$ for all $t \in [-\Delta, 0]$ for some $a > 0$, it can be seen that $x(t) \geq 0$ for all $t \geq 0$ (since $\dot{x}(t) = 0$ whenever $x(t) = 0$) and therefore $\dot{x}(t) \geq -2x(t)^3$. Invoking the comparison lemma, we see that $x(\cdot)$ does not converge exponentially to zero. Another way to see this is to rely on [3, Theorem 10] by noticing that the Fréchet derivative of the vector field vanishes at zero.

It is fair to note that the "exponential trick", as formalized in Proposition 1, can also be used to conclude ISS for this system, by noticing that $\alpha(Q(z)) = w_2(z)$, which makes (8) fulfilled. Nevertheless, [16, Lemma 1] cannot be invoked here as w_2 is not a quadratic function. △

Existing works, including [4, Theorem 8], have already provided sufficient conditions under which the existence of an ISS LKF with point-wise dissipation is enough to conclude ISS. The next example shows that our main results apply to a wider class of systems.

Example 2: Consider the bi-dimensional TDS

$$\dot{x}_1(t) = -2x_1(t) + x_2(t-\Delta)^3 + x_2(t)^3 + u(t) \tag{18a}$$

$$\dot{x}_2(t) = -x_2(t) - x_2(t)^9 - x_1(t)^3. \tag{18b}$$

Let W be the LKF defined in (6) with $w_1(z) := \frac{1}{4}(z_1^4 + z_2^4)$ and $w_2(z) := z_2^{12}$ for all $z = (z_1, z_2) \in \mathbb{R}^2$, namely:

$$W(\phi) = \frac{1}{4}\phi_1(0)^4 + \frac{1}{4}\phi_2(0)^4 + \int_{-\Delta}^{0} \phi_2(\tau)^{12} d\tau. \tag{19}$$

By letting $f(\phi, v) := (-2\phi_1(0) + \phi_2(-\Delta)^3 + \phi_2(0)^3 + v, -\phi_2(0) - \phi_2(0)^9 - \phi_1(0)^3)^\top$ for all $\phi = (\phi_1, \phi_2)^\top \in \mathscr{X}^2$ and all $v \in \mathbb{R}$, and $\dot{W} := D^+W(\phi, f(\phi, v))$ for short, we have

$$\begin{aligned}\dot{W} = &-2\phi_1(0)^4 + \phi_1(0)^3\phi_2(-\Delta)^3 + \phi_1(0)^3\phi_2(0)^3 + \phi_1(0)^3 v \\ &- \phi_2(0)^4 - \phi_2(0)^{12} - \phi_2(0)^3\phi_1(0)^3 + \phi_2(0)^{12} - \phi_2(-\Delta)^{12}.\end{aligned} \tag{20}$$

Using Young's inequality, it follows that

$$\begin{aligned}\dot{W} \leq &-2\phi_1(0)^4 + \frac{3}{4}\phi_1(0)^4 + \frac{1}{4}\phi_2(-\Delta)^{12} + \frac{3}{4}\phi_1(0)^4 + \frac{v^4}{4} \\ &-\phi_2(0)^4 - \phi_2(-\Delta)^{12} \\ \leq &-\alpha(Q(\phi(0))) + \gamma(|v|),\end{aligned} \tag{21}$$

where $\alpha(s) := s^2/2$, $\gamma(s) = s^2/2 + s^4/4$ for all $s \geq 0$ and $Q(z) = |z|^2/2$ for all $z \in \mathbb{R}^2$. Moreover, letting $\dot{Q} := \nabla Q(\phi(0)) f(\phi, v)$ for compactness, it holds that

$$\begin{aligned}\dot{Q} = &-2\phi_1(0)^2 + \phi_1(0)\phi_2(-\Delta)^3 + \phi_1(0)\phi_2(0)^3 + \phi_1(0)v \\ &-\phi_2(0)^2 - \phi_2(0)^{10} - \phi_2(0)\phi_1(0)^3 \\ \leq &-2\phi_1(0)^2 + \frac{1}{4}\phi_1(0)^4 + \frac{3}{4}\phi_2(-\Delta)^4 + \frac{1}{4}\phi_1(0)^4 + \frac{3}{4}\phi_2(0)^4 \\ &+ \frac{1}{2}(\phi_1(0)^2 + v^2) + \frac{1}{4}\phi_2(0)^4 + \frac{3}{4}\phi_1(0)^4 \\ \leq &\frac{5}{4}(\phi_1(0)^4 + \phi_2(0)^4 + \phi_2(-\Delta)^4) + \frac{v^2}{2} \\ \leq &\sigma\left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))\right) + \gamma(|v|),\end{aligned}$$

