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Initial Error Affection and Strategy Modification in Multi-Population LQ Mean Field Games under Erroneous Initial Distribution Information

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Abstract— **In this paper, the initial error affection and strategy modification in multi-population linear quadratic mean field games (MPLQMFGs) under erroneous initial distribution information are investigated. First, a MPLQMFG model is developed where agents in different populations are coupled by dynamics and cost functions. Next, by studying the evolutionary of MPLQMFGs under erroneous initial distribution information, the predicted and the actual evolutionary of mean field states are given. Furthermore, assume that each agent maintains observations only of its own state and control as well as the mean field terms of its own population, and agents are allowed to modify their strategies at an intermediate moment, two sufficient conditions are provided, where the game will reach the Nash equilibrium under correct information. Besides, the affection of the initial error on the game is discussed. Finally, simulations on the opinion evolutions of two groups are performed to verify above conclusions.**

I. INTRODUCTION

Mean field games (MFGs) are proposed independently by Huang et al.[1]-[3]and Lasry&Lions[4]-[6] in 2007. The model of Lasry&Lions describes a game with a large number of homogeneous players and its mean field equilibrium, in which each player tries to minimize the value of its own cost function and interacts with other players through a mean field term contained in the cost function or/and dynamic equation. In [7], a multi-population mean field game (MPMFG) was proposed, in which players are divided into several populations, each population contains a large number of homogeneous agents, and players interact through the mean field terms of all populations. In [8]-[11], the linear quadratic mean field games (LQMFGs) with different settings were studied, where cost functional is quadratic in all state variables, control variables, and the mean field terms, while the controlled dynamics are linear and also consist of mean field terms.

For multi-population linear-quadratic mean field games (MPLQMFGs), the equilibrium usually can be mathematically described as a set of equations with a forward-backward structure with a forward dynamic equation and a backward equation describes a player's optimal strategy. When a unique mean field equilibrium exists, given the initial mean field states, the cost functions of all populations and the dynamic equation settings, an agent can obtain the unique mean field equilibrium by solving the system of equations describing the equilibrium.. So at the initial moment, players can predict the equilibrium and give their optimal strategies for the whole time period.

MFGs have been widely applicated in many fields[13]-[22],such as swarm robotics, industrial engineering and crowd motion. In some scenarios, agents are required to give their own whole period strategies at the initial moment, such as unmanned control with a pre-set trajectory. When an agent obtains the initial distribution, cost functions and dynamic equations of all agents, it can calculate its optimal control for the whole time period at the initial moment.

However, in application, there is often the problem that there may be some errors in the initial information obtained by agents[23][24], especially in MPMFGs, where agents may obtain erroneous information about other populations. A series of questions that arise from this problem are: when an agent gives its strategy at the initial moment based on erroneous information, what is the difference between the equilibrium in its prediction (EP) and the equilibrium under correct information (EC)? How will the actual mean field states (EA) evolve? Is it possible to detect and correct errors through partial observation of mean field terms after evolving based on erroneous information for a period of time? How will the game evolve if agents are given the opportunity to modify their strategies at an intermediate moment (EM)? Currently, there is still a lack of research on these questions in the field of MFGs.

Based on the above questions, this paper studies MPLQMFGs under erroneous initial distribution information and the strategy modification therein. We assume the cost functions and dynamic equations of all agents are available, and each agent maintains clean observations only of its own state and control as well as the mean field terms of its own population. In addition, the agent obtains the initial mean field states of other populations that may have errors, and believes these

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initial mean field terms are correct. At the initial moment, agents give their strategies for the whole time period based on the above information.

In section 2, we present the MPLQMFG model where different populations are coupled through cost functions and dynamic equations, mathematically describe the mean field equilibrium under complete information, and introduce the definition of EC. In section 3, we give our assumptions and study MPLQMFGs under erroneous initial distribution information. In section 4, we discuss the strategy modification when agents are allowed to modify their strategies at an intermediate moment without observing other populations' trajectories. In section 5, we analysis the initial error affection on EC, EP, EA and EM. At last, we conduct simulations and verify our conclusions. The main contributions of this paper can be summarized as follows:

- We build a mathematic model of MPLQMFGs under erroneous initial distribution information, study and give a mathematical description of EP, EA and EM.
- We compare EC, EP, EA and EM. Three all-agents-known linear relationship between the initial error and the deviations of EP from EC, the deviations of EA from EC, the deviations of EM from EC are given.
- We provide two sufficient conditions for agents to calculate their optimal strategies during re-game and corresponding calculation methods. It is worth noting that even if the actual mean field states of other populations is unknown, under some conditions, an agent can still calculate its re-game optimal strategy under complete information.

II. MULTI-POPULATION LINEAR QUADRATIC MEAN FIELD GAME MODEL WITH COMPLETE INFORMATION

In this section, we introduce a MPLQMFG model where agents in different populations are coupled by dynamics and cost functions. We introduce the definition of EC, and give the conclusion that EC is consistent with the mean field equilibrium with complete information.

