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Distributed Optimal Solutions for Multiagent Pursuit-Evasion Games

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***Abstract*— In this paper, distributed optimal solutions are designed for networked multiagent pursuit-evasion (MPE) games for capture and formation control. In the games, the pursuers aim to minimize the distance from their target evaders while the evaders attempt to maximize it, and at the same time, all players desire to maintain cohesion with their teammates. The goals of agents are obviously reflected in the obtained optimal control strategies which consist of an attracting term and/or a repelling term. Nash equilibrium is obtained by means of optimal strategies using the solutions of the HJI equations. Furthermore, three scenarios are considered in the MPE game: one-pursuer-one-evader, multiple-pursuer-one-evader, and multiple-pursuer-multiple-evader, where sufficient conditions are given for pursuers in achieving capture or formation control with ultimate zero or bounded errors. It is shown that the conditions depend on the structure of the communication graph, the parameters in the controllers, and the expected formation configurations. Finally, both simulations and real flight experiments successfully demonstrate the effectiveness of the proposed strategies.**

I. INTRODUCTION

The last decade has witnessed wide development of multi-agent systems due to their high application values in cooperative transportation, localization [1], [2], security surveillance, and logistic delivery, to name just a few. Pursuit-evasion (PE) games are one of the most interesting research topics. They are widely used both in military implementations such as missile guidance and aircraft control [3], [4], and civilian areas like sport strategies. In nature, animal hunting behaviors are also pursuit-evasion differential games.

The study of PE games starts from the simplest case with a single pursuer and a single evader [5], [6]. The PE game in [6] is formulated as a zero-sum game which is solved using the Hamilton-Jacobi-Isaacs (HJI) equations. The result is extended to the cases of two pursuers versus one evader [7], [8], and multiple pursuers versus one evader [9], [10]. However, it is difficult to solve the HJI equations for nonlinear systems. Instead of solving the HJI equations, the control strategies were derived by differentiation of a particular value function in [11].

In recent years, more general multiple-pursuer-multiple-evader PE games [12] have gained much attention, owing to the increased interest in multi-agent problems. In [13], a distributed hybrid controller is proposed for each pursuer using both local coordination protocols and time-varying potential fields. Conditions for guaranteed capture or guaranteed evasion are analyzed in [14] for multiple nonlinear players. Suboptimal approaches for the multiplayer PE differential games were presented in [15] by decoupled player control strategies. In [16], distributed optimal strategies are obtained for all players by using a graph-theoretic approach which depends on the player's teammates and neighbors of the opposite team. The obstacle avoidance PE games are further studied in [17]. The framework of [16] was extended by [18] to search for an adaptive Nash equilibrium solution for the differential games.

In most PE games, the objective of pursuers is to capture the target evader, that is, to achieve position consensus [5]–[18]. However, this kind of perish-together strategy may lead to the ruin of pursuers. Instead, formation control or surrounding control is of more practical [19]–[22]. With surrounding the target, the pursuers can jet a mesh to capture the target and then carry it to a safety zone. Specifically, a distributed estimation-and-control hierarchical framework is developed in [19], [20] for, respectively, linear systems and surface vessels. The surrounding formation control can also be achieved by defining an expected displacement, under which the evaders lie in the convex hull formed by the pursuers. Besides, the evaders may also want to maintain some formation configurations to better complete their tasks.

In this paper, distributed optimal control strategies for MPE games for capture and formation control are designed over complex communication graphs. The contributions of this paper are summarized as follows. (1) We appropriately defined distinct local error variables and novel performance indices for players in both teams, based on which the obtained distributed optimal solutions consist of an attracting term and/or a repelling term that reflect the goals of agents. More importantly, when group cohesion is ignored, the solutions for the evaders are still valid for them to maximize their distance from the pursuers. (2) Formation control is also studied when developing optimal solutions for the MPE games. Besides capture, the formation control can achieve surrounding control of the target and also considers the case that the evaders desire to maintain some formation configurations. The results show that diverse expected formation configurations may result in zero or bounded formation control error. (3) We present conditions for capture and formation control for three scenarios: one-pursuer-one-evader, multiple-pursuer-one-evader, and multiple-pursuer-multiple-evader. Under a novel analysis, the results present that both the communication graph and the expected configuration will affect the capture and formation control. Due

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to the decoupling of the solutions in achieving the goals of agents, the interdependence of subsystems caused by the three communication graphs, and the existence of expected formation configurations, the closed-loop system is complex to analyze.

II. PRELIMINARIES

Consider a team of N pursuers who have dynamics

$$\dot{x}_i^p = Ax_i^p + Bu_i^p, \quad i = 1, \cdots, N \tag{1}$$

where $x_i^p \in \mathbb{R}^n$ and $u_i^p \in \mathbb{R}^m$ are, respectively, the state and input of the ith pursuer. Consider also a group of M evaders with dynamics

$$\dot{x}_j^e = Ax_j^e + Bu_j^e, \quad j = 1, \cdots, M \tag{2}$$

where $x_j^e \in \mathbb{R}^n$ and $u_j^e \in \mathbb{R}^m$ are, respectively, the state and input of the jth evader.

