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Open-Loop Saddle Points for Irregular Linear-Quadratic Two-Person Zero-Sum Games

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Abstract— **In this paper, we consider the feedback representation of open-loop saddle points for irregular linear-quadratic (LQ) two-person zero-sum games, where the control weighting matrices in the cost functional are only semidefinite. The existence of an open-loop saddle point is characterized by the solvability of a system of constrained linear forward-backward differential equations (FBDEs), together with a convexity-concavity condition. In classical zero-sum games, the feedback representation is obtained by decoupling the FBDEs through the regular solution to a Riccati differential equation. But the associated Riccati equation cannot be used to decouple the FBDEs to obtain the feedback representation of open-loop saddle points. The essential differences between regular and irregular LQ zero-sum games are investigated. The irregular feedback representation can be derived from two equilibrium conditions in two different layers by using the "two-layer optimization" approach.**

I. Introduction

This paper is concerned with the solution to a class of singular two-person zero-sum linear quadratic (LQ) differential games. Singular games have received sustained attention due to the widely application, including pursuit-evasion games [5], robust interception problems of maneuvering targets [9], robust trajectory tracking problems [6], and robust investment problems [7]. The zero-sum games has been solved when the weighting matrices in the cost functional related to the control of all players are definite. The main approaches include the Isaacs MinMax principle [12] and the Bellman–Isaacs equation method [13]. The open-loop and closed-loop saddle points for two-person LQ games are studied in [14], [15]. The existence of an open-loop saddle point is characterized by the solvability of a system of constrained linear forward-backward differential equations (FBDEs), together with a convexity-concavity condition. Since the weighting matrices in the cost functional related to the control are definite, the associated Riccati equation admits an unique solution such that the regular condition holds. Thus the feedback representation of the open-loop saddle point can be derived from the equilibrium condition by using the regular Riccati equation to decouple the FBDEs. Due to the importance of the regularity, we call this case as the regular two-person zero-sum game.

But in singular two-person zero-sum games, the regular condition does not hold even if the associated Riccati equation admits solutions. The associated Riccati equation cannot be used to decouple the FBDEs to obtain the feedback representation of open-loop saddle points. This is the main difficulty of singular games. In this paper we term this case of singular zero-sum games as *irregular* zero-sum games. Irregular two-person games have been extensively studied in the literature by different approaches. The regularization approach is used in [16]–[18]. The singular perturbation technique is applied to transform the irregular game to a new regular game with a partial cheap control. The saddle point is obtained by using the limitation of the solution to Riccati equation when the perturbation is approaching to zero. The second approach is the "transformation in state space" [8]. A regular differential game is obtained but an initial impulse control must be applied. In [19], [20], the higher-order optimality conditions are considered. However, the higher-order optimality conditions do not provide a candidate optimal control for the game, having no solution in the class of regular functions. In summary, although irregular games has been studied for more than 50 years, some fundamental problems still remain to be solved. The controllers are not analytical or cannot be derived in $L^2(t_0, T; \mathcal{R}^m)$, where $L^2(t_0, T; \mathcal{R}^m)$ is the space of the $\mathcal{R}^m$-valued, square integrable functions over (t_0, T). Recently, a new "two-layer optimization" approach is proposed to solve the irregular LQ optimal control problem in [1]–[3]. For any initial value, the optimal controller can be derived in $L^2(t_0, T; \mathcal{R}^m)$ in two different layers without the initial impulse control. Thus the controller is analytic and implementable.

In this paper, we study the irregular two-person zero-sum LQ differential games. By variational analysis, the existence of an open-loop saddle point is characterized by the solvability of a system of FBDEs. Inspired by the irregular optimal control, the "two-layer optimization" approach is used to further tackle the FBDEs. A new system of FBDEs characterizing the existence of open-loop saddle points is obtained. The irregular zero-sum game admits open-loop saddle points if and only if the new system of FBDEs is controllable. Moreover, the feedback representation can be divided into two parts designed in two different layers. The first part of feedback representation can be derived from a new equilibrium condition based on the solution to two

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Riccati equations in the first layer. The second part can be obtained by solving a controllability problem for one specific system in the second layer such that the final value of the system is zero under the controller. Compared with the existing works following the singular perturbation approach [16]–[18] and the "transformation in state space" approach [8], the designed controller can be derived in $L^2(t_0, T; \mathcal{R}^m)$ for arbitrary initial values. Therefore the controller is analytic and implementable.

The remainder of this paper is organized as follows. Section II introduces the irregular LQ two-person zero-sum games. Section III reviews the standard LQ zero-sum games and shows the difficulty of irregular zero-sum games. Section IV presents the solution to the irregular zero-sum games. A numerical example is given in Section V. Section VI concludes this paper.

