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The Stabilization Condition for Interval Type-2 Fuzzy Systems via Relaxed Membership-Parameter Matrix Inequalities

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Abstract— **This paper aims to investigate the relaxed stability condition for interval type-2 Takagi–Sugeno fuzzy systems via membership-parameter matrix inequalities. The membership framework of interval type-2 fuzzy sets is structured in convex polytopes with a straightforward method. The stabilization synthesis with a non-parallel distributed compensator controller incorporating lower and upper membership functions is achieved in the sense of a matrix. Moreover, this paper introduces the relaxation strategy for the orthogonal complement, effectively reducing the number of decision variables related to the linear matrix inequalities. In conclusion, examples are presented to demonstrate the effectiveness and applicability of the proposed methods.**

I. INTRODUCTION

Takagi–Sugeno (T–S) fuzzy models have the flexibility and applicability to handle the control problem of nonlinear systems [1]–[4]. Since its inception, the ability of the T–S fuzzy model to construct complex systems from combinations of linear systems using IF–THEN rule has garnered significant research interest. Due to the considerable level of interest, extensive research has been devoted to the study of fuzzy-model-based control systems from diverse perspectives, with a focus on deriving stabilization conditions expressed in the form of linear matrix inequalities (LMIs).

The utilization of interval type-2 (IT2) T–S fuzzy was subsequently proposed as a means to account for the intrinsic uncertainty associated with the modeling process by introducing lower and upper membership functions (MFs). The stabilization condition of IT2 fuzzy systems based on MF-dependent analysis was presented in [5] with the general quadratic Lyapunov function and parallel distributed compensator (PDC) control scheme.

In practice, MFs may exhibit discontinuous or angular behavior, taking the form of shapes such as parallelograms [6] or triangles [7]. Such cases are considered undifferentiable and, therefore, cannot benefit from derivative information. Nevertheless, incorporating derivative knowledge in IT2 fuzzy sets is feasible, provided appropriate lower and upper MFs are chosen. To further leverage the information contained in the MFs, the stabilization condition incorporating the derivatives of MFs was considered in [8] with a non-PDC strategy that increases the controller design flexibility.

An alternative approach was presented by [9], which proposed relaxed stability conditions that utilize the membership shape. Recently, it was demonstrated in [10] that the LMI condition can be established by the vertices of a convex polytope, which in turn capture the distribution of MFs. Utilizing a tighter convex polytope can result in less conservative conditions, which is a natural outcome.

Motivated by the mentioned discussions, the relaxed stabilization condition for IT2 fuzzy systems in terms of membership-parameter matrix inequalities with the non-PDC controller depending on the lower and upper MFs is achieved. Based on the bounding constraints of MFs, the intuitive illustration of the explicit distribution of both MFs and their upper and lower MFs is presented in the shape of convex polytopes. In contrast to the algorithm proposed by [10], this approach represents a notably more straightforward method to determine the vertices of the polytope. With the proposed control scheme, the stability conditions using the lower and upper membership-dependent Lyapunov function are analyzed. Moreover, this study introduces a novel relaxation strategy involving the orthogonal complement, which has the potential to decrease the number of decision variables associated with LMIs compared to the previous relaxation method [11].

In this paper, the following notations are used: $\mathbb{R}^n$ stands for the n-dimensional real vector space; $P > 0$ represents that the matrix P is a positive definite; 1_n stands for the n-dimensional column vector whose all entries are ones; $0_{n\times m}$ stands for the zero matrix with $n \times m$ order; I_n denotes the identity matrix with $n \times n$ order; $\otimes$ is the Kronecker product symbol; $\mathbf{He}\{A\}$ denotes $A + A^T$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. IT2 T–S Fuzzy Model

An IT2 T–S fuzzy model with n_r rules and m premise variables is considered. The k-th fuzzy rule is represented below:

$$\textbf{Rule } k : \textbf{IF } f_1(x(t)) \text{ is } \mathcal{F}_1^k \text{ and } \cdots \text{ and } f_m(x(t)) \text{ is } \mathcal{F}_m^k$$
$$\textbf{THEN } \dot{x}(t) = A_k x(t) + B_k u(t),$$

where $\mathcal{F}_i^k$ is the k-th IT2 fuzzy set corresponding to the premise variable $f_i(x(t))$ for $i = 1, ..., m$ and $k = 1, ..., n_r$;

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The k-th consequent yields the system matrix A_k and input matrix B_k which are given previously; $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$ denote the state and input vector, respectively. The k-th firing strength is in interval sets as follows:

$$\begin{aligned}\theta_k(x(t)) &\in \left[\ \Pi_{i=1}^{m}\rho^{L}_{\mathcal{F}_i^k}(f_i(x(t))),\ \Pi_{i=1}^{m}\rho^{U}_{\mathcal{F}_i^k}(f_i(x(t)))\ \right] \\ &= \left[\ \theta_k^L(x(t)),\ \theta_k^U(x(t))\ \right],\end{aligned}$$

where $\theta_k^L(x(t))$ and $\theta_k^U(x(t))$ are the given lower and upper MFs for the k-th rule; $\rho^{L}_{\mathcal{F}_i^k}(f_i(x(t)))$ and $\rho^{U}_{\mathcal{F}_i^k}(f_i(x(t)))$ stand for the grade of membership satisfying

$$0 \leq \rho^{L}_{\mathcal{F}_i^k}(f_i(x(t))) \leq \rho^{U}_{\mathcal{F}_i^k}(f_i(x(t))) \leq 1,$$

and then

$$0 \leq \theta_k^L(x(t)) \leq \theta_k^U(x(t)) \leq 1.$$

The k-th rule's MF is represented as $\theta_k(x(t)) = \sigma_k(x(t))\theta_k^L(x(t)) + (1 - \sigma_k(x(t)))\theta_k^U(x(t))$ in which $\sigma_k(x(t)) \in [0,1]$ is the nonlinear weight function and satisfies $\sum_{i=1}^{n_r}\theta_i(x(t)) = 1$. For simplicity in writing, the state vector notation in MFs is omitted, i.e., $\theta_k(x(t))$ is illustrated as θ_k.