The pursuers (1) and evaders (2) form a group of $N+M$ agents. Define $\mathcal{G}_p = (\mathcal{V}_p, \mathcal{E}_p)$ the communication graph among the N pursuers, where $\mathcal{V} = \{v_{p1}, \cdots, v_{pN}\}$ and $\mathcal{E}_p = \mathcal{V}_p \times \mathcal{V}_p$. $(v_{pk}, v_{pi}) \in \mathcal{E}_p$ if and only if pursuer i has access to the information of pursuer k, and we say agent k is a neighbor of agent i. Let a_{ik} be the communication weight of the graph $\mathcal{G}_p$, with $a_{ik} = 1$ if $(v_{pk}, v_{pi}) \in \mathcal{E}_p$, otherwise, $a_{ik} = 0$. Let $\mathcal{A}_p = [a_{ik}] \in \mathbb{R}^{N\times N}$ be the weighted adjacency matrix where $a_{ii} = 0$. Denote by $d_i^{pp} = \sum_{k=1}^N a_{ik}$ the in-degree of pursuer i and $\mathcal{D}_{pp} = \mathrm{diag}\{d_i^{pp}\}$ the in-degree matrix of the graph. Then, the Laplacian matrix can be defined as $\mathcal{L}_p = \mathcal{D}_{pp} - \mathcal{A}_p$. Similarly, the interaction topology among evaders is represented by $\mathcal{G}_e = (\mathcal{V}_e, \mathcal{E}_e)$ with the nodes $\mathcal{V}_e = \{v_{e1}, \cdots, v_{eM}\}$. The edge weights are b_{jl} with $b_{jl} = 1$ if $(v_{el}, v_{ej}) \in \mathcal{E}_e$ and $b_{jl} = 0$ otherwise. The in-degree of evader j is $d_j^{ee} = \sum_{l=1}^M b_{jl}$ and the in-degree matrix $\mathcal{D}_{ee} = \mathrm{diag}\{d_j^{ee}\}$. Define the matrices $\mathcal{A}_e = [b_{jl}]$ and $\mathcal{L}_e = D_{ee} - \mathcal{A}_e$.

Let $\mathcal{G}_{pe} = (\mathcal{V}_{pe}, \mathcal{E}_{pe})$ represent the communication topology among all the agents. Specifically, for $i \in \mathcal{V}_p$ and $j \in \mathcal{V}_e$, the edge weight $c_{ij} = 1$ if pursuer i can obtain the information of evader j, otherwise, $c_{ij} = 0$. Similarly, $e_{ji} = 1$ if evader j knows the information of pursuer i. The in-degree of pursuer i in the graph $\mathcal{G}_{pe}$ is defined as $d_i^{pe} = \sum_{j=1}^M c_{ij}$, and the in-degree of evader j is $d_j^{ep} = \sum_{i=1}^N e_{ji}$. The graph is undirected if $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$ with $i \neq j$. In this paper, we assume the graph $\mathcal{G}_{pe}$ is undirected.

Based on the above information, we define two local error variables for each pursuer, with respect to its pursuer neighbors and evader neighbors, respectively,

$$\tilde{x}_i^{pp} = \sum_{k=1}^N a_{ik}(x_k^p - x_i^p), \tilde{x}_i^{pe} = \sum_{j=1}^M c_{ij}(x_j^e - x_i^p + \Delta x_{ij}^{pe}) \tag{3}$$

where Δx_{ij}^{pe} is the expected displacement between the pursuer i and the evader j, and it can be a zero vector. The justification for the formation is that, in many practical applications, the team of pursuers may want to surround the target evader, instead of achieving state consensus to collide with it.

Similarly, we define another two local errors for each evader, with respect to its evader neighbors and pursuer neighbors, respectively,

$$\tilde{x}_j^{ee} = \sum_{l=1}^M b_{jl}(x_l^e - x_j^e + \Delta x_{jl}^{ee}), \quad \tilde{x}_j^{ep} = \sum_{i=1}^N e_{ji}(x_i^p - x_j^e) \tag{4}$$

where Δx_{jl}^{ee} denotes the expected displacement between evaders j and l. In many application scenarios, the evaders desire to move in formation to increase the opportunity to complete the tasks.

Remark 1: It is well known that, for $x_1, x_2 \in \mathbb{R}^n$, $x_2 - x_1$ is a vector pointing from x_1 to x_2. It physically represents an attracting force of agent 2 to agent 1, and also a repelling force of agent 1 to agent 2. Thus, $\tilde{x}_i^{pp}$ and $\tilde{x}_i^{pe}$ denote the attracting forces from the pursuer neighbors and evader neighbors, respectively, to pursuer i. Similarly, $\tilde{x}_j^{ee}$ and $\tilde{x}_j^{ep}$ are, respectively, the attracting forces from the evader neighbors and repelling forces from the pursuer neighbors to evader j.

III. PROBLEM FORMULATION AND SOLUTIONS FOR MPE GAMES

In the MPE game, the objective of pursuers is to minimize the distance from their neighboring evaders to intercept them or achieve the desired formation for the surrounding control. Moreover, the pursuers also intend to stay close to their teammates to keep the group cohesion. Therefore, the control strategy of each pursuer can be divided into two parts. The first part is for remaining close to its teammates, and the second part is for pursuing the evaders, that is, $u_i^p = u_i^{p1} + u_i^{p2}$.