Through the paper, for a vector or matrix M, M^T denotes its transpose, $M^\dagger$ denotes its (Moore-Penrose) pseudo-inverse, and $\mathcal{R}(M)$ denotes its range. We use $0_{n\times m}$ to denote the $n \times m$ zero matrix , $*_{n\times m}$ to denote the $n \times m$ non-zero irrelevant term, and I_n to denote the n-dimensional identity matrix.

II. Problem formulation

Consider a two-person zero-sum games where the linear system is given by the following differential equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t), \\ x(t_0) = x_0, \quad t \in [t_0, T], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of this system with the initial state x_0 at the initial time t_0; for $i = 1, 2$, $u_i(t) \in \mathbb{R}^{m_i}$ is the control of player $\mathcal{A}_i$. We assume that $A(\cdot), B_1(\cdot), B_2(\cdot)$ are deterministic matrix-valued functions with appropriate dimensions. To measure the performance of the controls $u_1(\cdot)$ and $u_2(\cdot)$, we introduce the following performance functional:

$$\begin{aligned} &J_T(t_0, x_0; u_1(\cdot), u_2(\cdot)) \\ =& \int_{t_0}^{T} [x^T(t)Q(t)x(t) + u_1^T(t)R_1(t)u_1(t) \\ &+ u_2^T(t)R_2(t)u_2(t)]dt + x^T(T)Hx(T), \end{aligned} \tag{2}$$

where $Q(t), R_1(t) \geq 0, R_2(t) \leq 0, H$ are deterministic symmetric matrices with appropriate dimensions. We assume that the performance functional (2) is a cost functional for player $\mathcal{A}_1$ and a payoff functional for player $\mathcal{A}_2$. Therefore, player $\mathcal{A}_1$ wishes to minimize (2) by selecting a control process $u_1(\cdot)$, while player $\mathcal{A}_2$ wishes to maximize (2) by selecting a control process $u_2(\cdot)$. The above described problem is referred to as a LQ two-person zero-sum game.

For notational simplicity, we let $m = m_1 + m_2$ and denote $B(\cdot) \triangleq \begin{bmatrix} B_1(\cdot) & B_2(\cdot) \end{bmatrix}$, and

$$u(\cdot) \triangleq \begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix}, \quad R(\cdot) \triangleq \begin{bmatrix} R_1(\cdot) & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & R_2(\cdot) \end{bmatrix}.$$

The associated Riccati equation for the state equation (1) with the performance functional (2) is given by

$$\begin{aligned} 0_{n\times n} =& \dot{P}(t) + P(t)A(t) + A^T(t)P(t) + Q(t) \\ &- P(t)B(t)R^\dagger(t)B^T(t)P(t), \quad P(T) = H. \end{aligned} \tag{3}$$

We introduce the following definitions.

Definition 1: The performance functional (2) is called regular if the associated Riccati equation (3) admits solution $P(\cdot)$ such that the regular condition

$$\mathcal{R}(B^T(t)P(t)) \subseteq \mathcal{R}(R(t)). \tag{4}$$

holds, otherwise it is called irregular if

$$\mathcal{R}(B^T(t)P(t)) \not\subseteq \mathcal{R}(R(t)). \tag{5}$$

Definition 2: For the system (1) with the performance functional (2), a control pair $(u_1^*(\cdot), u_2^*(\cdot))$ is called a saddle point for the initial pair $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ if

$$\begin{aligned} J_T(t_0, x_0; u_1^*(\cdot), u_2(\cdot)) &\leq J_T(t_0, x_0; u_1^*(\cdot), u_2^*(\cdot)) \\ &\leq J_T(t_0, x_0; u_1(\cdot), u_2^*(\cdot)). \end{aligned} \tag{6}$$

In this paper, we mainly study the following problem.

Problem 1 (P1): For the system (1) with the performance functional (2) and any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, find an open-loop saddle point $(u_1^*(\cdot), u_2^*(\cdot))$ such that (6) holds.

III. Comparison between regular and irregular zero-sum games

In this section, we will show that the difficulty of singular games is caused by the so-called irregularity. By the variational method, we have the following results for Problem (P1).