Assumption 1: The derivative of the lower and upper MFs exist with the following conditions:

$$\alpha_k^L \leq \dot{\theta}_k^L \leq \alpha_k^U, \tag{1}$$

$$\beta_k^L \leq \dot{\theta}_k^U \leq \beta_k^U, \tag{2}$$

where α_k^L, α_k^U, β_k^L and β_k^U are given scalar values for $k = 1, ..., n_r$.

In practice, when the MF is discontinuous or with the shapes of a parallelogram, a triangle, its derivative($\dot{\theta}(x(t))$) does not exist. Meanwhile, since the differentiable boundary MFs can be defined appropriately, the derivative information for the lower and upper MFs is admissible. Thus, assumption 1 is reasonable.

Here, the boundary conditions for the MFs are described:

$$0 \leq \gamma_k^L \leq \theta_k^L \leq \theta_k \leq \theta_k^U \leq \gamma_k^U \leq 1, \tag{3}$$

$$0 \leq \omega^L \leq \sum_{i=1}^{n_r}\theta_i^L \leq \sum_{i=1}^{n_r}\theta_i \leq \sum_{i=1}^{n_r}\theta_i^U \leq \omega^U, \tag{4}$$

where the extrema γ_k^L, γ_k^U, ω^L and ω^U are given as scalar values for $k = 1, ..., n_r$.

Let us define the following membership vectors:

$$\theta^L = \begin{bmatrix} \theta_1^L \\ \vdots \\ \theta_{n_r}^L \end{bmatrix},\ \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{n_r} \end{bmatrix},\ \theta^U = \begin{bmatrix} \theta_1^U \\ \vdots \\ \theta_{n_r}^U \end{bmatrix}.$$

The IT2 T–S fuzzy model described in the fuzzy rule becomes a closed-loop form as follows:

$$\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t), \tag{5}$$

where $A(\theta) = \sum_{i=1}^{n_r}\theta_i A_i$ and $B(\theta) = \sum_{i=1}^{n_r}\theta_i B_i$.

B. Non-PDC Membership Dependent Controller

Non-PDC control methods can lead to the enlargement of feasible solutions, which leads to acceptability for nonlinear systems with relatively inaccurate model descriptions. From this perspective, the non-PDC IT2 fuzzy controller depending on the lower and upper MFs is considered

$$u(t) = \mathcal{K}(\theta^L, \theta^U)x(t), \tag{6}$$

where $\mathcal{K}(\theta^L, \theta^U) \in \mathbb{R}^{n_u \times n_x}$ is described as follows:

$$\begin{aligned}\mathcal{K}(\theta^L, \theta^U) =& K + K^L(\theta^L \otimes I_{n_x}) + K^U(\theta^U \otimes I_{n_x}) \\ &+ (\theta^L \otimes I_{n_u})^T K^{LU}(\theta^U \otimes I_{n_x}) + (\theta^L \otimes I_{n_u})^T \\ &\times K^{LL}(\theta^L \otimes I_{n_x}) + (\theta^U \otimes I_{n_u})^T K^{UU}(\theta^U \otimes I_{n_x})\end{aligned}$$

with the proper-dimension gain matrices K, K^L, K^U, K^{LU}, K^{LL} and K^{UU}. The proposed control method describes the second-order polynomials for the lower and upper MFs resulting in the following quadratic expression for $\mathcal{K}(\theta^L, \theta^U)$:

$$\begin{bmatrix} 1 \otimes I_{n_u} \\ \theta^L \otimes I_{n_u} \\ \theta^U \otimes I_{n_u} \end{bmatrix}^T \begin{bmatrix} K & K^L & K^U \\ 0 & K^{LL} & K^{LU} \\ 0 & 0 & K^{UU} \end{bmatrix} \begin{bmatrix} 1 \otimes I_{n_x} \\ \theta^L \otimes I_{n_x} \\ \theta^U \otimes I_{n_x} \end{bmatrix}. \tag{7}$$

The state feedback control scheme (6) to stabilize (5) yields the closed-loop system

$$\dot{x}(t) = (A(\theta) + B(\theta)\mathcal{K}(\theta^L, \theta^U))x(t). \tag{8}$$

C. IT2 Fuzzy Membership Distribution

The boundary conditions of MFs in (3), (4) and the convex-sum property($\sum_{i=1}^{n_r}\theta_i(x(t)) - 1 = 0$) are exploited for discovering the membership distribution including lower and upper memberships. Let us define a (n_r-1)-dimensional simplex $\Delta_\sigma^{n_r-1}$ containing the n_r-dimensional distribution θ^* as:

$$\Delta_\sigma^{n_r-1} = \left\{\theta^* \in \mathbb{R}^{n_r} \mid \sum_{i=1}^{n_r}\theta_i^* = \sigma, 0 \leq \theta_i^* \leq 1\right\}, \tag{9}$$

where σ is positive. Obviously, $\Delta_1^{n_r-1}$ is a standard simplex enclosing the membership trajectory of $\theta \in \mathbb{R}^{n_r}$. Now, based on the boundary constraint (3), the l-dimensional extreme orthotope with 2^{n_r} points placed on the combination of γ_i^L and γ_j^U is defined as $E_\theta^{n_r}$ for $i, j \in \{1, ..., n_r\}$. Although the exact extreme values for θ exist, it is virtually impossible to calculate the extrema for θ. However, it can be concluded that the MF is captured by the (n_r-1)-dimensional convex polytope, which is the intersection spaces between $\Delta_1^{n_r-1}$ and $E_\theta^{n_r}$, and the set of vertices for the convex polytope is defined as $S_\theta^{n_r}$.