A. Dynamics and Cost Functions

We consider a stochastic game with M populations, $\{\mathcal{P}_m, 1 \le m \le M\}$, each population has N_m agents, $\{\mathcal{A}_{im}, 1 \le i \le N_m\}$, where the dynamics of the agents are given by the following controlled stochastic differential equations (SDEs):

$$\begin{aligned} dx_{im}(t) =& [A_m x_{im}(t) + B_m u_{im}(t) + \Sigma_{n=1}^{M} C_m^n z_n(t) + \\ & F_m \bar{u}_m(t)dt)]dt + D_m dW_{im}(t), \\ x_{im}(0) =& x_{im}^0, \end{aligned} \tag{1}$$

with terminal time $T \in (0, \infty)$ and initial conditions $x_{im}(0) = x_{im}^0, 1 \le m \le M, 1 \le i \le N_m$. Where $(x_{im}(t))_{0\le t\le T}, (u_{im}(t))_{0\le t\le T} \in \mathbb{R}^d$are the state and control input of agent $\mathcal{A}_{im}$, $z_m(t), u_m(t) \in \mathbb{R}^d$ are the mean field state and mean field control of $\mathcal{P}_m$, $(W_{im}(t))_{0\le t\le T}, 1 \le i \le N_m, 1 \le m \le M$ are independent d-dimensional standard Wiener processes, $A_m, B_m, C_m^n, F_m, D_m$ are matrices on $\mathbb{R}^d \times \mathbb{R}^d$, and B_m is invertible.

The cost function of $\mathcal{A}_{im}$ is given by

$$\begin{aligned} J_m(u_{im}) =& \frac{1}{2}\mathbb{E}[\int_0^T [\|x_{im}(t) - s_m\|^2_{Q_{Im}} + \|u_{im}(t)\|^2_{R_m} + \\ & \Sigma_{i=1}^M \|x_{im}(t) - (\Gamma_m^n z_n(t) + \eta_m^n)\|^2_{Q_m^n}]dt + \|x_{im}(T) - \\ & \bar{s}_m\|^2_{\bar{Q}_{Im}} + \Sigma_{i=1}^M \|x_{im}(T) - (\bar{\Gamma}_m^n z_n(T) + \bar{\eta}_m^n)\|^2_{\bar{Q}_m^n}], \end{aligned} \tag{2}$$

where we define $\|X\|_Q^2 = X^T Q X, \forall Q \in \mathbb{R}^{d\times d}, X \in \mathbb{R}^d$. $Q_{Im}, Q_m^n, R_m, \bar{Q}_{Im}, \bar{Q}_m^n$ are positive definite matrices on $\mathbb{R}^{d\times d}$.

B. the Optimal Control of $\mathcal{A}_{im}$

Theorem 2.1 For given $(z_m(t))_{0\le t\le T}, 1 \le m \le M$, the optimal control of $\mathcal{A}_{im}$ adapted to its admissible control set $L^2_{\mathcal{F}_{im}}(0, T; \mathbb{R}^d)$ is given by $u_{im}(t) = -R_m^{-1} B_m^T y_{im}(t)$, where $y_{im}(t) = \mathbb{E}[p_{im}(t)|\mathcal{F}_t^{im}]$, $\mathcal{F}_t^{im} := \sigma(x_{im}(0), W_{im}(s), 0 \le s \le t)$, $p_{im}(t)$ satisfies

$$\begin{aligned} & dp_{im}(t) = -\{(A_m^T p_{im}(t) + (Q_{Im} + \Sigma_{n=1}^M Q_m^n) x_{im}(t) - \\ & Q_{Im} s_m - \Sigma_{n=1}^M Q_m^n (\Gamma_m^n z_n(t) + \eta_m^n)\}dt + q_{im}(t) dW_{im}(t), \\ & p_{im}(T) = (\bar{Q}_{Im} + \Sigma_{n=1}^M \bar{Q}_m^n) x_{im}(T) - \bar{Q}_{Im} \bar{s}_m - \Sigma_{n=1}^M \bar{Q}_m^n \\ & (\bar{\Gamma}_m^n z_n(T) + \bar{\eta}_m^n), \end{aligned} \tag{3}$$

Proof of this theorem is given in Appendix.

Note that p_{im} can be represented by $p_{im}(t) = P_m(t) x_{im}(t) + g_{im}(t)$, where $P_m(t)$ satisfies a non-symmetric riccati equation

$$\begin{aligned} & -dP_m(t) = \{(P_m(t) A_m + A_m^T P_m(t) + (Q_{Im} + \Sigma_{n=1}^M Q_m^n) \\ & - P_m(t) B_m R_m^{-1} B_m^T P_m(t)\}dt, \\ & P_m(T) = \bar{Q}_{Im} + \Sigma_{n=1}^M \bar{Q}_m^n, \end{aligned} \tag{4}$$

and $g_{im}(t)$ satisfies the backward stochastic differential equation (BSDEs)

$$\begin{aligned} & dg_{im}(t) = -\{(A_m^T - P_m B_m R_m^{-1} B_m^T) g_{im}(t) - P_m F_m R_m^{-1} \\ & B_m^T \bar{g}_m + \Sigma_{n=1}^M (P_m C_m^n - Q_m^n \Gamma_m^n) z_n(t) - P_m F_m R_m^{-1} B_m^T P_m \\ & z_m(t) - Q_{Im} s_m - \Sigma_{n=1}^M Q_m^n \eta_m^n\}dt + (q_{im} - P_{im} D_m) dW_{im}, \\ & g_{im}(T) = -\bar{Q}_{Im} \bar{s}_m - \Sigma_{n=1}^M \bar{Q}_m^n (\bar{\Gamma}_m^n z_n(T) + \bar{\eta}_m^n), \end{aligned} \tag{5}$$

where $\bar{u}_m(t) = \Sigma_{i=1}^{N_m} u_{im}(t)/N_m, \bar{g}_m(t) = \Sigma_{i=1}^{N_m} g_{im}(t)/N_m$.