The goals of each pursuer can be formulated as a scalar function $J_{pi}(\tilde{x}_i^{pp}, \tilde{x}_i^{pe}, u_i^{p1}, u_i^{p2})$, regarded as the performance index for pursuer i which is defined as

$$\begin{aligned} J_{pi} = \int_0^\infty & \Big[(\tilde{x}_i^{pp})^{\mathrm{T}} Q_i^{pp} \tilde{x}_i^{pp} + (u_i^{p1})^{\mathrm{T}} R_i^{pp} u_i^{p1} \\ & + (\tilde{x}_i^{pe})^{\mathrm{T}} Q_i^{pe} \tilde{x}_i^{pe} + (u_i^{p2})^{\mathrm{T}} R_i^{pe} u_i^{p2} \Big] dt \end{aligned} \tag{5}$$

where Q_i^{pp}, Q_i^{pe}, R_i^{pp} and R_i^{pe} are positive definite matrices with appropriate dimension. Pursuer i is thus concerned with minimizing J_{pi}.

On the contrary, the goals of the evaders are to maximize the distance from their neighboring pursuers, and at the same time, to stay close to their teammates. Similarly, the control input of evader j consists of two parts, i.e., $u_j^e = u_j^{e1} + u_j^{e2}$. The performance index for evader j can be defined as

$$\begin{aligned} J_{ej} = \int_0^\infty & \big[(\tilde{x}_j^{ee})^{\mathrm{T}} Q_j^{ee} \tilde{x}_j^{ee} + (u_j^{e1})^{\mathrm{T}} R_j^{ee} u_j^{e1} \\ & - (\tilde{x}_j^{ep})^{\mathrm{T}} Q_j^{ep} \tilde{x}_j^{ep} + (u_j^{e2})^{\mathrm{T}} R_j^{ep} u_j^{e2} \big] dt \end{aligned} \tag{6}$$

where matrices Q_j^{ee}, Q_j^{ep}, R_j^{ee} and R_j^{ep} are positive definite. Notice that minimizing the third term $-(\tilde{x}_j^{ep})^{\mathrm{T}} Q_j^{ep} \tilde{x}_j^{ep}$ equals maximizing the distance from the pursuers, which implies escaping from them.

Based on the above definitions, we define the following MPE differential games on communication graphs $\mathcal{G}_{pe}$.

Definition 1. (MPE game): The MPE game is defined as

$$V_{pi} = \min_{u_i^{p1}, u_i^{p2}} J_{pi}(\tilde{x}_i^{pp}, \tilde{x}_i^{pe}, u_i^{p1}, u_i^{p2}) \tag{7}$$

$$V_{ej} = \min_{u_j^{e1}, u_j^{e2}} J_{ej}(\tilde{x}_j^{ee}, \tilde{x}_j^{ep}, u_j^{e1}, u_j^{e2}) \tag{8}$$

where V_{pi} and V_{ej} are the values of the MPE game for pursuer i and evader j, respectively.

Let u_{-i}^p and u_{-i}^e be the control strategies of the pursuer neighbors and evader neighbors of pursuer i, respectively, and u_{-j}^e and u_{-j}^p be the control strategies of the evader neighbors and pursuer neighbors of evader j. The Nash equilibrium is defined as follows.

Definition 2. (Nash equilibrium): Control strategies u_i^{p1*}, u_i^{p2*}, $i = 1, \cdots, N$, and u_j^{e1*}, u_j^{e2*}, $j = 1, \cdots, M$, form a Nash equilibrium if the inequalities

$$J_{pi}(u_i^{p1*}, u_i^{p2*}, u_{-i}^{p*}, u_{-i}^{e*}) \le J_{pi}(u_i^{p1}, u_i^{p2}, u_{-i}^{p*}, u_{-i}^{e*})$$
$$J_{ej}(u_j^{e1*}, u_j^{e2*}, u_{-j}^{e*}, u_{-j}^{p*}) \le J_{ej}(u_j^{e1}, u_j^{e2}, u_{-j}^{e*}, u_{-j}^{p*})$$

hold for all agents in the game.

The optimal control strategy of pursuer i can be obtained by the Hamiltonian function [9], [23]

$$\begin{aligned} H_i^p = {} & (\tilde{x}_i^{pp})^{\mathrm{T}} Q_i^{pp} \tilde{x}_i^{pp} + (u_i^{p1})^{\mathrm{T}} R_i^{pp} u_i^{p1} \\ & + (\tilde{x}_i^{pe})^{\mathrm{T}} Q_i^{pe} \tilde{x}_i^{pe} + (u_i^{p2})^{\mathrm{T}} R_i^{pe} u_i^{p2} \\ & + \nabla V_{pi}^T(\tilde{x}_i^{pp}) \dot{\tilde{x}}_i^{pp} + \nabla V_{pi}^T(\tilde{x}_i^{pe}) \dot{\tilde{x}}_i^{pe} \end{aligned}$$

where V_{pi} is the value defined in (7). Following (3), it satisfies that

$$\dot{\tilde{x}}_i^{pp} = -\sum_{k=1}^{N} a_{ik}(Ax_i^p + Bu_i^{p1} + Bu_i^{p2}) + \sum_{k=1}^{N} a_{ik} \dot{x}_k^p$$
$$\dot{\tilde{x}}_i^{pe} = -\sum_{j=1}^{M} c_{ij}(Ax_i^p + Bu_i^{p1} + Bu_i^{p2}) + \sum_{j=1}^{M} c_{ij} \dot{x}_j^e.$$