In the same way, when σ is ω^L or ω^U, two intersection spaces between $\Delta_\sigma^{n_r-1}$ and $E_\theta^{n_r}$ can be described. Each set of vertices for those convex polytopes is defined as $S_{\theta^L}^{n_r^*}$ and $S_{\theta^U}^{n_r^*}$, respectively. Then the distribution of the lower MF is included by the n_r-dimensional convex polytope with vertices in $S_{\theta^L}^{n_r}$ where the set $S_{\theta^L}^{n_r}$ is $S_{\theta^L}^{n_r^*} \cup S_\theta^{n_r}$. Similarly, the convex polytope with vertices in $S_{\theta^U}^{n_r}$ encloses the upper

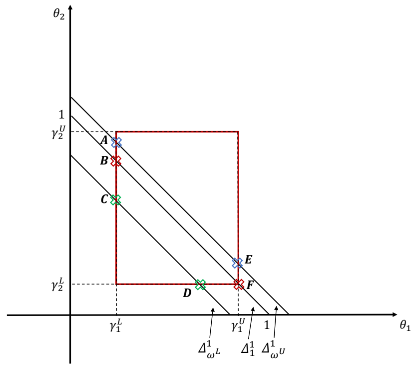

Fig. 1. Type-2 fuzzy membership distribution in the two-dimensional space. The red rectangle stands for E_{θ}^{2}. The crossing points sets are illustrated as follows: $S_{\theta}^{2}=\{B,F\}$, $S_{\theta^L}^{2}=\{B,C,D,F\}$ and $S_{\theta^U}^{2}=\{A,B,E,F\}$.

membership distribution where $S_{\theta^U}^{n_r}=S_{\theta^U}^{n_r^*}\cup S_{\theta}^{n_r}$. In Fig. 1, the membership distributions with $n_r=2$ are illustrated. The line segment between points B and F includes the distribution for θ. The lower and upper membership trajectories are captured by the convex polytopes with vertices in $S_{\theta^L}^{n_r}$ and $S_{\theta^U}^{n_r}$, respectively.

III. Stability Analysis

A. Stability Analysis with Relaxation Technique

For extracting the lower and upper membership knowledge, let us consider the lower and upper membership-dependent Lyapunov function:

$$V(x(t))=x^T(t)P(\theta^L,\theta^U)x(t), \tag{10}$$

where $P(\theta^L,\theta^U)\in\mathbb{R}^{n_x\times n_x}$ is a positive and symmetric matrix, and defined as

$$P(\theta^L,\theta^U)=\mathcal{P}^L(\theta^L\otimes I_{n_x})+\mathcal{P}^U(\theta^U\otimes I_{n_x}),$$

where $\mathcal{P}^L\triangleq\begin{bmatrix}P_1^L & \cdots & P_{n_r}^L\end{bmatrix}$ and $\mathcal{P}^U\triangleq\begin{bmatrix}P_1^U & \cdots & P_{n_r}^U\end{bmatrix}$ with positive and symmetric matrices P_i^L and P_i^U for $i=1,...,n_r$. For guaranteeing the stability of (8), the derivative of the abstract energy function (10),

$$\begin{aligned}\dot{V}(x(t))=x^T(t)\dot{P}(\theta^L,\theta^U)x(t)+x^T(t)\mathbf{He}\Big\{P(\theta^L,\theta^U)\\ \times A(\theta)+P(\theta^L,\theta^U)B(\theta)\mathcal{K}(\theta^L,\theta^U)\Big\}x(t),\end{aligned} \tag{11}$$

should be negative.

Theorem 1: It is supposed that there exist matrices X_{ij},$Y_{ij}\in\mathbb{R}^{n_x\times n_x}$, $M\in\mathbb{R}^{3n_rn_x\times(5n_r+1)n_x}$, $\bar{K}\in\mathbb{R}^{n_u\times n_x}$, $\bar{K}^L,\bar{K}^U\in\mathbb{R}^{n_u\times n_rn_x}$, $\bar{K}^{LL},\bar{K}^{LU},\bar{K}^{UU}\in\mathbb{R}^{n_rn_u\times n_rn_x}$ and symmetric matrices $0<\bar{P}_i^L,\bar{P}_i^U\in\mathbb{R}^{n_x\times n_x}$ for $i,j=1,...,n_r$ such that the conditions

$$X_{ij}+X_{ij}^T>0,\ Y_{ij}+Y_{ij}^T>0, \tag{12}$$

for $1\le i,j\le n_r$, and the triple-parameter LMI;

$$\mathbf{He}\{\Psi+\Phi+\Xi\}<0, \tag{13}$$

at the all combinations among $S_{\theta^L}^{n_r},S_{\theta}^{n_r},S_{\theta^U}^{n_r}$ where

$$\Psi=\begin{bmatrix}\psi_{11} & 0 & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \psi_{33} & \psi_{34} & 0 & 0\\ 0 & 0 & 0 & \psi_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