C. Mean Field Approximation

On the one hand, a Nash equilibrium is reached if and only if each agent's control is the optimal response to the current mean field terms $z_m(t), 1 \le m \le M$, on the other hand, $z_m(t)$ is defined as $z_m(t) = \Sigma_{i=1}^{N_m} x_{im}(t)/N_m$. When $N_m \to \infty$, we have $z_m(t) = \mathbb{E}(x_m(t))$, where $x_m(t)$ is the state of a representative agent $\mathcal{A}_m$ of population m. We define $y_m(t) = \Sigma_{i=1}^{N_m} y_{im}(t)/N_m$, $\mathcal{X}(t) = (z_1^T(t), ..., z_M^T(t))^T, \mathcal{Y}(t) = (y_1^T(t), ..., y_M^T(t))^T$. When a

Nash equilibrium is reached, conditions $\mathcal{X}(t), \mathcal{Y}(t)$ need to satisfy are given by the following theorem

Theorem 2.2 The equilibrium mean field process $(z_m)_{1\leq t\leq T}$ satisfy the following equations

$$d\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix} = \left\{\begin{pmatrix}\mathcal{A}+\mathcal{C} & -(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T\\ -\mathcal{Q} & -\mathcal{A}\end{pmatrix}\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix} - Z\right\}dt,$$
$$\mathcal{X}(0) = \mathcal{X}^0,$$
$$\mathcal{Y}(T) = \bar{\mathcal{Q}}\mathcal{X}(T) + \bar{\nu}.$$
$$\mathcal{C} = \begin{pmatrix}C_1^1 & C_1^2 & \dots & C_1^M\\ \vdots & \vdots & \vdots & \vdots\\ C_M^1 & C_M^2 & \dots & C_M^M\end{pmatrix}, \mathcal{A} = \begin{pmatrix}A_1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & A_M\end{pmatrix},$$
$$\mathcal{B} = \begin{pmatrix}B_1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & B_M\end{pmatrix}, \mathcal{R} = \begin{pmatrix}R_1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & R_M\end{pmatrix},$$
$$\mathcal{F} = \begin{pmatrix}F_1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & F_M\end{pmatrix},$$
$$\mathcal{Q} = \begin{pmatrix}\mathcal{Q}_1^1 & Q_1^2\Gamma_1^2 & \dots & Q_1^M\Gamma_1^M\\ \vdots & \vdots & \vdots & \vdots\\ Q_M^1\Gamma_M^1 & Q_M^2\Gamma_M^2 & \dots & \mathcal{Q}_M^M\end{pmatrix},$$
$$\bar{\mathcal{Q}} = \begin{pmatrix}\bar{\mathcal{Q}}_1^1 & \bar{Q}_1^2\bar{\Gamma}_1^2 & \dots & \bar{Q}_1^M\bar{\Gamma}_1^M\\ \vdots & \vdots & \vdots & \vdots\\ \bar{Q}_M^1\bar{\Gamma}_M^1 & \bar{Q}_M^2\bar{\Gamma}_M^2 & \dots & \bar{\mathcal{Q}}_M^M\end{pmatrix}, Z = \begin{pmatrix}0\\ \nu\end{pmatrix},$$
$$\mathcal{Q}_m^m = -(Q_{Im} + \Sigma_{n=1}^M Q_m^n) + Q_m^m\Gamma_m^m,$$
$$\bar{\mathcal{Q}}_m^m = -(\bar{Q}_{Im} + \Sigma_{n=1}^M \bar{Q}_m^n) + \bar{Q}_m^m\bar{\Gamma}_m^m, 1\leq m\leq M,$$
$$\nu = -(\nu_1^T, ..., \nu_M^T)^T, \nu_m = Q_{Im}s_m + \Sigma_{n=1}^M Q_m^m\eta_m^n,$$
$$\bar{\nu} = -(\bar{\nu}_1^T, ..., \bar{\nu}_M^T)^T, \bar{\nu}_m = \bar{Q}_{Im}\bar{s}_m + \Sigma_{n=1}^M \bar{Q}_m^m\bar{\eta}_m^n. \tag{6}$$

We notice that $\mathcal{Y}(t) = P(t)\mathcal{X}(t) + \mathcal{G}(t)$, where $P(t)$ satisfies a non-symmetric riccati equation

$$\begin{aligned}-dP =& P(\mathcal{A}+\mathcal{C}) + \mathcal{A}^T P + \mathcal{Q} - P(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P dt,\\ P(T) =& \bar{\mathcal{Q}},\end{aligned} \tag{7}$$

and $\mathcal{G}(t)$ satisfies the backward ordinary differential equations (BODEs)

$$\begin{aligned}d\mathcal{G} &= -(\mathcal{A}^T - P(t)(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T)\mathcal{G} - \nu dt,\\ \mathcal{G}(T) &= \bar{\nu}.\end{aligned} \tag{8}$$

D. Existence and Uniqueness Condition

In this section, we provide a sufficient condition for the existence and uniqueness of the mean field equilibrium, which is equivalent to (6) has a unique solution. With respect to the affine form $\mathcal{Y}(t) = P(t)\mathcal{X}(t) + \mathcal{G}(t)$, (6) admits a unique solution if and only if (7) admits a unique solution. In accordance with Theorem 4.1 and Theorem 4.2 in [12], we have the following proposition.