where $\sigma(s) := 10s^2$. Thus, $\liminf_{s \to +\infty} \alpha(s)/\sigma(e^\Delta s) = e^{-2\Delta}/20 > 0$ and the growth condition (13) is fulfilled. Combining this with (21), we conclude with Theorem 1 that (18) is ISS.

It turns out that the LKF W cannot be used to conclude ISS of system (18) using [4, Theorem 8]. Indeed, for any $\phi \in \mathscr{X}^2$, it holds from (19) that

$$\frac{1}{8}|\phi(0)|^4 \leq W(\phi) \leq \frac{1}{4}|\phi(0)|^4 + \int_{-\Delta}^{0} \alpha_1(|\phi(\tau)|) d\tau$$

with $\alpha_1(s) = s^{12}$ for all $s \geq 0$ and any such upper bound would involve a function α_1 under the integral sign which is at least of order s^{12} at infinity. As pointed out in [3, Theorem 25], a necessary condition to apply [4, Theorem 8] is to have

$$\liminf_{s \to +\infty} \frac{\alpha(s)}{\alpha_1(s)} > 0. \tag{22}$$

But, in view of (20), we see that this is not possible as $\alpha(s)$ is at most of order s^4. To see this more clearly, consider $v = 0$ and any $\varphi = (\varphi_1, \varphi_2)^\top \in \mathscr{X}^2$ satisfying $\varphi_2(-\Delta) = \varphi_1(0)^{1/3}$. Then it holds from (20) that $\dot{W} = -2\varphi_1(0)^4 - \varphi_2(0)^4 \geq -2|\varphi(0)|^4$. Since $\varphi_1(0)$ and $\varphi_2(0)$ are arbitrary, this shows that any ISS point-wise dissipation rate indeed necessarily satisfies $\alpha(s) \leq 2s^4$ for all $s \geq 0$, which in turn violates (22) and makes [4, Theorem 8] inapplicable. △

VI. Proofs

A. Proof of Proposition 1

For brevity, given $\phi \in \mathscr{X}^n$ and $v \in \mathbb{R}^m$, we let $\dot{W} := D^+W(\phi, f(\phi, v))$ and $\dot{\tilde{W}} := D^+\tilde{W}(\phi, f(\phi, v))$. Proceeding as in [3, Example 1], the Driver derivative of W and $\tilde{W}$ along the solutions of (1) reads, for all $\phi \in \mathscr{X}^n$ and all $v \in \mathbb{R}^m$,

$$\begin{aligned}\dot{W} &= \nabla w_1(\phi(0)) f + w_2(\phi(0)) - w_2(\phi(-\Delta)), \\ \dot{\tilde{W}} &= \nabla w_1(\phi(0)) f + k w_2(\phi(0)) - k e^{-\Delta c} w_2(\phi(-\Delta)) - \mathscr{I}(\phi),\end{aligned}$$

where $\mathscr{I}(\phi) := kc \int_{-\Delta}^{0} e^{c\tau} w_2(\phi(\tau)) d\tau$ and some arguments were omitted. Combining these two expressions, we get that $\dot{\tilde{W}} = \dot{W} - (1-k)w_2(\phi(0)) - (ke^{-\Delta c} - 1)w_2(\phi(-\Delta)) - \mathscr{I}(\phi)$. Using the point-wise dissipation estimate (5), it follows that

$$\begin{aligned}\dot{\tilde{W}} \leq &-\alpha(|\phi(0)|) - (1-k)w_2(\phi(0)) \\ &-(ke^{-\Delta c} - 1)w_2(\phi(-\Delta)) - \mathscr{I}(\phi) + \gamma(|v|).\end{aligned}$$