with $\psi_{11}=B(\theta)\bar{K}$, $\psi_{13}=A(\theta)\bar{\mathcal{P}}_L+B(\theta)\bar{K}^L$, $\psi_{14}=A(\theta)\bar{\mathcal{P}}_U+B(\theta)\bar{K}^U$, $\psi_{15}=-\frac{1}{2}\bar{\mathcal{P}}_L$, $\psi_{16}=-\frac{1}{2}\bar{\mathcal{P}}_U$, $\psi_{33}=(I_{n_r}\otimes B(\theta))\bar{K}^{LL}$, $\psi_{34}=(I_{n_r}\otimes B(\theta))\bar{K}^{LU}$, $\psi_{44}=(I_{n_r}\otimes B(\theta))\bar{K}^{UU}$, and

$$\Phi=\begin{bmatrix}\phi_{11} & 0 & 0 & 0 & \phi_{15} & \phi_{16}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \phi_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \phi_{66}\end{bmatrix},$$

with

$$\phi_{11}=-\sum_{i=1}^{n_r}\sum_{j=1}^{n_r}(X_{ij}\alpha_i^L\alpha_j^U+Y_{ij}\beta_i^L\beta_j^U),$$

$$\begin{aligned}\phi_{15}=\Bigg[\sum_{i=1}^{n_r}(X_{i1}\alpha_i^L+X_{1i}\alpha_i^U)\ \sum_{i=1}^{n_r}(X_{i2}\alpha_i^L+X_{2i}\alpha_i^U)\\ \cdots\ \sum_{i=1}^{n_r}(X_{in_r}\alpha_i^L+X_{n_ri}\alpha_i^U)\Bigg]\end{aligned}$$

$$\begin{aligned}\phi_{16}=\Bigg[\sum_{i=1}^{n_r}(Y_{i1}\beta_i^L+Y_{1i}\beta_i^U)\ \sum_{i=1}^{n_r}(Y_{i2}\beta_i^L+Y_{2i}\beta_i^U)\\ \cdots\ \sum_{i=1}^{n_r}(Y_{in_r}\beta_i^L+Y_{n_ri}\beta_i^U)\Bigg]\end{aligned}$$

$$\phi_{55}=-\begin{bmatrix}X_{11} & X_{12} & \cdots & X_{1n_r}\\ X_{21} & X_{22} & \cdots & X_{2n_r}\\ \vdots & \vdots & \ddots & \vdots\\ X_{n_r1} & X_{n_r2} & \cdots & X_{n_rn_r}\end{bmatrix}$$

$$\phi_{66}=-\begin{bmatrix}Y_{11} & Y_{12} & \cdots & Y_{1n_r}\\ Y_{21} & Y_{22} & \cdots & Y_{2n_r}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{n_r1} & Y_{n_r2} & \cdots & Y_{n_rn_r}\end{bmatrix},$$

and

$$\Xi=\begin{bmatrix}-(\theta^L)^T\otimes I_{n_x} & -\theta^T\otimes I_{n_x} & -(\theta^U)^T\otimes I_{n_x}\\ 0 & I_{n_r}\otimes I_{n_x} & 0\\ I_{n_r}\otimes I_{n_x} & 0 & 0\\ 0 & 0 & I_{n_r}\otimes I_{n_x}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}M. \tag{14}$$

The matrices whose i-th entries are $\bar{P}_i^L,\bar{P}_i^U$, respectively, defined as $\bar{\mathcal{P}}^L$, $\bar{\mathcal{P}}^U$ (i.e., $\bar{\mathcal{P}}^L\triangleq\begin{bmatrix}\bar{P}_1^L & \cdots & \bar{P}_{n_r}^L\end{bmatrix}$, $\bar{\mathcal{P}}^U\triangleq\begin{bmatrix}\bar{P}_1^U & \cdots & \bar{P}_{n_r}^U\end{bmatrix}$) and $\bar{P}(\theta^L,\theta^U)=\bar{\mathcal{P}}^L(\theta^L\otimes I_{n_x})+\bar{\mathcal{P}}^U(\theta^U\otimes I_{n_x})$. Then the IT2 T–S fuzzy system (8) is asymptotically stable. And, the control gain $\mathcal{K}(\theta^L,\theta^U)$ in

(6) is obtained by the following structured gain matrices: $K = \bar{K}(\bar{P}(\theta^L,\theta^U))^{-1}$, $K^L = \bar{K}^L(I_{n_r}\otimes\bar{P}(\theta^L,\theta^U))^{-1}$, $K^U = \bar{K}^U(I_{n_r}\otimes\bar{P}(\theta^L,\theta^U))^{-1}$, $K^{LL} = \bar{K}^{LL}(I_{n_r}\otimes\bar{P}(\theta^L,\theta^U))^{-1}$, $K^{LU} = \bar{K}^{LU}(I_{n_r}\otimes\bar{P}(\theta^L,\theta^U))^{-1}$, $K^{UU} = \bar{K}^{UU}(I_{n_r}\otimes\bar{P}(\theta^L,\theta^U))^{-1}$.