Lemma 1: Let $x^*(\cdot)$ be the state process under the controls $(u_1^*(\cdot), u_2^*(\cdot))$. Then the pair $(u_1^*(\cdot), u_2^*(\cdot))$ is an open-loop saddle point for the system (1) with the performance functional (2) if and only if the following equilibrium condition holds for $t \in [t_0, T]$:

$$0 = R(t)u^*(t) + B^T(t)p^*(t), \tag{7}$$

where $(x^*(\cdot), p^*(\cdot))$ is the solution to the following system of FBDEs:

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t), \quad x^*(t_0) = x_0, \tag{8a}$$

$$\dot{p}^*(t) = -A^T(t)p^*(t) - Q(t)x^*(t), \ p^*(T) = Hx^*(T), \tag{8b}$$

and the following convexity-concavity condition holds:

$$\begin{aligned} J_T(t_0, 0; u_1(\cdot), 0) =& \int_{t_0}^{T} \big[\langle Q(t)x_1(t), x_1(t)\rangle \\ &+ \langle R_1(t)u_1(t), u_1(t)\rangle\big]dt + \langle Hx_1(T), x_1(T)\rangle \geq 0, \end{aligned} \tag{9a}$$

$$\begin{aligned} J_T(t_0, 0; 0, u_2(\cdot)) =& \int_{t_0}^{T} \big[\langle Q(t)x_2(t), x_2(t)\rangle \\ &+ \langle R_2(t)u_2(t), u_2(t)\rangle\big]dt + \langle Hx_2(T), x_2(T)\rangle \leq 0, \end{aligned} \tag{9b}$$

where $x_i(\cdot)$ is the solution to the following differential equation for $i = 1, 2$:

$$\dot{x}_i(t) = A(t)x_i(t) + B_i(t)u_i(t), \quad x_i(t_0) = 0_{n\times 1}. \tag{10}$$

Proof. This result can be obtained directly from [14], [15].

A. The review of regular zero-sum games

In regular two-person zero-sum games ($R_1(t) > 0$ and $R_2(t) < 0$), the feedback representation can be analytically derived from the equilibrium condition (7) by using the Riccati equation (3). If we replace $p^*(t)$ with $P(t)x^*(t)$, then the equilibrium condition (7) becomes

$$0 = R(t)u^*(t) + B^T(t)P(t)x^*(t). \tag{11}$$

Thus when the regular condition (4) holds, the linear equation (11) is solvable for any $x^*(t) \in \mathbb{R}^n$, which implies the optimization problem is solvable as well. The results of feedback representation in the regular case are summarized as follows.

Proposition 1: Assume the convexity-concavity condition (9) holds. The Problem (P1) is solvable if the Riccati equation (3) admits solution $P(t)$ on $t \in [t_0, T]$ such that the regular condition (4) holds. In this case, the state-feedback representation of saddle points $u^*(\cdot)$ is given by

$$u^*(t) = -R^\dagger(t)B^T(t)P(t)x^*(t). \tag{12}$$

The value function is given by

$$J_T^*(t_0, x_0; u_1^*, u_2^*) = x_0^T P(t_0)x_0. \tag{13}$$

Proof. This result can be obtained directly from [14].

The convexity-concavity condition can be checked by Riccati equations with regular conditions.

Proposition 2: The convexity-concavity condition (9) holds if the following Riccati equations

$$\begin{aligned}0_{n\times n} =&\dot{\Pi}_1(t) + \Pi_1(t)A(t) + A^T(t)\Pi_1(t) + Q(t) \quad (14a)\\ &- \Pi_1(t)B_1(t)R_1^\dagger(t)B_1^T(t)\Pi_1(t), \quad \Pi_1(T) = H,\\ 0_{n\times n} =&\dot{\Pi}_2(t) + \Pi_2(t)A(t) + A^T(t)\Pi_2(t) - Q(t) \quad (14b)\\ &+ \Pi_2(t)B_2(t)R_2^\dagger(t)B_2^T(t)\Pi_2(t), \quad \Pi_2(T) = -H,\end{aligned}$$

admit solutions $\Pi_1(\cdot), \Pi_2(\cdot)$ such that the following regular conditions hold:

$$[I - R_1(t)R_1^\dagger(t)]B_1^T(t)\Pi_1(t) = 0_{m_1\times n}, \tag{15a}$$

$$[I - R_2(t)R_2^\dagger(t)]B_2^T(t)\Pi_2(t) = 0_{m_2\times n}. \tag{15b}$$

Proof. This result can be obtained directly from [15].

B. The difficulty of irregular two-person zero-sum games

But in the singular case, the regular condition (4) may not be true. Conversely, when the regular condition dose not holds, i.e., (5) holds, the linear equation (11) is unsolvable for arbitrary $x(\cdot)$. This case of (5) is termed as irregular games. This inspires us to study the irregular zero-sum games.