Using condition (8), we obtain that $\dot{\tilde{W}} \leq -(p+1-k)w_2(\phi(0)) - (ke^{-\Delta c} - 1)w_2(\phi(-\Delta)) - \mathscr{I}(\phi) + \gamma(|v|)$. Consider any $c > 0$ such that $e^{\Delta c} < 1 + p$. Then, for any $k \in (e^{\Delta c}, 1+p)$, we have that $\overline{p} := p + 1 - k > 0$ and $ke^{-\Delta c} - 1 > 0$. Consequently, it holds that $\dot{\tilde{W}} \leq -\overline{p} w_2(\phi(0)) - \mathscr{I}(\phi) + \gamma(|v|)$. As w_1 and w_2 are positive definite and radially unbounded, there exists $\alpha_0 \in \mathscr{K}_\infty$ such that $w_2(\cdot) \geq \alpha_0(w_1(\cdot))$. Hence

$$\begin{aligned}\dot{\tilde{W}} &\leq -\overline{p}\alpha_0(w_1(\phi(0))) - \mathscr{I}(\phi) + \gamma(|v|) \\ &\leq -\overline{\alpha}_0(w_1(\phi(0))) - \overline{\alpha}_0\left(k\int_{-\Delta}^{0} e^{c\tau} w_2(\phi(\tau)) d\tau\right) + \gamma(|v|),\end{aligned}$$

where $\overline{\alpha}_0(s) := \min\{\overline{p}\alpha_0(s), cs\}$. Since $\overline{\alpha}_0 \in \mathscr{K}_\infty$, [10, Lemma 9] ensures that $\overline{\alpha}_0(2r) + \overline{\alpha}_0(2s) \geq \overline{\alpha}_0(r+s)$ for all $r, s \geq 0$. Thus, we get from (7) that $\dot{\tilde{W}} \leq -\overline{\alpha}_0(\tilde{W}(\phi)/2) + \gamma(|v|)$, meaning that $\tilde{W}$ is indeed an ISS LKF with LKF-wise dissipation. ISS then follows from [18, Theorem 3.4].

B. Proof of Proposition 2

For all $\phi \in \mathscr{X}$ and all $v \in \mathbb{R}$, let

$$f(\phi, v) := -\phi(0) - \frac{\phi(0)}{1+\phi(0)^2} + \frac{\phi(-1)^4}{1+|\phi(0)|^3} + \frac{v}{1+\phi(0)^2}.$$

Then f is Lipschitz on bounded sets and satisfies $f(0,0)=0$. We proceed to the proof of the three items of the statement.

i) System (9) *is ISS.* We rely on the Razumikhin approach for ISS, as recalled in [3, Theorem 26]. To that aim, consider the function $V_0 \in C^1(\mathbb{R}, \mathbb{R}_{\geq 0})$ defined as $V_0(z) := \frac{1}{4}z^4$, for all $z \in \mathbb{R}$. Then V_0 is positive definite and radially unbounded and, for all $\phi \in \mathscr{X}$ and all $v \in \mathbb{R}$, it holds that

$$\begin{aligned}
&\dot{V}_0 := \nabla V_0(\phi(0)) f(\phi, v) \\
&= -\phi(0)^4 - \frac{\phi(0)^4}{1+\phi(0)^2} + \frac{\phi(0)^3\phi(-1)^4}{1+|\phi(0)|^3} + \frac{\phi(0)^3 v}{1+\phi(0)^2} \\
&\leq -\phi(0)^4\left(1+\frac{1/4}{1+\phi(0)^2}\right) + \phi(-1)^4 + \frac{v^4/4}{1+\phi(0)^2}. \quad (23)
\end{aligned}$$

Let us define $\rho \in \mathscr{K}_\infty$ by the inverse of the $\mathscr{K}_\infty$ function ρ^{-1} defined as $\rho^{-1}(s) := s\left(1+\frac{1/8}{1+2\sqrt{s}}\right)$ for all $s \geq 0$. Let us also consider the function $\gamma \in \mathscr{K}_\infty$ defined as $\gamma(s) := s^4$, for all $s \geq 0$. By setting $\|V_0(\phi(\cdot))\| := \max_{\tau \in [-\Delta, 0]} V_0(\phi(\tau))$, the following implications hold:

$$\begin{aligned}
V_0(\phi(0)) \geq \rho\left(\|V_0(\phi(\cdot))\|\right) &\Rightarrow \frac{\phi(-1)^4}{4} \leq \rho^{-1}\left(V_0(\phi(0))\right) \\
V_0(\phi(0)) \geq \gamma(|v|) &\Rightarrow v^4 \leq V_0(\phi(0)).
\end{aligned}$$