Proof: Based on the replacement technique, the derivative of Lyapunov function given in (11) is rewritten as,

$$\begin{aligned}\dot{V}(x(t)) &= x^TP(\theta^L,\theta^U)\Big\{-\dot{\bar{P}}(\theta^L,\theta^U)+\mathbf{He}\Big\{A(\theta)\\ &\quad\times\bar{P}(\theta^L,\theta^U)+B(\theta)\bar{\mathcal{K}}(\theta^L,\theta^U)\Big\}\Big\}P(\theta^L,\theta^U)x\\ &= x^TP(\theta^L,\theta^U)\eta^T\mathbf{He}\{\Psi\}\eta P(\theta^L,\theta^U)x, \qquad (15)\end{aligned}$$

where $\bar{P}(\theta^L,\theta^U) = P^{-1}(\theta^L,\theta^U)$, $\bar{\mathcal{K}}(\theta^L,\theta^U) = \mathcal{K}(\theta^L,\theta^U)\bar{P}(\theta^L,\theta^U)$, $\dot{\bar{P}}(\theta^L,\theta^U) = -\bar{P}(\theta^L,\theta^U)\dot{P}(\theta^L,\theta^U)\bar{P}(\theta^L,\theta^U)$ and the outer matrix η is denoted as $[I_{n_x}|\theta^T\otimes I_{n_x}|(\theta^L)^T\otimes I_{n_x}|(\theta^U)^T\otimes I_{n_x}|(\dot{\theta}^L)^T\otimes I_{n_x}|(\dot{\theta}^U)^T\otimes I_{n_x}]^T$. By the property of kronecker products, it holds that

$$B(\theta)(\theta^L\otimes I_{n_u})^T\bar{K}^{LL} = (\theta^L\otimes I_{n_x})^T(I_{n_r}\otimes B(\theta))\bar{K}^{LL},$$

this switching method was applied to $\bar{K}^{LU},\bar{K}^{UU}$ in the same manner and yields ψ_{33},ψ_{34} and ψ_{44}.

Nonetheless, $\mathbf{He}\{\Psi\}<0$ at every vertex of $S_\theta^{n_r}$ can imply that the derivative of Lyapunov functions is negative, the LMI is not solvable due to its zero diagonal terms. Using relaxed conditions by membership constraints, the feasible stabilization condition can be achieved in the shape of the LMI with triple parameters. The orthogonal complements yield that $\eta^T\mathbf{He}\{\Xi\}\eta = 0$ with slack matrix M described in (14). Furthermore, Ξ has triple-membership parameters θ^L,θ and θ^U.

From the relations in (1) and (2), the relaxed condition is facilitated by introducing the slack matrices $X_{ij},Y_{ij}\in\mathbb{R}^{n_x\times n_x}$ such that (12) for $1\le i,j\le n_r$. Then it is satisfied with the following description:

$$\begin{cases}-(\dot{\theta}_i^L-\alpha_i^L)(\dot{\theta}_j^L-\alpha_j^U)(X_{ij}+X_{ij}^T)>0,\\ -(\dot{\theta}_i^U-\beta_i^L)(\dot{\theta}_j^U-\beta_j^U)(Y_{ij}+Y_{ij}^T)>0.\end{cases} \qquad (16)$$

for $1\le i,j\le n_r$. Here, the relaxed relations (16) leads that $\eta^T\mathbf{He}\{\Phi\}\eta>0$.

Thus, we have

$$\dot{V}(x(t)) < x^TP(\theta^L,\theta^U)\eta^T\mathbf{He}\{\Psi+\Phi+\Pi\}\eta P(\theta^L,\theta^U)x,$$

and the relaxed stabilization condition is derived in the triple-membership-parameter matrix inequalities (13). ∎

While the upper and lower MFs can be obtained in advance, computing the control gain matrices depending $\bar{P}(\theta^L,\theta^U)$ requires times to solve LMIs. One strategy for circumventing the need to perform online updates of the control gain involves selecting (10) as a general quadratic Lyapunov function.

Corollary 1: It is supposed that there exist matrices $M\in\mathbb{R}^{3n_rn_x\times(3n_r+1)n_x}$, $\bar{K}\in\mathbb{R}^{n_u\times n_x}$, $\bar{K}^L,\bar{K}^U\in\mathbb{R}^{n_u\times n_rn_x}$, $\bar{K}^{LL},\bar{K}^{LU},\bar{K}^{UU}\in\mathbb{R}^{n_rn_u\times n_rn_x}$ and symmetric matrices $0<\bar{P}\in\mathbb{R}^{n_x\times n_x}$ such that the conditions

$$\mathbf{He}\{\tilde{\Psi}+\tilde{\Xi}\}<0, \qquad (17)$$

at the all combinations among $S_{\theta^L}^{n_r},S_\theta^{n_r},S_{\theta^U}^{n_r}$ where

$$\tilde{\Psi} = \begin{bmatrix}\tilde{\psi}_{11} & 0 & \tilde{\psi}_{13} & \tilde{\psi}_{14}\\ 0&0&0&0\\ 0&0&\tilde{\psi}_{33}&\tilde{\psi}_{34}\\ 0&0&0&\tilde{\psi}_{44}\end{bmatrix},$$

with $\tilde{\psi}_{11}=B(\theta)\bar{K}+A(\theta)\bar{P}$, $\tilde{\psi}_{13}=B(\theta)\bar{K}^L$, $\tilde{\psi}_{14}=B(\theta)\bar{K}^U$, $\tilde{\psi}_{33} = (I_{n_r}\otimes B(\theta))\bar{K}^{LL}$, $\tilde{\psi}_{34} = (I_{n_r}\otimes B(\theta))\bar{K}^{LU}$, $\tilde{\psi}_{44} = (I_{n_r}\otimes B(\theta))\bar{K}^{UU}$, and

$$\tilde{\Xi} = \begin{bmatrix}-(\theta^L)^T\otimes I_{n_x} & -\theta^T\otimes I_{n_x} & -(\theta^U)^T\otimes I_{n_x}\\ 0 & I_{n_r}\otimes I_{n_x} & 0\\ I_{n_r}\otimes I_{n_x} & 0 & 0\\ 0 & 0 & I_{n_r}\otimes I_{n_x}\end{bmatrix}M. \qquad (18)$$