IV. THE COMPLETE SOLUTION TO PROBLEM (P1)

In this section we seek the solution to the two-person zero-sum game with irregular performance. The performance is irregular, namely $\mathcal{R}(B^T(t)P(t)) \nsubseteq \mathcal{R}(R(t))$. Without loss of generality, let $rank(R_1(t)) = \bar{m}_1(t) < m_1$ and $rank(R_2(t)) = \bar{m}_2(t) < m_2$. Thus $rank(I - R_1^\dagger(t)R_1(t)) = \tilde{m}_1 = m_1 - \bar{m}_1 > 0$ and $rank(I - R_2^\dagger(t)R_2(t)) = \tilde{m}_2 = m_2 - \bar{m}_2 > 0$. Let $\tilde{m} = \tilde{m}_1 + \tilde{m}_2$. There are two elementary row transformation matrices $T_1(t) \in \mathbb{R}^{m_1\times m_1}$ and $T_2(t) \in \mathbb{R}^{m_2\times m_2}$ such that

$$\begin{aligned}T_1(t)[I - R_1^\dagger(t)R_1(t)] &= \begin{bmatrix}0_{\bar{m}_1\times m_1}\\ \Upsilon_{T_1}(t)\end{bmatrix},\\ T_2(t)[I - R_2^\dagger(t)R_2(t)] &= \begin{bmatrix}0_{\bar{m}_2\times m_2}\\ \Upsilon_{T_2}(t)\end{bmatrix},\end{aligned} \tag{16}$$

where $\Upsilon_{T_1}(t) \in \mathbb{R}^{\tilde{m}_1\times m_1}$ and $\Upsilon_{T_2}(t) \in \mathbb{R}^{\tilde{m}_2\times m_2}$ are full row rank. Or equivalently, there is also an elementary row transformation matrix

$$T_0(t) \triangleq \begin{bmatrix}T_1(t) & 0_{m_1\times m_2}\\ 0_{m_2\times m_1} & T_2(t)\end{bmatrix} \tag{17}$$

such that

$$T_0(t)[I - R^\dagger(t)R(t)] = \begin{bmatrix}0_{\bar{m}_1\times m_1} & 0_{\bar{m}_1\times m_2}\\ \Upsilon_{T_1}(t) & 0_{\tilde{m}_1\times m_2}\\ 0_{\bar{m}_2\times m_1} & 0_{\bar{m}_2\times m_2}\\ 0_{\tilde{m}_2\times m_1} & \Upsilon_{T_2}(t)\end{bmatrix}. \tag{18}$$

In addition, let

$$\begin{aligned}&\begin{bmatrix}*_{n\times\bar{m}_1} & C_1^T(t) & *_{n\times\bar{m}_2} & C_2^T(t)\end{bmatrix}\\ &\triangleq P(t)B(t)[I - R^\dagger(t)R(t)]T_0^{-1}(t),\\ &\begin{bmatrix}*_{n\times\bar{m}_1} & \bar{B}_1(t) & *_{n\times\bar{m}_2} & \bar{B}_2(t)\end{bmatrix}\\ &\triangleq B(t)[I - R^\dagger(t)R(t)]T_0^{-1}(t),\\ A_0(t) &\triangleq A(t) - B(t)R^\dagger(t)B^T(t)P(t),\\ D_0(t) &\triangleq -B(t)R^\dagger(t)B^T(t),\\ T_0^{-1}(t) &\triangleq \begin{bmatrix}*_{m_1\times\bar{m}_1} & G_1(t) & 0_{m_1\times\bar{m}_2} & 0_{m_1\times\tilde{m}_2}\\ 0_{m_2\times\bar{m}_1} & 0_{m_2\times\tilde{m}_1} & *_{m_2\times\bar{m}_1} & G_2(t)\end{bmatrix},\\ G_0(t) &\triangleq \begin{bmatrix}G_1(t) & 0_{m_1\times\tilde{m}_2}\\ 0_{m_2\times\tilde{m}_1} & G_2(t)\end{bmatrix},\\ B_0(t) &\triangleq \begin{bmatrix}\bar{B}_1(t) & \bar{B}_2(t)\end{bmatrix},\\ C_0^T(t) &\triangleq \begin{bmatrix}C_1^T(t) & C_2^T(t)\end{bmatrix},\\ \Upsilon_{T_0} &\triangleq \begin{bmatrix}\Upsilon_{T_1} & 0_{\tilde{m}_1\times\tilde{m}_2}\\ 0_{\tilde{m}_2\times\tilde{m}_1} & \Upsilon_{T_2}\end{bmatrix},\end{aligned}$$

and define

$$0_{n\times n} = \dot{P}_1(t) + P_1(t)A_0(t) + A_0^T(t)P_1(t) + P_1(t)D_0(t)P_1(t), \tag{20}$$

where the terminal value $P_1(T)$ is to be determined.

We are now in the position to give the main results of this section as follows.