For all $\phi \in \mathscr{X}$ for which $V_0(\phi(0)) \geq \max\{\rho\left(\|V_0(\phi(\cdot))\|\right), \gamma(|v|)\}$, it follows from (23) that

$$\begin{aligned}
\dot{V}_0 &\leq -\phi(0)^4\left(1+\frac{1/4}{1+\phi(0)^2}\right) + 4\rho^{-1}(V_0) + \frac{V_0/4}{1+\phi(0)^2} \\
&\leq -\phi(0)^4\left(1+\frac{1/4}{1+\phi(0)^2}\right) + \phi(0)^4\left(1+\frac{1/8}{1+\phi(0)^2}\right) \\
&\quad + \frac{\phi(0)^4/16}{1+\phi(0)^2} \leq -\frac{\phi(0)^4/16}{1+\phi(0)^2}.
\end{aligned}$$

Since $\rho^{-1}(s) > s$, it also holds that $\rho(s) < s$ for all $s > 0$. ISS then follows from [3, Theorem 26] or [20, Theorem 1].

ii) W dissipates point-wisely. In view of (10), it holds that

$$\begin{aligned}
&D^+W(\phi, f(\phi, v)) \\
&= -\frac{\phi(0)^4}{1+\phi(0)^2} - \phi(-1)^4\left(1-\frac{\phi(0)^3}{1+|\phi(0)|^3}\right) + \frac{v\phi(0)^3}{1+\phi(0)^2} \\
&\leq -\frac{\phi(0)^4}{1+\phi(0)^2} + \frac{1}{1+\phi(0)^2}\left(\frac{3}{4}\phi(0)^4 + \frac{1}{4}v^4\right) \\
&\leq -\frac{\phi(0)^4}{4(1+\phi(0)^2)} + \frac{1}{4}v^4 \leq -\alpha(|\phi(0)|) + \gamma(|v|),
\end{aligned}$$

where $\alpha(s) := s^4/4(1+s^2)$ and $\gamma(s) := s^4/4$ for all $s \geq 0$. Observing that $\alpha, \gamma \in \mathscr{K}_\infty$, we conclude that W is an ISS LKF with point-wise dissipation.

iii) $\tilde{W}$ does not dissipate LKF-wisely. In the absence of an input (meaning $u \equiv 0$), the TDS (9) reads

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3}. \quad (24)$$

The LKF $\tilde{W}$ resulting from the "exponential trick" is $\tilde{W}(\phi) := \frac{\phi(0)^4}{4} + k\int_{-1}^{0} e^{c\tau}\phi(\tau)^4 d\tau$, with $k, c > 0$. For all $\phi \in \mathscr{X}$, its derivative $\dot{\tilde{W}} := D^+\tilde{W}(\phi, f(\phi, 0))$ along (24) reads

$$\begin{aligned}
\dot{\tilde{W}} &= -\phi(0)^4 - \frac{\phi(0)^4}{1+\phi(0)^2} + \frac{\phi(0)^3\phi(-1)^4}{1+|\phi(0)|^3} \\
&\quad + k\phi(0)^4 - ke^{-c}\phi(-1)^4 - kc\int_{-1}^{0} e^{c\tau}\phi(\tau)^4 d\tau \\
&= -\phi(0)^4\left(1+\frac{1}{1+\phi(0)^2} - k\right) \quad (25) \\
&\quad - \phi(-1)^4\left(-ke^{-c} - \frac{\phi(0)^3}{1+|\phi(0)|^3}\right) - kc\int_{-1}^{0} e^{c\tau}\phi(\tau)^4 d\tau.
\end{aligned}$$

We see from this expression that k needs to be smaller than 1 to get the first term negative. In that case, for $|\phi(0)|$ large enough, the second term becomes positive and cannot be compensated by the third term if $\phi(\cdot)^4$ happens to have a small integral over $[-1,0]$. Hence, $\tilde{W}$ does not decrease everywhere along the solutions of the input-free system, no matter how k and c are chosen. This is the intuition behind the proof, which is fully provided in [12].