Then the IT2 T–S fuzzy system (8) is asymptotically stable. Furthermore, the control gain $\mathcal{K}(\theta^L,\theta^U)$ in (6) is obtained with the following structured gain matrices: $K = \bar{K}\bar{P}^{-1}$, $K^L = \bar{K}^L(I_{n_r}\otimes\bar{P})^{-1}$, $K^U = \bar{K}^U(I_{n_r}\otimes\bar{P})^{-1}$, $K^{LL} = \bar{K}^{LL}(I_{n_r}\otimes\bar{P})^{-1}$, $K^{LU} = \bar{K}^{LU}(I_{n_r}\otimes\bar{P})^{-1}$, $K^{UU} = \bar{K}^{UU}(I_{n_r}\otimes\bar{P})^{-1}$.

Proof : Instead of the membership-dependent Lyapunov function in (10), the general quadratic Lyapunov function $V(x(t)) = x^T(t)Px(t)$ is considered where a symmetric matrix $0<P\in\mathbb{R}^{n_x\times n_x}$. When $\bar{P}=P^{-1}$, the proof is proceeded in the same way of the theorem 1. ∎

Remark 1: The conventional relaxed technique [12] employing orthogonal complementation necessitates a larger slack matrix for more outer variables (in theorem 1, the outer variable is η). However, it can not be asserted that all slack variables corresponding to the orthogonal matrix are necessary. The orthogonal complements are rewritten as follows:

$$\begin{aligned}\bar{\Xi}_1 &= \begin{bmatrix}I_{n_x}\\ \theta^L\otimes I_{n_x}\end{bmatrix}^T\begin{bmatrix}-(\theta^L)^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_1\\ \bar{\Xi}_2 &= \begin{bmatrix}I_{n_x}\\ \theta\otimes I_{n_x}\end{bmatrix}^T\begin{bmatrix}-\theta^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_2\\ \bar{\Xi}_3 &= \begin{bmatrix}I_{n_x}\\ \theta^U\otimes I_{n_x}\end{bmatrix}^T\begin{bmatrix}-(\theta^U)^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_3\end{aligned}$$

where $\bar{\Xi}_i=0$ for $i=1,2,3$ with slack matrices $M_1,M_2,M_3\in\mathbb{R}^{n_rn_x\times(n_r+1)n_x}$. In the light of this perspective, while exploiting knowledge of the orthogonality, the relaxed condition can be described as $\bar{\Xi}=0$ where

$$\begin{aligned}\bar{\Xi} = \begin{bmatrix}-(\theta^L)^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_1 + \begin{bmatrix}-\theta^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_2\\ + \begin{bmatrix}-(\theta^U)^T\otimes I_{n_x}\\ I_{n_r}\otimes I_{n_x}\end{bmatrix}M_3. \qquad (19)\end{aligned}$$

Thus another candidate for relaxing orthogonal complement (19), which can reduce the number of decision variables, can be deemed on behalf of (14) and (18) with proper matrix manipulations.

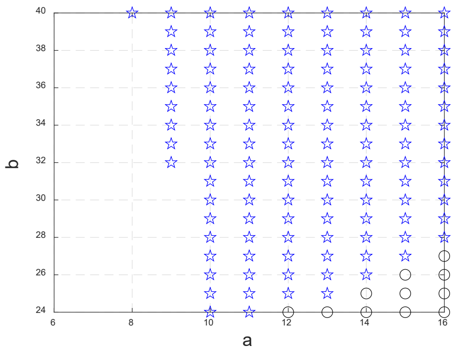

Fig. 2. Stability region of the fuzzy system with different system parameters a and b.

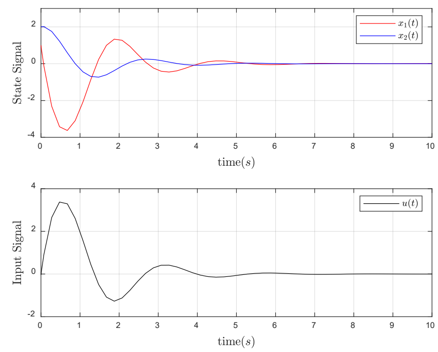

Fig. 3. State and Input histories of the fuzzy system.

TABLE I

THE NUMBER OF DECISION VARIABLES

Method	[8]	Thm1	Thm1*
# variables	140	734	302

IV. SIMULATION

The simulation ran on a laptop with 2.4GHz Intel Core i5-1135G7 and 8GB of RAM. For solving LMI in MATLAB 2020b, CVX is used with the option *"precision best"* to obtain more strict solutions.