Lemma 2: Assume the Riccati equation (3) admits solution $P(t)$ on $t \in [t_0, T]$ such that the condition (5) holds and the convexity-concavity condition (9) holds. Problem (P1) is solvable if and only if there exists a pair $\bar{u}(t) \triangleq (\bar{u}_1^T(t), \bar{u}_2^T(t))^T \in \mathbb{R}^{\tilde{m}_1} \times \mathbb{R}^{\tilde{m}_2}$ such that

$$0 = C_0(t)x^*(t) + B_0^T(t)\Delta^*(t), \quad \Delta^*(T) = 0_{n\times 1}, \tag{21}$$

where $\bar{u}(t), x^*(t)$, and $\Delta^*(t)$ satisfy the FBDEs

$$\dot{x}^*(t) = A_0(t)x^*(t) + D_0(t)\Delta^*(t) + B_0(t)\bar{u}(t), \tag{22a}$$

$$\dot{\Delta}^*(t) = -A_0^T(t)\Delta^*(t) - C_0^T(t)\bar{u}(t). \tag{22b}$$

Proof. The proof is given in Appendix A.

Next we decouple the FBDEs (22) to obtain the feedback representation in the irregular case.

Theorem 1: Assume the convexity-concavity condition (9) holds and the Riccati equation (3) admits solution $P(t)$ on $t \in [t_0, T]$ such that the condition (5) holds. Problem (P1) is solvable if there exists a $P_1(t)$ in (20) with terminal value $P_1(T)$ such that

$$0 = C_0(t) + B_0^T(t)P_1(t), \quad t_0 \leq t \leq T, \tag{23}$$

and a $\bar{u}(t)$ that achieves

$$P_1(T)x^*(T) = 0, \tag{24}$$

where $x^*(t)$ obeys

$$\dot{x}^*(t) = [A_0(t)+D_0(t)P_1(t)]x^*(t)+B_0(t)\bar{u}(t),\ x^*(t_0) = x_0. \tag{25}$$

In this case, the feedback representation $u^*(\cdot)$ is given by

$$u^*(t) = -R^\dagger(t)B^T(t)[P(t)+P_1(t)]x^*(t)+G_0(t)\bar{u}(t). \tag{26}$$

Proof. The proof is given in Appendix B.

Based on the above discussion, the sufficient condition for the solvability of Problem (P1) is given as follows.

Theorem 2: Assume the convexity-concavity condition (9) holds. Problem (P1) is solvable if there exists matrices $P(t)$ and $P_1(t)$ satisfying $0 = B_0^T(T)[P(T) + P_1(T)]$ such that the following changed performance functional

$$\bar{J}_T(t_0, x_0; u_1, u_2) = J_T(t_0, x_0; u_1, u_2) + x^T(T)P_1(T)x(T) \tag{27}$$

is regular and $P_1(T)x(T) = 0$ is achieved with the saddle point for the performance functional (27). In this case, the value function is given by

$$J_T^*(t_0, x_0; u_1^*, u_2^*) = x_0^T[P(t_0) + P_1(t_0)]x_0. \tag{28}$$

Proof. The proof is given in Appendix C.

At this time, the controller has not yet been fully explicitly given and a new problem about the controllability of $P_1(T)x(T)$ appears, i.e., find $\bar{u}(\cdot)$ to make $P_1(T)x^*(T) = 0_{n\times 1}$, where $P_1(\cdot)x^*(\cdot)$ satisfies

$$\frac{d}{dt}P_1(t)x^*(t) = -A_0^T(t)P_1(t)x^*(t) - C_0^T(t)\bar{u}(t). \tag{29}$$

Below, we study the controllability problem and give the complete solution of Problem (P1).

Theorem 3: Given condition (5), Problem (P1) admits open-loop saddle point if there exists a $P_1(t)$ in (20) such that (23) holds and

$$\mathcal{R}(P_1(t_0)) \subseteq \mathcal{R}(G_1[t_0, T]), \tag{30}$$

where the Gramian matrix $G_1[t_0, T]$ is defined by

$$G_1[t_0, T] \triangleq \int_{t_0}^{T} \Phi(t_0, t)C_0^T(t)C_0(t)\Phi^T(t_0, t)dt \tag{31}$$

and $\Phi(t_0, t)$ satisfies

$$\dot{\Phi}(t, s) = -A_0^T(t)\Phi(t, s), \quad \Phi(t, t) = I. \tag{32}$$

In this case, the open-loop solution is given by (26) with

$$\bar{u}(t) = C_0(t)\Phi^T(t_0, t)G_1^\dagger[t_0, T]P_1(t_0)x_0. \tag{33}$$

Proof. See Theorem 3 in [1].

V. A NUMERICAL EXAMPLE

In this section, we give the following example to show the efficiency of the results in this paper.