We set $\mathcal{Q} = \mathcal{Q}_p + \mathcal{S}, \bar{\mathcal{Q}} = \bar{\mathcal{Q}}_p + \bar{\mathcal{S}}$, where $\mathcal{Q}_p, \bar{\mathcal{Q}}_p$ are positive matrices. We represent the fundamental solution to $\mathcal{A}$ with $\phi(s,t)$, which means $\phi(s,t)$ is a fundamental solution of the equation

$$dx = \mathcal{A}xdt, \tag{9}$$

and $x(s) = \phi(s,t)x(t)$.

Proposition 2.1 Let $\phi(s,t)$ be the fundamental solution to $\mathcal{A}$. Suppose that

$$(1+\sqrt{T}\|\phi\|_T \cdot \|\mathcal{C}\mathcal{Q}_p^{-\frac{1}{2}}\|)(1+N(S)) < 2. \tag{10}$$

Where $\|\cdot\|$ stands for usual Euclidean norm. Then there exists a unique mean field equilibrium. Here,

$$\|\phi\|_T := \sup_{0\leq t\leq T}\sqrt{\|\phi^*(T,t)\bar{\mathcal{Q}}^{-\frac{1}{2}}\|^2 + \int_t^T \|\phi^*(s,t)\bar{\mathcal{Q}}^{\frac{1}{2}}\|^2 ds}$$
$$N(S) = max\{\|\bar{\mathcal{Q}}^{-\frac{1}{2}}\bar{\mathcal{S}}\bar{\mathcal{Q}}^{-\frac{1}{2}}\|, \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\}.$$

E. Equilibrium under Correct Information

When $\mathcal{A}_{im}$ gives its strategy $(u_{im}(t))_{0\leq t\leq T}$ at $t=0$ according to the correct information, and conditions in Proposition 2.1 are satisfied, we call the respective mean field equilibrium as the equilibrium under correct information (EC). Under this setting, agents can calculate the unique mean field equilibrium described in (6) and their respective optimal controls at time $t=0$, hence EC is consistent with the mean field equilibrium with complete information.

III. Multi-Population Linear Quadratic Mean Field Games under Erroneous Initial Distribution Information

In this section, we consider the situation where agents in $\mathcal{P}_n$ obtain erroneous initial mean field states of other populations, where each agent gives its strategy for $t\in[0,T]$ based on the erroneous information, and then evolves according to this control.

Based on the above discussion, we can know that each agent gives its control corresponding to an equilibrium of its predictions. Therefore, we consider the equilibrium in $\mathcal{P}_n$'s prediction(EPn) and the actual evolution(EA).

A. Assumptions

- The correct $z_m(0), x_{im}(0)$, dynamics of $\{\mathcal{A}_{jn}, 1\leq j\leq N_n, 1\leq n\leq M\}$ and $\{J_n(u), 1\leq n\leq M\}$ are accessible for $\mathcal{A}_{im}, 1\leq N_m, 1\leq m\leq M$.
- $z_m^n(0) = z_n(0) + E_m^n$ is the initial mean field state of $\mathcal{P}_n$ as observed by $\mathcal{A}_{im}, 1\leq i\leq N_m, n\neq m, 1\leq n\leq M$ with error $E_m^n\in\mathbb{R}^d$.
- At $t=0$, $\mathcal{A}_{im}$ gives its strategy $(u_{im}(t))_{0\leq t\leq T}$ and evolves according to this strategy during $0\leq t\leq T$.

We define $(z_m^n(t))_{0\leq t\leq T}, 1\leq n\leq M$ as the mean field trajectory of $\mathcal{P}_n$ predicted by $\mathcal{A}_{im}$.

B. the Strategy of $\mathcal{A}_{im}$

Theorem 3.1 For given $(z_m^n(t))_{0\le t\le T}, (\bar{u}_m(t))_{0\le t\le T}$, $1\le m\le M$, the control of $\mathcal{A}_{im}$ adapted to $L^2_{\mathcal{F}_0^{im}}(0,T;\mathbb{R}^d)$ is given by $u_{im} = -R_m^{-1}B_m^T y_{im}$, where $y_{im}(t) = \mathbb{E}[p_{im}(t)|\mathcal{F}_0^{im}]$, and $p_{im}(t)$ satisfies

$$\begin{aligned}
&dp_{im}(t) = -\{(A_m^T p_{im}(t) + (Q_{Im} + \Sigma_{n=1}^M Q_m^n)x_{im}(t) - \\
&Q_{Im}s_m - \Sigma_{n=1}^M Q_m^n(\Gamma_m^n z_m^n(t) + \eta_m^n)\}dt + q_{im}(t)dW_{im}(t), \\
&p_{im}(T) = (\bar{Q}_{Im} + \Sigma_{n=1}^M \bar{Q}_m^n)x_{im}(T) - \bar{Q}_{Im}\bar{s}_m - \Sigma_{n=1}^M \bar{Q}_m^n \\
&(\bar{\Gamma}_m^n z_m^n(T) + \bar{\eta}_m^n),
\end{aligned} \tag{11}$$

Proof of this theorem is similar to Theorem 2.1.