Example 1: In [5], [8], the numerical IT2 T–S fuzzy system under three rules is described as follows:

$$\textbf{Rule } k \textbf{ :IF } x_1(t) \text{ is } \mathcal{F}_1^k,$$
$$\textbf{THEN } \dot{x}(t) = A_k x(t) + B_k u(t), \qquad k = 1, 2, 3$$
$$A_1 = \begin{bmatrix} 2.78 & -5.63 \\ 0.01 & 0.33 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.2 & -3.22 \\ 0.35 & 0.12 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -a & -6.63 \\ 0.45 & 0.15 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix},$$

where $x(t) = [x_1(t)\ x_2(t)]^{\mathrm{T}}$, $x_1(t) \in [-10, 10]$ and a, b are the constant parameters of a nonlinear plant to explore a stability region. The lower and upper MFs are given as follows: $\theta_1^L = 0.95 - 0.925/(1 + exp(-x_1(t) - 4.5)/8)$, $\theta_1^U = 0.95 - 0.925/(1 + exp(-x_1(t) - 3.5)/8)$, $\theta_3^L = 0.025 + 0.925/(1 + exp(-x_1(t) + 4.5)/8)$, $\theta_3^U = 0.025 + 0.925/(1 + exp(-x_1(t) + 3.5)/8)$, and $\theta_2^L = 1 - (\theta_1^U + \theta_3^U)$, $\theta_2^U = 1 - (\theta_1^L + \theta_3^L)$.

For comparison purposes, the stabilization conditions based on the corollary 1 with algorithm 2 ($q = 7$) proposed in [8] is shown. The proposed theorem is described as Thm1. And, as discussed in remark 1, the structured relaxation method using the orthogonal complement (19) is depicted for comparison as Thm1*. For the abovementioned methods, the number of decision variables "# variables" is given in Table I. It is noteworthy that [8], which studied polynomially homogeneous matrices, did not consider relaxation techniques, resulting in smaller # variables.

The stability region of the fuzzy system with different values of a and b is shown in Fig. 2. The feasible solution regions for the method in [8] are illustrated as $\circ$. The feasible solution regions by Thm1 and Thm1* are denoted as $\star \cup \circ$. The results verifying the stability region show that the proposed method can drastically enlarge stability regions. Significantly, the stability region of Thm1* is equivalent to that of Thm1, despite a reduction of nearly half in its decision variables. This observation underscores the remarkable efficacy of the relaxation strategy proposed in remark 1.

The points set $S_{\theta^L}^{3^*}, S_\theta^3$ and $S_{\theta^U}^{3^*}$ are defined as

$$\begin{aligned} S_{\theta^L}^{3^*} = \{&(0.6657, 0.1795, 0.1548)(0.1548, 0.1649, 0.6259) \\ &(0.1548, 0.2785, 0.5124)(0.5124, 0.2785, 0.1548) \\ &(0.6259, 0.1649, 0.1548)\}, \\ S_\theta^3 = \{&(0.1548, 0.1795, 0.6657)(0.1548, 0.2785, 0.5667) \\ &(0.1694, 0.1649, 0.6657)(0.5667, 0.2785, 0.1548) \\ &(0.6657, 0.1649, 0.1694)(0.6657, 0.1795, 0.1548)\}, \\ S_{\theta^U}^{3^*} = \{&((0.1548, 0.2338, 0.6657)(0.1548, 0.2785, 0.6210) \\ &(0.2237, 0.1649, 0.6657)(0.6210, 0.2785, 0.1548) \\ &(0.6657, 0.1649, 0.2237)(0.6657, 0.2338, 0.1548))\}. \end{aligned}$$

It is notable that $S_{\theta^L}^3 = S_{\theta^L}^{3^*} \cup S_\theta^3$ and $S_{\theta^U}^3 = S_{\theta^U}^{3^*} \cup S_\theta^3$.

Fig. 3 shows the trajectories for the signal of states and input using Cor1 with the system parameter $a = 16$ and $b = 24$. The control gains are given as follows:

$$\begin{aligned} K &= \begin{bmatrix} -0.7877 & 0.3531 \end{bmatrix}, \\ K^L &= \begin{bmatrix} -32.9 & 11.3 & -32.7 & -3.8 & -32.9 & 11.3 \end{bmatrix} \text{e-03}, \\ K^U &= \begin{bmatrix} -37.3 & 12.6 & -30.6 & 0.2 & -37.3 & 12.6 \end{bmatrix} \text{e-03}, \\ K^{LL} &= \begin{bmatrix} 13.7 & 24.6 & -1.8 & 3.9 & 0.7 & 2.2 \\ 4.3 & 6.8 & 10.1 & 8.3 & 4.3 & 6.8 \\ 0.7 & 2.2 & -1.8 & 3.9 & 13.7 & 24.6 \end{bmatrix} \text{e-03}, \\ K^{LU} &= \begin{bmatrix} 4.4 & 12.2 & 2.2 & 6.4 & 4.4 & 12.2 \\ 2.2 & 6.3 & 1.1 & 3.3 & 2.2 & 6.3 \\ 4.4 & 12.2 & 2.2 & 6.4 & 4.4 & 12.2 \end{bmatrix} \text{e-03}, \\ K^{UU} &= \begin{bmatrix} 14.1 & 27.9 & -1.0 & 5.7 & 0.9 & 4.3 \\ 3.7 & 7.9 & 10.1 & 9.0 & 3.7 & 7.9 \\ 0.9 & 4.3 & -1.0 & 5.7 & 14.1 & 27.9 \end{bmatrix} \text{e-03}. \end{aligned}$$

Example 2: Through the inverted pendulum of [5], the proposed method is validated. The state equation of the pendulum is described as follows with the