Example 1. Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} u_1(t) + u_2(t), \quad x(0) = 1, \tag{34}$$

with the performance functional

$$J_{\frac{1}{2}}(0, 1; u_1, u_2) = \int_0^{\frac{1}{2}} \left\{ u_1^T(t) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} u_1(t) - |u_2(t)|^2 \right\} dt + |x(\frac{1}{2})|^2. \tag{35}$$

The convexity-concavity condition for this zero-sum game is

$$J_{\frac{1}{2}}(0, 0; u_1(\cdot), 0_{2\times 1}) = \int_0^{\frac{1}{2}} u_1^T(t) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} u_1(t)dt + |x_1(\frac{1}{2})|^2, \tag{36a}$$

$$J_{\frac{1}{2}}(0, 0; 0_{2\times 1}, u_2(\cdot)) = -\int_0^{\frac{1}{2}} |u_2(t)|^2 dt + |x_2(\frac{1}{2})|^2. \tag{36b}$$

The convexity condition $J_{\frac{1}{2}}(0, 0; u_1(\cdot), 0) \geq 0$ for any $u_1(\cdot)$. Next we show the concavity condition $J_{\frac{1}{2}}(0, 0; 0_{2\times 1}, u_2(\cdot)) \leq 0$. Consider the following Riccati equation

$$\dot{\Pi}(t) - \Pi^2(t) = 0, \quad \Pi(\frac{1}{2}) = -1. \tag{37}$$

The solution to the above Riccati equation is

$$\Pi(t) = -\frac{2}{2t + 1}, \quad 0 \leq t \leq \frac{1}{2}. \tag{38}$$

Thus the regular condition $\mathcal{R}(\Pi(t)) \subseteq \mathcal{R}(-1)$ holds. It follows by Proposition 2 that the concavity condition $J_{\frac{1}{2}}(0, 0; 0_{2\times 1}, u_2(\cdot)) \leq 0$. Thus the convexity-concavity condition (36) holds. The zero-sum game admits open-loop saddle points. The open-loop saddle points can be derived from the equilibrium condition

$$0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} u^*(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} p^*(t), \tag{39}$$

where $(x^*(\cdot), p^*(\cdot))$ is the solution to the following system of FBDEs:

$$\dot{x}^*(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} u^*(t), \quad x^*(0) = 1, \tag{40a}$$

$$\dot{p}^*(t) = 0, \quad p^*(T) = x^*(T). \tag{40b}$$

The Riccati equation for the system (34) with the performance functional (35) is given as follows.

$$\dot{P}(t) - P^2(t) = 0, \quad P(\frac{1}{2}) = 1. \tag{41}$$

The solution to the above Riccati equation is

$$P(t) = -\frac{2}{2t - 3}, \quad 0 \leq t \leq \frac{1}{2}. \tag{42}$$

The curves of $P(\cdot)$ is shown in Fig. 1. Then, it holds that

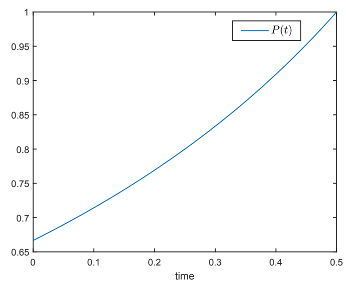

Fig. 1. The numerical solution of $P(\cdot)$.

$\mathcal{R}(B^T P(t)) \nsubseteq \mathcal{R}(R)$. Thus this is an irregular zero-sum game. By Theorem 1, we introduce the following Riccati equation

$$\dot{P}_1(t) - 2P_1(t)P(t) - P_1^2(t) = 0, \quad P_1(\frac{1}{2}) = -1. \tag{43}$$

By (42), we can obtain $P_1(t) = -P(t)$, which further gives $\bar{P}(t) = 0, t \in [0, \frac{1}{2}]$. Thus we design the following controllers

$$u_1^*(t) = -\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad u_2^*(t) = 0, \quad 0 \le t \le \frac{1}{2}. \tag{44}$$

Under the controllers (44), the state $x^*(t)$ is given by

$$x^*(t) = -2t + 1. \tag{45}$$

Thus we have $x(\frac{1}{2}) = 0$. Then $(1 - 2t, 0)$ is the solution to the FBDEs (40). Therefore the controllers (44) are an open-loop saddle point for the system (34) with the performance function (35) and the corresponding value function is 0.

VI. Conclusion

In this paper, we have studied the irregular LQ two-person zero-sum games by the "two-layer optimization" approach. The existence of an open-loop saddle point is characterized by the solvability of a system of constrained controlled FBDEs, together with a convexity-concavity condition. The feedback representation of open-loop saddle points is obtained by two Riccati equations together with solving a terminal controllability problem. In future work, we will consider the irregular LQ two-person zero-sum stochastic games and nonzero-sum games.