Letting the partial derivative of H_i^p

$$\frac{\partial H_i^p}{\partial u_i^{p1}} = 2R_i^{pp} u_i^{p1} - d_i^{pp} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pp}) - d_i^{pe} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pe}) = 0$$
$$\frac{\partial H_i^p}{\partial u_i^{p2}} = 2R_i^{pe} u_i^{p2} - d_i^{pp} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pp}) - d_i^{pe} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pe}) = 0$$

gives

$$u_i^{p1*} = \frac{1}{2}(R_i^{pp})^{-1}\big(d_i^{pp} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pp}) + d_i^{pe} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pe})\big) \tag{9}$$

$$u_i^{p2*} = \frac{1}{2}(R_i^{pe})^{-1}\big(d_i^{pp} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pp}) + d_i^{pe} B^{\mathrm{T}} \nabla V_{pi}(\tilde{x}_i^{pe})\big) \tag{10}$$

which are the optimal control strategies for pursuer i. V_{pi} is the solution of the coupled HJI

$$\begin{aligned} 0 = {} & (\tilde{x}_i^{pp})^{\mathrm{T}} Q_i^{pp} \tilde{x}_i^{pp} + (u_i^{p1*})^{\mathrm{T}} R_i^{pp} u_i^{p1*} + (\tilde{x}_i^{pe})^{\mathrm{T}} Q_i^{pe} \tilde{x}_i^{pe} \\ & + (u_i^{p2*})^{\mathrm{T}} R_i^{pe} u_i^{p2*} \\ & + \nabla V_{pi}^T(\tilde{x}_i^{pp}) \left(-\sum_{k=1}^{N} a_{ik}(Ax_i^p + Bu_i^{p1*} + Bu_i^{p2*}) + \dot{x}_k^p \right) \\ & + \nabla V_{pi}^T(\tilde{x}_i^{pe}) \left(-\sum_{j=1}^{M} c_{ij}(Ax_i^p + Bu_i^{p1*} + Bu_i^{p2*}) + \dot{x}_j^e \right). \end{aligned} \tag{11}$$

Similarly, the optimal control strategies for evader j are given by

$$u_j^{e1*} = \frac{1}{2}(R_j^{ee})^{-1}\big(d_j^{ee} B^{\mathrm{T}} \nabla V_{ej}(\tilde{x}_j^{ee}) + d_j^{ep} B^{\mathrm{T}} \nabla V_{ej}(\tilde{x}_j^{ep})\big) \tag{12}$$

$$u_j^{e2*} = \frac{1}{2}(R_j^{ep})^{-1}\big(d_j^{ee} B^{\mathrm{T}} \nabla V_{ej}(\tilde{x}_j^{ee}) + d_j^{ep} B^{\mathrm{T}} \nabla V_{ej}(\tilde{x}_j^{ep})\big). \tag{13}$$

Theorem 1: Considering the pursuers (1) and evaders (2) with local errors (3)–(4). Let (9)–(13) be the control strategies for pursuer i and evader j where V_{pi} and V_{ej} are the values of the game for pursuer i and evader j, respectively. Then, the MPE game (7)-(8) is in Nash equilibrium. Moreover, the values of the game for pursuer i and evader j are given by $V_{pi}(t_0)$ and $V_{ej}(t_0) - V_{ej}(t_\infty)$, respectively.

IV. Conditions for Capture and Formation Control in Three Scenarios

In this section, we consider the MPE game in three scenarios: one-purser-one-evader, multiple-pursuer-one-evader, and multiple-pursuer-multiple-evader.

Suppose that for $i = 1, \cdots, N$ and $j = 1, \cdots, M$, the value functions V_{pi} and V_{ej} have the form

$$V_{pi} = \alpha_{i1}(\tilde{x}_i^{pp})^{\mathrm{T}} P_i^{pp} \tilde{x}_i^{pp} + \alpha_{i2}(\tilde{x}_i^{pe})^{\mathrm{T}} P_i^{pe} \tilde{x}_i^{pe} \tag{14}$$
$$V_{ej} = \beta_{j1}(\tilde{x}_j^{ee})^{\mathrm{T}} P_j^{ee} \tilde{x}_j^{ee} - \beta_{j2}(\tilde{x}_j^{ep})^{\mathrm{T}} P_j^{ep} \tilde{x}_j^{ep} \tag{15}$$

where P_i^{pp}, P_i^{pe}, P_j^{ee} and P_j^{ep} are positive definite matrices. Taking ∇V_{pi} into (9)–(13) and following the fact that $u_i^p = u_i^{p1} + u_i^{p2}$ and $u_j^e = u_j^{e1} + u_j^{e2}$, we thus have

$$\begin{aligned} u_i^p = {} & \big((R_i^{pp})^{-1} + (R_i^{pe})^{-1}\big) \\ & \cdot (\alpha_{i1} d_i^{pp} B^{\mathrm{T}} P_i^{pp} \tilde{x}_i^{pp} + \alpha_{i2} d_i^{pe} B^{\mathrm{T}} P_i^{pe} \tilde{x}_i^{pe}) \end{aligned} \tag{16}$$
$$\begin{aligned} u_j^e = {} & \big((R_j^{ee})^{-1} + (R_j^{ep})^{-1}\big) \\ & \cdot (\beta_{j1} d_j^{ee} B^{\mathrm{T}} P_j^{ee} \tilde{x}_j^{ee} - \beta_{j2} d_j^{ep} B^{\mathrm{T}} P_j^{ep} \tilde{x}_j^{ep}). \end{aligned} \tag{17}$$