TABLE II

MFS OF AN INVERTED PENDULUM SYSTEM

$\rho^L_{\mathcal{F}^1_1}(f_1(x(t))) = \rho^L_{\mathcal{F}^2_1}(f_1(x(t))) = \frac{-f_1(x(t))+f_1^{max}}{f_1^{max}-f_1^{min}}$ $\rho^U_{\mathcal{F}^3_1}(f_1(x(t))) = \rho^U_{\mathcal{F}^4_1}(f_1(x(t))) = \frac{f_1(x(t))-f_1^{min}}{f_1^{max}-f_1^{min}}$ with $x_2(t) = 0, m_p = 3$kg and $M_c = 8$kg
$\rho^U_{\mathcal{F}^1_1}(f_1(x(t))) = \rho^U_{\mathcal{F}^2_1}(f_1(x(t))) = \frac{-f_1(x(t))+f_1^{max}}{f_1^{max}-f_1^{min}}$ $\rho^L_{\mathcal{F}^3_1}(f_1(x(t))) = \rho^L_{\mathcal{F}^4_1}(f_1(x(t))) = \frac{f_1(x(t))-f_1^{min}}{f_1^{max}-f_1^{min}}$ with $x_2(t) = 5, m_p = 3$kg and $M_c = 8$kg
$\rho^L_{\mathcal{F}^1_2}(f_2(x(t))) = \rho^L_{\mathcal{F}^3_2}(f_2(x(t))) = \frac{-f_2(x(t))+f_2^{max}}{f_2^{max}-f_2^{min}}$ $\rho^U_{\mathcal{F}^2_2}(f_2(x(t))) = \rho^U_{\mathcal{F}^4_2}(f_2(x(t))) = \frac{f_2(x(t))-f_2^{min}}{f_2^{max}-f_2^{min}}$ with $m_p = 3$kg and $M_c = 12$kg
$\rho^U_{\mathcal{F}^1_2}(f_2(x(t))) = \rho^U_{\mathcal{F}^3_2}(f_2(x(t))) = \frac{-f_2(x(t))+f_2^{max}}{f_2^{max}-f_2^{min}}$ $\rho^L_{\mathcal{F}^2_2}(f_2(x(t))) = \rho^L_{\mathcal{F}^4_2}(f_2(x(t))) = \frac{f_2(x(t))-f_2^{min}}{f_2^{max}-f_2^{min}}$ with $m_p = 2$kg and $M_c = 8$kg

state vector $x(t) = [x_1(t)\ \ x_2(t)]^T$: $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = \frac{gsin(x_1(t))-am_plx_2(t)^2sin(2x_1(t))/2-acos(x_1(t))u(t)}{4l/3-am_pl(cos(x_1(t)))^2}$, where $x_1(t) \in [-5\pi/12, 5\pi/12]$ is the angular displacement (rad) of the pendulum, $x_2(t) \in [-5, 5]$ is the angular velocity (rad/s) of the pendulum, the control input $u(t)$ is the force applied to the cart (N), $g = 9.8\ m/s^2$ is the acceleration due to gravity, the masses of the pendulum are $m_p \in$ [2kg, 3kg]. The masses of the cart were $M_c \in$ [8kg, 12kg]. The length of the pendulum is $l = 0.5$ m, and $a = \frac{1}{m_p+M_c}$.

An IT2 T–S fuzzy system (5) with four rules can describe the inverted pendulum system with the uncertain modeling parameters m_p and M_c.

Rule k :**IF** $f_1(x(t))$ is $\mathcal{F}^k_1$ and $f_2(x(t))$ is $\mathcal{F}^k_2$

THEN $\dot{x}(t) = A_kx(t) + B_ku(t), \qquad k = 1,2,3,4$

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ f_1^{min} & 0 \end{bmatrix}, \quad B_1 = B_3 = \begin{bmatrix} 0 \\ f_2^{min} \end{bmatrix},$$
$$A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ f_1^{max} & 0 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 0 \\ f_2^{max} \end{bmatrix},$$

where $x(t) = [x_1(t)\ \ x_2(t)]^{\mathrm{T}}$, $f_1^{min} = 10.0078, f_1^{max} = 18.4800, f_2^{min} = -0.1765$, and $f_2^{max} = -0.0261$. The MFs of the inverted pendulum for the proposed system are listed in Table II, where $f_1(x(t))$ and $f_2(x(t))$ are denoted as $\frac{g-am_plx_2(t)^2\cos(x_1(t))\sin(x_1(t))}{4l/3-am_pl\cos^2(x_1(t))x_1(t)}$ and $\frac{-a\cos(x_1(t))}{4l/3-am_pl\cos^2(x_1(t))}$, respectively.

Fig. 4 illustrates the phase portraits subject to the different initial states in the operating domain with corollary 1. The red represents the case where $m_p = 2$ kg and $M_c = 8$ kg, and the blue represents the case where $m_p = 3$ kg and $M_c = 12$ kg.

V. CONCLUSIONS

This paper presents a study of stabilization synthesis using a non-PDC MF-dependent controller for IT2 T–S fuzzy systems. The stability conditions were derived using membership-dependent matrix inequalities and valuable relaxation techniques. The proposed approaches exhibited less conservatism and applicability, as demonstrated by examples. In the future, the presented idea can be studied with the polynomially homogenous MFs to obtain the generalized stability conditions in the sense of a quadratic matrix form.

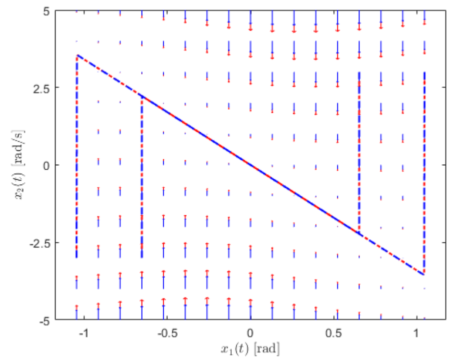

Fig. 4. Phase portraits of the states in the operating domain.

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