Appendix

A. Proof of Lemma 2

Proof of necessity. By Lemma 1, Problem (P1) is solvable then the FBSDE (8) admit solution $(x^*(t), p^*(t))$. We will show the FBDEs (22) admit solution $(x^*(t), \Delta^*(t))$ on $t \in [t_0, T]$. Based on the discussion of (11), we can see that $p^*(t) \neq P(t)x^*(t)$ under the condition (5), where $P(t)$ is the solution to (3). We therefore define a new variable $\Delta^*(t)$ as

$$p^*(t) = P(t)x^*(t) + \Delta^*(t), \tag{46}$$

where it is clear that $\Delta^*(T) = 0$. Next, we aim to derive the new FBDEs (22) under the solvability of Problem (P1).

First, we take the derivative of (46), obtaining

$$\dot{p}^*(t) = \dot{P}(t)x^*(t) + P(t)[A(t)x^*(t) + B(t)u^*(t)] + \dot{\Delta}^*(t). \tag{47}$$

From (8b) and (46), we then find that

$$\dot{p}^*(t) = -A^T(t)[P(t)x^*(t) + \Delta^*(t)] - Q(t)x^*(t). \tag{48}$$

By comparing (47) and (48), we obtain

$$\begin{aligned} &\dot{P}(t)x^*(t) + P(t)A(t)x^*(t) + P(t)B(t)u^*(t) + \dot{\Delta}^*(t) \\ &+ A^T(t)P(t)x^*(t) + A^T(t)\Delta^*(t) + Q(t)x^*(t) = 0_{n\times 1}. \end{aligned} \tag{49}$$

Second, we aim to find the saddle point $u^*(t)$ and the new equilibrium condition (21). By using (46), we can formulate the equilibrium condition (7) as

$$0 = R(t)u^*(t) + B^T(t)P(t)x^*(t) + B^T(t)\Delta^*(t). \tag{50}$$

Thus we have

$$\begin{aligned} u^*(t) = &- R^\dagger(t)B^T(t)[P(t)x(t) + \Delta^*(t)] \\ &+ [I - R^\dagger(t)R(t)]z(t), \end{aligned} \tag{51}$$

where $z(t)$ is an arbitrary vector with compatible dimension such that the following equality holds:

$$0 = [I - R(t)R^\dagger(t)][B^T(t)P(t)x^*(t) + B^T(t)\Delta^*(t)]. \tag{52}$$

Using (18) we have

$$\begin{aligned} &T_0(t)[I - R^\dagger(t)R(t)]z(t) \\ &= \begin{bmatrix} 0_{\bar{m}_1\times m_1} & 0_{\bar{m}_1\times m_2} \\ \Upsilon_{T_1}(t) & 0_{\tilde{m}_1\times m_2} \\ 0_{\bar{m}_2\times m_1} & 0_{\bar{m}_2\times m_2} \\ 0_{\tilde{m}_2\times m_1} & \Upsilon_{T_2}(t) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0_{\bar{m}_1\times 1} \\ \bar{u}_1(t) \\ 0_{\bar{m}_2\times 1} \\ \bar{u}_2(t) \end{bmatrix}, \end{aligned} \tag{53}$$

where $\bar{u}_1(t) = \Upsilon_{T_1}(t)z_1(t) \in \mathbb{R}^{\tilde{m}_1}$, $\bar{u}_2(t) = \Upsilon_{T_2}(t)z_2(t) \in \mathbb{R}^{\tilde{m}_2}$. Then

$$T_0^{-1}(t)T_0(t)[I - R^\dagger(t)R(t)]z(t) = G_0(t)\bar{u}(t). \tag{54}$$

Now, we can rewrite (51) as

$$u^*(t) = -R^\dagger(t)B^T(t)[P(t)x^*(t) + \Delta^*(t)] + G_0(t)\bar{u}(t). \tag{55}$$

Note that

$$\begin{aligned} &I - R(t)R^\dagger(t) \\ &= \begin{bmatrix} 0_{m_1\times\bar{m}_1} & \Upsilon_{T_1}^T(t) & 0_{m_1\times\bar{m}_2} & 0_{m_1\times\tilde{m}_2} \\ 0_{m_2\times\bar{m}_1} & 0_{m_2\times\tilde{m}_1} & 0_{m_2\times\bar{m}_2} & \Upsilon_{T_2}^T(t) \end{bmatrix} \\ &\times T_0^{-1}(t)^T[I - R(t)R^\dagger(t)]. \end{aligned} \tag{56}$$

Thus we can rewrite (52) as

$$\begin{aligned} 0 &= [I - R(t)R^\dagger(t)][B^T(t)P(t)x^*(t) + B^T(t)\Delta^*(t)] \\ &= \Upsilon_{T_0}^T(t)[C_0(t)x^*(t) + B_0^T(t)\Delta^*(t)]. \end{aligned} \tag{57}$$

Note that $\Upsilon_{T_0}^T(t)$ is of full column rank, and thus (57) can be directly rewritten as (21).