Theorem 3.2 For given $(z_m^n(t))_{0\le t\le T}, (\bar{u}_m(t))_{0\le t\le T}$, $1\le m\le M$, the optimal trajectory of $\mathcal{A}_{im}$ in the prediction of $\mathcal{A}_{im}$ is given by

$$\begin{aligned}
&dx_{im}(t) = \{A_m x_{im}(t) - B_m R_m^{-1}B_m^T y_{im}(t) + \Sigma_{n=1}^M C_m^n \\
&z_m^n(t) - F_m R_m^{-1}B_m^T y_m(t)\}dt + D_m dW_{im}(t), \\
&x_{im}(0) = x_{im}^0, \\
&dp_{im}(t) = -\{(A_m^T p_{im}(t) + (Q_{Im} + \Sigma_{n=1}^M Q_m^n)x_{im}(t) - \\
&Q_{Im}s_m - \Sigma_{n=1}^M Q_m^n(\Gamma_m^n z_m^n(t) + \eta_m^n)\}dt + q_{im}(t)dW_{im}, \\
&p_{im}(T) = (\bar{Q}_{Im} + \Sigma_{n=1}^M \bar{Q}_m^n)x_{im}(T) - \bar{Q}_{Im}\bar{s}_m - \\
&\Sigma_{n=1}^M \bar{Q}_m^n(\bar{\Gamma}_m^n z_m^n(T) + \bar{\eta}_m^n),
\end{aligned} \tag{12}$$

where $y_m(t) = -(B_m^T)^{-1}R_m\bar{u}_m(t)$.

C. Mean Field Approximation

When $N_m\to\infty, 1\le m\le M$, by taking expectations on $u_m^n(t)$ and $x_m^n(t)$, and requiring $z_n^m(t) = \mathbb{E}(x_m^n(t)), \bar{u}_n^m(t) = \mathbb{E}(u_m^n(t)), 1\le m\le M$, where $x_m^n(t), u_m^n(t)$ are the state and control of a representative agent $\mathcal{A}_m$ of $\mathcal{P}_m$ in the prediction of agents in $\mathcal{P}_n$, respectively. We have

Theorem 3.3 The equilibrium mean field states $(z_n^m)_{1\le t\le T}$ and mean field control $(\bar{u}_n^m)_{1\le t\le T}$ in EPn satisfy the following equations

$$\begin{aligned}
&d\begin{pmatrix}\mathcal{X}_n\\ \mathcal{Y}_n\end{pmatrix} = \left\{\begin{pmatrix}\mathcal{A}+\mathcal{C} & -(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T\\ -\mathcal{Q} & -\mathcal{A}\end{pmatrix}\begin{pmatrix}\mathcal{X}_n\\ \mathcal{Y}_n\end{pmatrix} - Z\right\}dt,\\
&\mathcal{X}_n(0) = \mathcal{X}_n^0,\\
&\mathcal{Y}_n(T) = \bar{\mathcal{Q}}\mathcal{X}_n(T) + \bar{\nu},
\end{aligned} \tag{13}$$

where $y_n^m(t) = -(B_m^T)^{-1}R_m\bar{u}_n^m(t)$, $\mathcal{X}_n(t) = (z_n^1(t)^T,...,z_n^M(t)^T)^T, \mathcal{Y}_n(t) = (y_n^1(t)^T,...y_n^M(t)^T)^T$.

We notice that $\mathcal{Y}_n(t) = P(t)\mathcal{X}_n(t) + \mathcal{G}(t)$, where $P(t)$ satisfies the non-symmetric riccati equation (7) and $\mathcal{G}(t)$ satisfies the BODEs (8). For an agent $\mathcal{A}_{jn}$, by solving the two known equations, it can get $(\mathcal{X}_n(t))_{0\le t\le T}$ and $(y_n(t))_{0\le t\le T}$, and solve (12) to calculate its optimal control $u_{jn}(t)$.

Since the unique existence condition for the equilibrium given in section 2 is solely depends on the coefficients of the mean field game model and is not related to $\mathcal{X}_n(0)$, if the condition in Proposition 2.1 is satisfied, there exists a unique EPn$(n=1,2,...,M)$.