C. Proof of Theorem 1

We will make use of the following observation, whose proof is omitted due to space constraints.

Lemma 2: Let $\eta, \mu \in \mathscr{K}_\infty$ be such that $\liminf_{s \to +\infty} \frac{\mu(s)}{\eta(s)} > 0$. Then, there exists a positive definite, non-decreasing, continuous and bounded function $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\omega(s) \leq \frac{\mu(s)}{\eta(s)}$ for all $s \geq 0$.

Since $V(0) = 0$ and V is Lipschitz on bounded sets, it can be checked that

$$V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathscr{X}^n, \quad (26)$$

for some $\overline{\alpha} \in \mathscr{K}_\infty$. Consider the functional defined for all $\phi \in \mathscr{X}^n$ as $V_1(\phi) := \max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))e^{\tau}$. Then V_1 is Lipschitz on bounded sets by Lemma 1 and

$$e^{-\Delta}\|Q(\phi(\cdot))\| \leq V_1(\phi) \leq \|Q(\phi(\cdot))\|, \quad (27)$$

where $\|Q(\phi(\cdot))\| := \max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))$. By (13) and Lemma 2, there exists a continuous, positive definite, nondecreasing and bounded function ω_0 such that

$$\frac{\alpha(s)}{2\sigma(e^{\Delta}s)} \geq \omega_0(s), \quad \forall s \geq 0. \quad (28)$$

Let $\omega \in \mathscr{K}_\infty$ be defined as $\omega(s) := \int_0^s \omega_0(r)dr$ for all $s \geq 0$, and define $\tilde{V} : \mathscr{X}^n \to \mathbb{R}_{\geq 0}$ as $\tilde{V}(\phi) := V(\phi) + \omega(V_1(\phi))$. Since ω is continuously differentiable, $\tilde{V}$ is Lipschitz on bounded sets. By (26)-(27), we have that

$$\omega\left(e^{-\Delta}\|Q(\phi(\cdot))\|\right) \leq \tilde{V}(\phi) \leq \bar{\alpha}(\|\phi\|) + \omega\left(\|Q(\phi(\cdot))\|\right).$$

In addition, since Q is continuous, positive definite and radially unbounded, there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ such that $\alpha_1(|z|) \leq Q(z) \leq \alpha_2(|z|)$ for all $z \in \mathbb{R}^n$, which ensures in particular that

$$\alpha_1(\|\phi\|) \leq \|Q(\phi(\cdot))\| \leq \alpha_2(\|\phi\|), \quad \forall \phi \in \mathscr{X}^n. \quad (29)$$

Combining the two above bounds, we get that

$$\omega\left(e^{-\Delta}\alpha_1(\|\phi\|)\right) \leq \tilde{V}(\phi) \leq \bar{\alpha}(\|\phi\|)+\omega\left(\alpha_2(\|\phi\|)\right). \quad (30)$$

Thus, $\tilde{V}$ is a coercive LKF. Using (11) and [7, Lemma 7],

$$\dot{\tilde{V}} \leq -\alpha(Q(\phi(0)))+\gamma(|v|)+\omega_0(V_1)\dot{V}_1, \quad (31)$$

where $\dot{\tilde{V}} := D^+\tilde{V}(\phi, f(\phi,v))$ and $\dot{V}_1 := D^+V_1(\phi, f(\phi,v))$. From Lemma 1, V_1 satisfies

$$V_1(\phi) > Q(\phi(0)) \ \Rightarrow\ \dot{V}_1 \leq -V_1(\phi) \quad (32)$$
$$V_1(\phi) = Q(\phi(0)) \ \Rightarrow\ \dot{V}_1 \leq \max\left\{-V_1(\phi), \dot{Q}\right\}, \quad (33)$$

with $\dot{Q} := \nabla Q(\phi(0)) f(\phi,v)$. Hence, we consider two cases. Case 1: $V_1(\phi) > Q(\phi(0))$. Then we have from (31) -(32) that

$$\begin{aligned}\dot{\tilde{V}} &\leq -\alpha(Q(\phi(0)))+\gamma(|v|)-\omega_0(V_1(\phi))V_1(\phi)\\ &\leq -\omega_0(V_1(\phi))V_1(\phi)+\gamma(|v|).\end{aligned}$$