Remark 2: One can note that the control strategy u_i^p for pursuer i reflects the two attracting forces from its neighboring teammates and evaders, which will drive it to stay close to its teammates and meanwhile capture the target. On the contrary, the repelling force $-\tilde{x}_j^{ep}$ in u_j^e for evader j prevents it from being intercepted by its neighboring pursuers. The repelling force still holds when $d_j^{ee} = 0$ for each evader j, that is, no group cohesion in the evader team. In [16], when $d_j^{ee} = 0$, it becomes that $u_j^e = \beta_{j2} d_j^{ep} (R_j^{ep})^{-1} B^{\mathrm{T}} P_j^{ep} \tilde{x}_j^{ep}$ which is the attracting forces from its neighboring pursuers.

Without loss of generality, the R matrices in (16)–(17) are selected as identity matrices, and the P matrices are solutions of the Lyapunov equation

$$PA + A^{\mathrm{T}}P - PBB^{\mathrm{T}}P = -I. \tag{18}$$

Note that the equation is solvable if all eigenvalues of A have nonpositive real parts. The control strategies thus become, for $i = 1, \cdots, N$ and $j = 1, \cdots, M$,

$$u_i^p = 2(\alpha_{i1} d_i^{pp} B^{\mathrm{T}} P \tilde{x}_i^{pp} + \alpha_{i2} d_i^{pe} B^{\mathrm{T}} P \tilde{x}_i^{pe}) \tag{19}$$

$$u_j^e = 2(\beta_{j1} d_j^{ee} B^{\mathrm{T}} P \tilde{x}_j^{ee} - \beta_{j2} d_j^{ep} B^{\mathrm{T}} P \tilde{x}_j^{ep}). \tag{20}$$

Now the control policies depend on the coefficients α_{i1}, α_{i2}, β_{j1} and β_{j2}, whose values indicate the priority of keeping close to the teammate and staying close to or far away from the other side agents. In the following, we will analyze how the coefficients affect the PE games in three scenarios.

A. PE Game for One-pursuer-one-evader Problem

When the evaders increase their distance with respect to each other to separate the pursuers, each pursuer must select a single evader as its target. Suppose that pursuer i has selected evader i as the target using the target selection algorithm (see [16] for example). In such a case, the local error for pursuer i with respect to the evader is defined as $\tilde{x}_i^{pe} = x_i^e - x_i^p$. Similarly, $\tilde{x}_i^{ep} = -\tilde{x}_i^{pe}$. Following the steps in Section III, the control strategies for pursuer i and evader i are, respectively,

$$u_i^p = \alpha_{i2} B^{\mathrm{T}} P \tilde{x}_i^{pe}, u_i^e = -\beta_{i2} B^{\mathrm{T}} P \tilde{x}_i^{ep} = \beta_{i2} B^{\mathrm{T}} P \tilde{x}_i^{pe}. \tag{21}$$

Theorem 2: Consider the multiagent system with N pursuers and N evaders with dynamics (1) and (2), respectively, and with the control policies (21). Assume that pursuer i selects evader i as its target. Then, if $\alpha_{i2} \geq \frac{1}{2} + \beta_{i2}$, we have $\lim_{t\to\infty} \tilde{x}_i^{pe} = 0$ exponentially for any initial conditions.

Proof. See Theorem 2 of [26].

B. MPE Game for Multiple-pursuer-one-evader Problem

When there are multiple pursuers and one evader, the pursuers may want to intercept the target evader or to achieve the surrounding formation control, and the evader aims to maximize the distance from all the pursuers. In this case, we have $d_i^{pe} = 1$, $R_j^{ee} = 0$, $d_j^{ee} = 0$ and $d_j^{ep} = N$ for the unique evader $j = 1$. For simplicity, we denote $\beta_{j2} = \beta$. Thus,

$$u_1^e = -\beta N B^{\mathrm{T}} P \tilde{x}_1^{ep}. \tag{22}$$

Theorem 3: Consider the multiagent system with N pursuers and one evader with dynamics (1) and (2), respectively, and with control policies (19) and (22), respectively. Then, if $\alpha_{i2} \geq (2\beta N^2 + 1)/4$ for all $i = 1, \cdots, N$, and

(i) if $\Delta x_{i1}^{pe} = 0$ for all i, we have $\lim_{t\to\infty} \tilde{x}_i^{pe} = 0$ exponentially for any initial conditions;
(ii) if $\exists i$ such that $\Delta x_{i1}^{pe} \neq 0$ but $\sum_{i=1}^{N} \Delta x_{i1}^{pe} = 0$, and $\alpha_{i1} d_i^{pp} = 0$ for all i, we have $\lim_{t\to\infty} \tilde{x}_i^{pe} = 0$ exponentially for any initial conditions;
(iii) if $\sum_{i=1}^{N} \Delta x_{i1}^{pe} \neq 0$, the equilibrium of the closed system is globally exponentially input-to-state stable (ISS) with input Δx^{pe}.