Third, we derive the dynamics of $\Delta^*(t)$. Substituting (51) into (49) and using (3) yields

$$\begin{aligned}0 =&\dot{\Delta}^*(t) + [A^T(t) - P(t)B(t)R^\dagger(t)B^T(t)]\Delta^*(t) \\ &+ P(t)B(t)[I - R^\dagger(t)R(t)]z(t).\end{aligned} \tag{58}$$

Note that

$$P(t)B(t)[I - R^\dagger(t)R(t)]z(t) = C_0^T(t)\bar{u}(t). \tag{59}$$

Thus (22b) follows by substituting (59) into (58).

Finally, we derive the dynamics equation (22a). By substituting (55) into (8a) and combining this with the fact that

$$B(t)[I - R^\dagger(t)R(t)]z(t) = B_0(t)\bar{u}(t), \tag{60}$$

we can derive the state dynamics (22a).

Proof of sufficiency. If (21) is true then (51) and (52) can be jointly rewritten as (50). Further, by reversing the process for (46)-(50), we can easily verify that $p^*(t) = P(t)x^*(t) + \Delta^*(t)$, where $x^*(t)$ and $\Delta^*(t)$ satisfy the FBDE (22), solving (7)-(8). Thus, Problem (P1) is solvable, completing the proof.

B. Proof of Theorem 1

Based on Lemma 2, Problem (P1) is solvable if the FBDEs (22) admits solution $(x^*(t), \Delta^*(t))$. It is sufficient to verify that $(x^*(t), \Delta^*(t)) = (x^*(t), P_1(t)x^*(t))$ is the solution to the FBDEs (22). Taking the derivative of $P_1(t)x^*(t)$ yields

$$\begin{aligned}\frac{d}{dt}P_1(t)x^*(t) =& - A_0^T(t)P_1(t)x^*(t) - P_1(t)D_0(t)P_1(t)x^*(t) \\ &+ P_1(t)D_0(t)\Delta^*(t) - C_0^T(t)\bar{u}(t)\end{aligned} \tag{61}$$

where (23) is used in the above equality. By comparing (61) and (22b), we have the following corresponding:

$$\Delta^*(t) = P_1(t)x^*(t). \tag{62}$$

Thus $(x^*(t), \Delta^*(t)) = (x^*(t), P_1(t)x^*(t))$ is the solution to the FBDEs (22). Thus Problem (P1) is solvable. The feedback representation (26) follows by (55) with (62). This completes the proof.

C. Proof of Theorem 2

The aim is to prove if there exists matrices $P(t)$ and $P_1(t)$ satisfying $0 = B_0^T(T)[P(T) + P_1(T)]$ such that (27) is regular and $P_1(T)x(T) = 0$ is achieved, then the FBDEs (8) can be solved in this case. The first step is to prove that the Riccati equation $P(t) + P_1(t)$ is regular. From the regularity of (27), we have the following Riccati equation is regular:

$$\begin{aligned}0 =&\dot{\bar{P}}(t) + A^T(t)\bar{P}(t) + \bar{P}(t)A(t) + Q(t) \\ &- \bar{P}(t)B(t)R^\dagger(t)B^T(t)\bar{P}(t), \quad \bar{P}(T) = H + P_1(T).\end{aligned} \tag{63}$$

That is,

$$[I - R(t)R^\dagger(t)]B^T(t)\bar{P}(t) = 0. \tag{64}$$

By some elementary computation, we can obtain

$$\bar{P}(t) = P(t) + P_1(t). \tag{65}$$

It follows by (64) that

$$[I - R(t)R^\dagger(t)]B^T(t)[P(t) + P_1(t)] = 0. \tag{66}$$

Note that by using (56) we obtain

$$\begin{aligned}&[I - R(t)R^\dagger(t)]B^T(t)[P(t) + P_1(t)] \\ =& \Upsilon_{T_0}^T(t)B_0^T(t)[P(t) + P_1(t)],\end{aligned} \tag{67}$$

which further gives

$$0 = C_0(t) + B_0^T(t)P_1(t). \tag{68}$$

Note that $P_1(T)x(T) = 0$ is achieved. The value function (28) can be proved by completing square. The proof is completed by Theorem 1.

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