D. Actual Trajectories

Agents in $\mathcal{P}_n$ calculate their optimal control based on EPn, so that the mean field control $(\bar{u}_n(t))_{0\le t\le T}$ in EPn is consistent with the actual mean field control $(\bar{u}_n^A(t))_{0\le t\le T}$. Then the actual mean field trajectories $(z_n^A(t))_{0\le t\le T}, 1\le n\le M$ satisfy

$$\begin{aligned}
dz_n^A(t) =&\{A_n z_n^A(t) - B_n R_n^{-1}B_n^T y_n^A(t) + \Sigma_{k=1}^M C_n^k z_k^A(t)\\
&- F_n R_n^{-1}B_n^T y_n^A(t)\}dt,\\
z_n^A(0) =& z_n^A(0),
\end{aligned} \tag{14}$$

where $y_n^A(t) = -(B_n^T)^{-1}R_n\bar{u}_n^A(t) = y_n^n(t)$. That is, agents in $\mathcal{P}_n$ evolve according to their optimal controls based on EPn. When $z_k^A(t), k\ne n$ are different from those in EPn, $z_n^A(t)$ will be different from $z_n^n(t)$.

IV. STRATEGY MODIFICATION

In this section, we allow each agent in $\mathcal{P}_n$ to obtain its actual mean field trajectory $(z_n^A(t))_{0\le t\le t_0}$ and its own state at time t_0, and modify their control for $t\in[t_0,T]$. We call this actual equilibrium with strategy modification as EM.

A. Information obtained by $\mathcal{P}_n$ at t_0

In this subsection, we discuss what information can be obtained by $\mathcal{P}_n$ from the analysis of $(z_n^A(t))_{0\le t\le t_0}$. According to (14), we have

$$\Sigma_{k=1}^M C_n^k z_k^A(t) = \frac{dz_n^A(t)}{dt} - A_n z_n^A(t) + (B_n+F_n)R_n^{-1}B_n^T y_n^A(t),$$

where $y_n^A(t) = y_n^n(t)$ is known by $\mathcal{P}_n$, $t\in[0,t_0]$.

So $(\Sigma_{k=1}^M C_n^k z_k^A(t)))_{0\le t\le t_0}$ and $(z_n^A(t))_{0\le t\le t_0}$ can be obtained by $\mathcal{P}_n$ at t_0.

B. Strategy Modification when Re-Game at t_0

Let's assume that all agents look back at the mean field trajectory of their own population at time t_0 and decide their control for $t\in[t_0,T]$, and all agents know this rule.

Obviously, when the actual mean field states $\mathcal{X}^A(t_0) = (z_1^A(t_0), z_2^A(t_0),...,z_M^A(t_0))$ are available for all agents at time t_0, agents can compute their actual optimal controls, modify their strategies and generate a new equilibrium. According to section 2, The new equilibrium is given by the following equations

$$\begin{aligned}
&d\begin{pmatrix}\mathcal{X}^A\\ \mathcal{Y}^A\end{pmatrix} = \left\{\begin{pmatrix}\mathcal{A}+\mathcal{C} & -(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T\\ -\mathcal{Q} & -\mathcal{A}\end{pmatrix}\begin{pmatrix}\mathcal{X}^A\\ \mathcal{Y}^A\end{pmatrix} - Z\right\}dt,\\
&\mathcal{X}^A(t_0) = \mathcal{X}^A(t_0),\\
&\mathcal{Y}^A(T) = \bar{\mathcal{Q}}\mathcal{X}^A(T) + \bar{\nu},
\end{aligned} \tag{15}$$

where $\mathcal{Y}^A(t) = P(t)\mathcal{X}^A(t) + \mathcal{G}(t)$, $P(t)$ satisfies the non-symmetric riccati equation (7) and $\mathcal{G}(t)$ satisfies (8).

C. Sufficient Conditions for EM

When the existence and uniqueness condition in Proposition 2.1 is satisfied, We give two sufficient conditions for agents to compute their optimal controls and inform the above equilibrium.

Theorem 4.1 When $M=2$ and C_1^2, C_2^1 is invertible, the actual re-game mean field trajectories are consistent with that in the equilibrium described in (15), and all agents take their optimal controls.

Proof:

For $\mathcal{P}_n$, $C_n^m z_m^A(t_0)+C_n^n z_n^A(t_0)$ and $z_n^A(t_0)$ is available, where $n=1,2$. Since C_n^m is invertible, $z_m^A(t_0)$ can be calculated by $\mathcal{P}_n$. So the actual mean field states $\mathcal{X}^A(t_0)$ is available for $\mathcal{P}_n$, and the actual re-game mean field trajectories are consistent with that in the equilibrium described in (15), which means agents can compute the actual equilibrium and their optimal controls.□

Theorem 4.2 When $\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t)=v(t)I$, where I is an identity matrix and $v(t)$ is a scalar function, the actual re-game mean field trajectories are consistent with that in the equilibrium described in (15), and all agents take their optimal controls.