Proof. This theorem presents sufficient conditions for capture. We thus analyze the result from the viewpoint of the pursuers. Let $\hat{x}_i^p = x_i^p - \Delta x_{i1}^{pe}$. It follows from (3) that $\tilde{x}_i^{pe} = x_1^e - \hat{x}_i^p$, whose dynamics satisfies

$$\begin{aligned}\dot{\tilde{x}}_i^{pe} =& A(x_1^e - \hat{x}_i^p) - \beta N BB^{\mathrm{T}} P \tilde{x}_1^{ep} - Bu_i^p \\ & - 2B(\alpha_{i1} d_i^{pp} B^{\mathrm{T}} P \tilde{x}_i^{pp} + \alpha_{i2} B^{\mathrm{T}} P \tilde{x}_i^{pe}) \\ =& (A - 2\alpha_{i2} BB^{\mathrm{T}} P)\tilde{x}_i^{pe} + \beta N BB^{\mathrm{T}} P \sum_{i=1}^{N} \tilde{x}_i^{pe} \\ & - 2\alpha_{i1} d_i^{pp} BB^{\mathrm{T}} P \tilde{x}_i^{pp} - \beta N BB^{\mathrm{T}} P \sum_{i=1}^{N} \Delta x_{i1}^{pe}. \end{aligned} \tag{23}$$

On the one hand, from the definition of $\tilde{x}_i^{pp}$ in (3), we have

$$\begin{aligned}\tilde{x}_i^{pp} =& \sum_{k=1}^{N} a_{ik}(x_k^p - x_1^e - \Delta x_{k1}^{pe} - x_i^p + x_1^e + \Delta x_{i1}^{pe} + \Delta x_{k1}^{pe} - \Delta x_{i1}^{pe}) \\ =& \sum_{k=1}^{N} a_{ik}(x_i^{pe} - x_k^{pe}) + \sum_{k=1}^{N} a_{ik}(\Delta x_{k1}^{pe} - \Delta x_{i1}^{pe}). \end{aligned} \tag{24}$$

Denote $\tilde{x}^{pp} = \mathrm{col}(\tilde{x}_1^{pp}, \cdots, \tilde{x}_N^{pp})$. Then, it follows that

$$\tilde{x}^{pp} = (\mathcal{L}_p \otimes I_n)\tilde{x}^{pe} - (\mathcal{L}_p \otimes I_n)\Delta x^{pe}. \tag{25}$$

Define the Lyapunov function candidate for the closed-loop system $\tilde{x}^{pe}$ as $V = (\tilde{x}^{pe})^{\mathrm{T}}(I_N \otimes P)\tilde{x}^{pe}$. Its derivative along the trajectory of (23) gives

$$\begin{aligned}\dot{V} = \sum_{i=1}^{N} \Big[& (\tilde{x}_i^{pe})^{\mathrm{T}}(A^{\mathrm{T}}P + PA - 4\alpha_{i2} PBB^{\mathrm{T}}P)\tilde{x}_i^{pe} \\ & + 2\beta N (\tilde{x}_i^{pe})^{\mathrm{T}} PBB^{\mathrm{T}} P \sum_{i=1}^{N} \tilde{x}_i^{pe} - 4\alpha_{i1} d_i^{pp} (\tilde{x}_i^{pe})^{\mathrm{T}} PBB^{\mathrm{T}} P \tilde{x}_i^{pp} \\ & - 2\beta N (\tilde{x}_i^{pe})^{\mathrm{T}} PBB^{\mathrm{T}} P \sum_{i=1}^{N} \Delta x_{i1}^{pe} \Big]. \end{aligned} \tag{26}$$

Denote $\alpha_2 = \min\{\alpha_{i2}\}$ for $i = 1, \cdots, N$. By (18), it follows that

$$\begin{aligned}\dot{V} \leq & -(4\alpha_2 - 1 - 2\beta N^2)(\tilde{x}^{pe})^{\mathrm{T}}(I_N \otimes PBB^{\mathrm{T}}P)\tilde{x}^{pe} \\ & + (\tilde{x}^{pe})^{\mathrm{T}} \mathrm{diag}\{4\alpha_{i1} d_i^{pp} PBB^{\mathrm{T}}P\}(\mathcal{L}_p \otimes I_n)\Delta x^{pe} \\ & - 2\beta N (\tilde{x}^{pe})^{\mathrm{T}}(1_N 1_N^{\mathrm{T}} \otimes PBB^{\mathrm{T}}P)\Delta x^{pe} - (\tilde{x}^{pe})^{\mathrm{T}}\tilde{x}^{pe} \end{aligned} \tag{27}$$

where the last inequality holds because $\mathcal{L}_p$ and $PBB^{\mathrm{T}}P$ are positive semi-definite.