Using (27) and (29), it follows that

$$\begin{aligned}\dot{\tilde{V}} &\leq -\omega_0\left(e^{-\Delta}\max_{\tau\in[-\Delta,0]}Q(\phi(\tau))\right)e^{-\Delta}\max_{\tau\in[-\Delta,0]}Q(\phi(\tau))+\gamma(|v|)\\ &\leq -\omega_0\left(e^{-\Delta}\alpha_1(\|\phi\|)\right)e^{-\Delta}\alpha_1(\|\phi\|)+\gamma(|v|). \quad (34)\end{aligned}$$

Case 2: $V_1(\phi) = Q(\phi(0))$. Then we have from (33) and (31):

$$\dot{\tilde{V}} \leq -\alpha(Q(\phi(0)))+\gamma(|v|+\omega_0(V_1(\phi))\max\{-V_1(\phi),\dot{Q}\}.$$

If $-V_1(\phi) \geq \dot{Q}$ then, proceeding as in Case 1, we get that

$$\begin{aligned}\dot{\tilde{V}} &\leq -\alpha(Q(\phi(0)))+\gamma(|v|)-\omega_0(V_1(\phi))V_1(\phi)\\ &\leq -\omega_0\left(e^{-\Delta}\alpha_1(\|\phi\|)\right)e^{-\Delta}\alpha_1(\|\phi\|)+\gamma(|v|). \quad (35)\end{aligned}$$

On the other hand, if $-V_1(\phi) < \dot{Q}$, we get from (12) that

$$\begin{aligned}\dot{\tilde{V}} \leq& -\alpha(Q(\phi(0)))+\gamma(|v|)+\omega_0(V_1(\phi))\dot{Q}\\ \leq& -\alpha(V_1(\phi))+\omega_0(V_1(\phi))\sigma\left(\|Q(\phi(\cdot))\|\right)\\ &+\left(\omega_0(V_1(\phi))+1\right)\gamma(|v|).\end{aligned}$$

Since ω_0 is bounded, there exists $\bar{\omega}_0 > 0$ such that $\omega_0(\cdot) \leq \bar{\omega}_0$. Using also (27), we obtain that $\dot{\tilde{V}} \leq -\alpha(V_1(\phi)) + \omega_0(V_1(\phi))\sigma\left(e^{\Delta}V_1(\phi)\right)+(\bar{\omega}_0+1)\gamma(|v|)$. Using successively (28), (27) and (29), it follows that

$$\begin{aligned}\dot{\tilde{V}} &\leq -\frac{1}{2}\alpha(V_1(\phi))+(\bar{\omega}_0+1)\gamma(|v|)\\ &\leq -\frac{1}{2}\alpha\left(e^{-\Delta}\max_{\tau\in[-\Delta,0]}Q(\phi(\tau))\right)+(\bar{\omega}_0+1)\gamma(|v|)\\ &\leq -\frac{1}{2}\alpha\left(e^{-\Delta}\alpha_1(\|\phi\|)\right)+(\bar{\omega}_0+1)\gamma(|v|). \quad (36)\end{aligned}$$

Combining (34), (35) and (36), we obtain that $\dot{\tilde{V}} \leq -\tilde{\alpha}(\|\phi\|)+\tilde{\gamma}(|v|)$, with the $\mathcal{K}_\infty$ functions $\tilde{\alpha}(s) := \min\left\{\omega_0\left(e^{-\Delta}\alpha_1(s)\right)e^{-\Delta}\alpha_1(s), \frac{1}{2}\alpha(e^{-\Delta}\alpha_1(s))\right\}$ and $\tilde{\gamma}(s) := (\bar{\omega}_0+1)\gamma(s)$. In view of (30), we finally get that $\dot{\tilde{V}} \leq -\tilde{\alpha}\circ(\bar{\alpha}+\omega\circ\alpha_2)^{-1}(\tilde{V}(\phi))+\tilde{\gamma}(|v|)$, meaning that $\tilde{V}$ is a coercive ISS LKF with LKF-wise dissipation.

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