Proof:

Set $C_n=(C_n^{1T},...,C_n^{MT})^T, 1\le n\le M$, then according to (15) and $\mathcal{Y}^A(t)=P(t)\mathcal{X}^A(t)+\mathcal{G}(t)$, we have

$$\begin{aligned}&dC_n^T\mathcal{X}^A(t)=C_n^T(\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t))\mathcal{X}^A(t)\\&-C_n^T(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t)\mathcal{G}(t)dt,\\&C_n^T\mathcal{X}^A(t_0)=C_n^T\mathcal{X}^A(t_0).\end{aligned}$$

Since $\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t)=v(t)I$, it can be rewrite as

$$\begin{aligned}dC_n^T\mathcal{X}^A(t)=&v(t)C_n^T\mathcal{X}^A(t)-C_n^T(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P\mathcal{G}dt,\\C_n^T\mathcal{X}^A(t_0)=&C_n^T\mathcal{X}^A(t_0).\end{aligned}$$

As $C_n^T\mathcal{X}^A(t_0)=\Sigma_{k=1}^M C_n^k z_n^A(t_0)$ is available to $\mathcal{P}_n$, $(\Sigma_{k=1}^M C_n^k z_n^A(t))_{t_0\le t\le T}$ can be solved from above equations. Set $C_{In}=(I_n^1,...,I_n^M)^T, 1\le n\le M$, where$I_n^m=\delta_{nm}I_{d\times d}$, then we have

$$\begin{aligned}&dC_{In}^T\mathcal{X}^A(t)=v(t)C_{In}^T\mathcal{X}^A(t)-C_{In}^T(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P\mathcal{G}dt,\\&C_{In}^T\mathcal{X}^A(t_0)=C_{In}^T\mathcal{X}^A(t_0).\end{aligned}$$

As $C_{In}^T\mathcal{X}^A(t_0)=z_n^A(t_0)$ is available to $\mathcal{P}_n$, $(z_n^A(t))_{t_0\le t\le T}$ can be solved from above equations, and $(y_n^A(t))_{t_0\le t\le T}$ can be solved from (14), where $(P_n(t))_{t_0\le t\le T}$ can be solved from (4), $(\bar{g}_n(t))_{t_0\le t\le T}$ can be solved from $y_n^A(t)=P_n(t)z_n^A(t)+\bar{g}_n(t)$.

Then for agent $\mathcal{A}_{jn}$, according to Theorem 3.2, its optimal control and trajectory predicted at time t_0 satisfy

$$\begin{aligned}&dx_{jn}(t)=\{A_n x_{jn}(t)-B_n R_n^{-1}B_n^T y_{jn}(t)+\Sigma_{k=1}^M C_n^k\\&z_k^A(t)-F_n R_n^{-1}B_n^T y_n^A(t)\}dt+D_n dW_{jn}(t),\\&x_{jn}(t_0)=x_{jn}(t_0),\\&dp_{jn}(t)=-\{(A_n^T p_{jn}(t)+(Q_{In}+\Sigma_{k=1}^M Q_n^k)x_{jn}(t)-\\&Q_{In}s_n-\Sigma_{k=1}^M Q_n^k(\Gamma_n^k z_k^A(t)+\eta_n^k)\}dt+q_{jn}(t)dW_{jn},\\&p_{jn}(T)=(\bar{Q}_{In}+\Sigma_{k=1}^M\bar{Q}_n^k)x_{in}(T)-\bar{Q}_{In}\bar{s}_n-\\&\Sigma_{k=1}^M\bar{Q}_n^k(\bar{\Gamma}_n^k z_n^k(T)+\bar{\eta}_n^k).\end{aligned}$$

Since $(z_n^A(t))_{t_0\le t\le T}, (\Sigma_{k=1}^M C_n^k z_n^A(t))_{t_0\le t\le T}, x_{jn}(t_0)$ are available for $\mathcal{A}_{jn}$, $(u_{jn}(t))_{t_0\le t\le T}$ can be solved from above equations, which means each agent can calculate its optimal control under the equilibrium in (15), so this equilibrium is the actual equilibrium.

Above all, the actual re-game mean field trajectories are consistent with that in the equilibrium described in (15), and all agents take their optimal controls.□

V. INITIAL ERROR AFFECTIONS

In this section, we discuss the initial error affection on EPn, EA, EM and the affection of re-game time t_0. We set $\mathcal{P}_n$'s initial error as $\mathcal{E}_n=\mathcal{X}_n(0)-\mathcal{X}^0$.

A. The Deviations of EPn from EC

According to (6) and (13), for the deviation of the equilibrium in the prediction of $\mathcal{P}_n$(EPn) and that under correct information(EC), $\mathcal{X}_n^E(t)=\mathcal{X}_n(t)-\mathcal{X}(t), \mathcal{Y}_n^E(t)=\mathcal{Y}_n(t)-\mathcal{Y}(t)$, $\mathcal{Y}_n^E(t)=P(t)\mathcal{X}_n^E(t)$, we have

$$\begin{aligned}&d\mathcal{X}_n^E(t)=(\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t))\mathcal{X}_n^E(t)dt,\\&\mathcal{X}_n^E(0)=\mathcal{E}_n.\end{aligned}\tag{16}$$

We set $H(t)=\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t)$, then a basis solution matrix $\Phi(t)$ of (16) can be solved according to $H(t)$. Since $(H(t))_{0\le t\le T}$ can be calculated without knowing the information of initial states, we have the following theorem