Note that if $\alpha_{i2} \geq (2\beta N^2 + 1)/4$ for all $i = 1, \cdots, N$, the first two terms of (27) are negative, then whether or not V decreases to zero depends on the last two terms. It is obvious that $\dot{V} \leq -(\tilde{x}^{pe})^{\mathrm{T}}\tilde{x}^{pe}$ under the conditions in (i) and (ii), which finally results in $\lim_{t\to\infty} \tilde{x}_i^{pe} = 0$. If $\sum_{i=1}^{N} \Delta x_i^{pe} \neq 0$, the last term in (27) is nonzero. Then, we have

$$\begin{aligned}\dot{V} \leq & -\Big(1 - \frac{\kappa_1}{2}\lambda_{\max}^2\big(\mathrm{diag}\{4\alpha_{i1} d_i^{pp} PBB^{\mathrm{T}}P\}(\mathcal{L}_p \otimes I_n)\big) \\ & - \kappa_2 \lambda_{\max}^2(1_N 1_N^{\mathrm{T}} \otimes PBB^{\mathrm{T}}P)\Big)\|\tilde{x}^{pe}\|^2 + (\frac{1}{2\kappa_1} + \frac{1}{\kappa_2})\|\Delta x^{pe}\|^2 \end{aligned}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a symmetric matrix. Choose κ_1 and κ_2 small enough such that $1 - \frac{\kappa_1}{2}\lambda^2_{\max}\big(\mathrm{diag}\{4\alpha_{i1}d^{pp}_i PBB^{\mathrm{T}}P\}(\mathcal{L}_p \otimes I_n)\big) - \kappa_2\lambda^2_{\max}(1_N 1_N^{\mathrm{T}} \otimes PBB^{\mathrm{T}}P) > 0$. By the ISS Lyapunov theorem (Theorem 1 of [24] and Lemma 3.2 of [25]), the equilibrium of (23) is globally exponentially ISS. ■

Remark 3: The result (i) in Theorem 3 indicates that pursuers can achieve intercept if they put more effort than the evader. The condition $\alpha_{i1}d^{pp}_i = 0$ implies that the pursuers are not influenced by their neighbors but to intercept the evader. The condition $\sum_{i=1}^{N}\Delta x^{pe}_{i1} = 0$ implies a symmetric formation, under which the sum of repelling forces of the pursuers to the evader is zero. The two conditions thus contribute to interception. In (iii), the asymmetric formation leads to asymmetric forces from the pursuers' neighbor, and the forces do not align with the attractive force from the evader, which thus leads to a bounded formation error. Moreover, the greater the asymmetry, the larger the error.

C. MPE for Multiple-pursuer-multiple-evader Problem

In the multiple-pursuer-multiple-evader case, each pursuer desires to intercept its target individually or cooperatively with its neighbors. On the contrary, the evaders will try their best to prevent themselves from being intercepted, and simultaneously achieve a desired formation.

We assume that the numbers of pursuers and evaders are the same, i.e. $M = N$. If there are more pursuers, the problem can be decoupled into several multiple-pursuer-one-evader cases, and the results follow Theorem 3. If there are more evaders, some of them would be able to escape not unexpectedly. In this section, each pursuer aims to capture the target, and it is trivial to form a formation, we thus assume that $\Delta x^{pe}_{ij} = 0$ for $i = 1, \cdots, N$ and j denotes the target evader. For simplicity, we also assume that pursuer i selects evader i as its target.

Theorem 4: Consider the multiagent system with N pursuers and N evaders with dynamics (1) and (2), respectively, and with control policies (19) and (20), respectively. Then, for any $\beta_{j1} \geq \frac{1}{4\min\{d^{ee}_j\}\lambda_{\min}(\mathcal{L}_e)}$ for each evader j, there exists a $\alpha^*_2(\beta_{j2})$, such that if $\alpha_{i2} \geq \alpha^*_2(\beta_{j2})$, and

(i) if $\Delta x^{ee}_{jl} = 0$ for any evaders j and l, we have $\lim_{t\to\infty}\tilde{x}^{pe}_i = 0$ exponentially for any initial conditions;

(ii) if $\exists j, l$ such that $\Delta x^{ee}_{jl} \neq 0$ but $\alpha_{i1}d^{pp}_i = 0$ for all $i = 1, \cdots, N$, we have $\lim_{t\to\infty}\tilde{x}^{pe}_i = 0$ and $\lim_{t\to\infty}\tilde{x}^{ee}_j = 0$ exponentially for any initial conditions;

(iii) if $\exists j, l$ such that $\Delta x^{ee}_{jl} \neq 0$ and $\exists i$ such that $\alpha_{i1}d^{pp}_i \neq 0$, the equilibrium of the closed system is globally exponentially ISS with input Δx^{ee}_1.

Proof. See Theorem 4 of [26] for reference.

V. Simulation and Experimental Results

In this section, both simulation and experimental results are presented to verify our control strategies. Players are double-integrator systems described by (1) and (2) with $A = \begin{bmatrix}0_{2\times2} & I_2\\ 0_{2\times2} & 0_{2\times2}\end{bmatrix}$ and $B = \begin{bmatrix}0_{2\times2}\\ I_2\end{bmatrix}$.

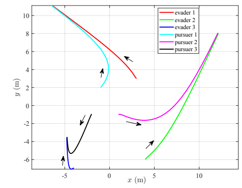

Fig. 1: PE game for one-pursuer-one-evader problem.

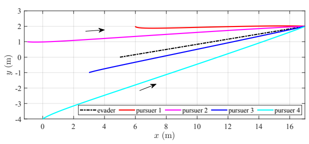

Fig. 2: Capture occurs under the conditions in (i) of Theorem 3.

Fig. 1 shows capture results that verify the control law (21) designed for the one-pursuer-one-evader PE game. In the control law (21), for $i = 1, 2, 3$, we set $\alpha_{i2} = 3$ and $\beta_{i2} = 1$, which obviously satisfies the condition $\alpha_{i2} \geq \frac{1}{2} + \beta_{i2}$.

For the case that multiple pursuers try to capture one evader, the pursuers and evader use control strategies (19) and (22), respectively. In this example, we assume that there are four pursuers, therefore, $N = 4$. To satisfy the sufficient condition $\alpha_{i2} \geq (2\beta N^2 + 1)/4$ for target capture in Theorem 2, we choose $\alpha_{i2} = 9$ and $\beta = 1$. We firstly consider the case that $\Delta x^{pe}_{i1} = 0$. The value of α_{i1} is chosen randomly. Fig. 2 shows that capture occurs, which verifies the result (i) in Theorem 3. Next, we assume that the pursuers try to achieve surrounding control of the evader, rather than capture it. Let the desired state displacement between the pursuers and the evader be $\begin{bmatrix}\Delta x^{pe}_{11} & \Delta x^{pe}_{21} & \Delta x^{pe}_{31} & \Delta x^{pe}_{41}\end{bmatrix} = \begin{bmatrix}0 & -2 & 2 & 0\\ 2 & 0 & 0 & -2\\ & 0_{2\times4} & & \end{bmatrix}$. Let $\alpha_{i1} = 0$ for $i = 1, 2, 3, 4$. It is obvious that the above settings satisfy the condition in (ii) of Theorem 3. The trajectories of players and the formation errors are

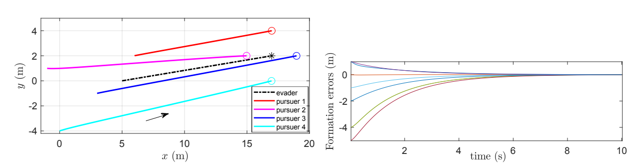

Fig. 3: Trajectories and formation errors for the result (ii) of Theorem 3.

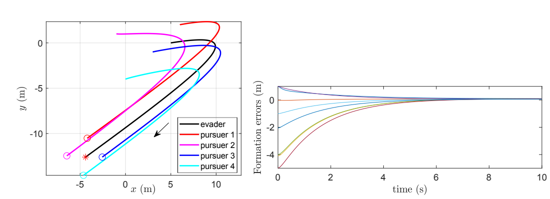

Fig. 4: Trajectories and formation errors for the result (iii) of Theorem 3.

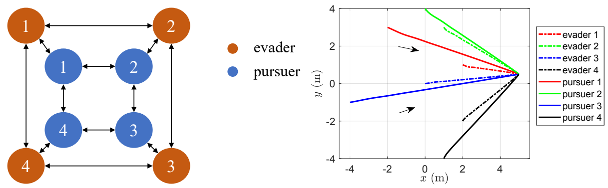

Fig. 5: (Left) Communication topology for the PE game with multiple pursuers and multiple evaders. (Right) Capture occurs under the conditions in (i) of Theorem 4.

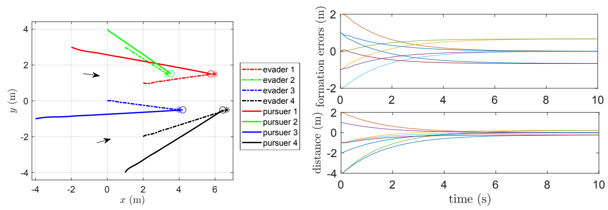

Fig. 6: Trajectories of players, formation errors of evaders, and x and y distances between each pair of pursuer and evader under the conditions (iii) of Theorem 4.

presented in Fig. 3, which shows that the formation error finally converges to zero. Now we modify the desired state displacement between the pursuers and the evader as $\left[\Delta x_{11}^{pe} \quad \Delta x_{21}^{pe} \quad \Delta x_{31}^{pe} \quad \Delta x_{41}^{pe}\right] = \begin{bmatrix} 0 & -1.9 & 2 & 0 \\ 2.01 & 0 & 0 & -1.94 \\ & 0_{2\times 4} & & \end{bmatrix}$ which does not satisfy $\sum_{i=1}^{4} \Delta x_{i1}^{pe} \neq 0$. The simulation results in Fig. 4 show the bounded formation error.

Finally, we consider the PE game for multiple pursuers and multiple evaders, who cooperate with their teammates to achieve their objectives. Suppose that there are four pursuers and four evaders. The communication topology is shown in Fig. 5. We thus have $d^{ee} = 2$ and $\lambda_{\min}(\mathcal{L}_e) = 1$. A value of $\beta_{j1} = 1$, $j = 1, 2, 3, 4$, is used to satisfy the condition $\beta_{j1} \geq \frac{1}{4\min\{d_j^{ee}\}\lambda_{\min}(\mathcal{L}_e)}$ of Theorem 4. The values of other parameters are set as $\beta_{j2} = 6$, $\alpha_{i1} = 1$ and $\alpha_{i2} = 10$ for $i = 1, 2, 3, 4$. Firstly, let $\Delta x_{jl}^{ee} = 0$. Fig. 5 shows that capture occurs. Further, let the desired formation among evaders be $\left[\Delta x_{12}^{ee} \quad \Delta x_{23}^{ee} \quad \Delta x_{34}^{ee} \quad \Delta x_{41}^{ee}\right] = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ & 0_{2\times 4} & & \end{bmatrix}$. Fig. 6 displays the simulation result, where the distances between each pair of pursuer and evader and formation errors of evaders are bounded. If we further set $\alpha_{i1} = 0$, it will cause capture and zero formation errors.

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