Theorem 5.1 $(\mathcal{X}_n^E(t))_{0\le t\le T}$ has a *linear relationship* with $\mathcal{E}_n$, and this linear relationship can be computed by all agents without knowing $\mathcal{E}_n$, which is

$$\mathcal{X}_n^E(t)=\Phi(t)\Phi(0)^{-1}\mathcal{E}_n.\tag{17}$$

B. The Deviations of EA from EC

Consider the deviation $\bar{u}_n^E(t)=\bar{u}_n^A(t)-u_n^C(t)$, where $(u_n^C(t))_{0\le t\le T}$ is the mean field control of $\mathcal{P}_n$ in EC. Let $U_n^E(t)=(\bar{u}_n^{1E}(t)^T,...,\bar{u}_n^{ME}(t)^T)^T$, where $\bar{u}_n^{mE}(t)=\bar{u}_n^m(t)-u_m^C(t)$ and $\bar{u}_n^E(t)=\bar{u}_n^{nE}(t)$, then we have

$$U_n^E(t)=-\mathcal{R}^{-1}\mathcal{B}^T P(t)\Phi(t)\Phi(0)^{-1}\mathcal{E}_n.\tag{18}$$

Let $\Delta\mathcal{X}^A(t)=(\Delta z_1^A(t)^T,...,\Delta z_M^A(t)^T)^T$, $\Delta z_n^A(t)=z_n^A(t)-z_n^C(t)$, we have

$$\begin{aligned}d\Delta\mathcal{X}^A=&(\mathcal{A}+\mathcal{C})\Delta\mathcal{X}^A-(\mathcal{B}+\mathcal{F})\mathcal{B}_0(t)\mathcal{E}dt\\\Delta\mathcal{X}^A=&0\end{aligned}$$

where $\mathcal{E}=(\mathcal{E}_1^T,...,\mathcal{E}_M^T)^T$, $C_{In}=(I_n^1,...,I_n^M)^T, I_n^m=\delta_{nm}I_{d\times d}$,

$$\mathcal{B}_0(t)=\begin{pmatrix}\mathcal{M}_1(t)&...&0\\\vdots&\vdots&\vdots\\0&...&\mathcal{M}_M(t)\end{pmatrix},$$
$$\mathcal{M}_n(t)=C_{In}^T\mathcal{R}^{-1}\mathcal{B}^T P(t)\Phi(t)\Phi(0)^{-1}.$$

The above equation corresponds a homogeneous linear equation

$$d\Delta\mathcal{X}^A=(\mathcal{A}+\mathcal{C})\Delta\mathcal{X}^A dt.$$

Define $\Phi_1(t)$ as a basis solution of this equation, then $\Phi_1(t)$ can be solved according to $H_1(t) = \mathcal{A}+\mathcal{C}$. Using the method of variation of parameters, we have the following theorem

Theorem 5.2 $(\Delta\mathcal{X}^A(t))_{0\le t\le T}$ has a *linear relationship* with $\mathcal{E}$, and this linear relationship can be computed by all agents without knowing $\mathcal{E}$, which is

$$\begin{aligned} &\Delta\mathcal{X}^A(t) = \mathcal{M}(t)\mathcal{E}, \\ &\mathcal{M}(t) = -\Phi_1(t)\int_0^t \Phi_1^{-1}(s)(\mathcal{B}+\mathcal{F})\mathcal{B}_0(s)ds. \end{aligned} \tag{19}$$

C. The Deviations of EM from EC

Represent the mean field trajectories under EM as $\mathcal{X}^M(t) = (z_1^T(t), ..., z_M^T(t))^T$, the deviation of EM from EC as $\Delta\mathcal{X}^M = (\Delta z_1^T(t), ..., \Delta z_M^T(t))$. According to (15),

$$\begin{aligned} &d\Delta\mathcal{X}^M(t) = (\mathcal{A}+\mathcal{C}-(\mathcal{B}+\mathcal{F})\mathcal{R}^{-1}\mathcal{B}^T P(t))\Delta\mathcal{X}^M(t)dt, \\ &\Delta\mathcal{X}^M(0) = \mathcal{M}(t_0)\mathcal{E}. \end{aligned}$$

Then according to theorem 5.1 and theorem 5.2, we have the following conclusion

Theorem 5.3 $(\Delta\mathcal{X}^M(t))_{t_0\le t\le T}$ has a *linear relationship* with $\mathcal{E}$, and this linear relationship can be computed by all agents without knowing $\mathcal{E}$, which is

$$\Delta\mathcal{X}^M(t) = \Phi(t)\Phi(t_0)^{-1}\mathcal{M}(t_0)\mathcal{E}. \tag{20}$$

D. The Affection of Re-Game Time t_0 on EM

Consider the situation where $t_0 \to 0$. Applying the continuity of $P(t), \Phi_1(t)$ and $\Phi(t)$, we have $\lim_{t_0\to 0}\mathcal{M}(t_0) = 0$. Then applying the conclusion of theorem 5.3 and the boundedness of $\Phi_1(t)$ over interval $t \in [0, T]$, we have

$$\lim_{t_0\to 0}\Delta\mathcal{X}^M(t) = 0, \lim_{t_0\to 0} EM = EC.$$

VI. SIMULATIONS

A. Model Formulations

We set $M = 2$, $N_1 = N_2 = 100$, two populations $\mathcal{P}_1$ get $z_1^1(0), z_1^2(0)$, $\mathcal{P}_2$ get $z_2^1(0), z_2^2(0)$ and coefficients of the model at $t = 0$, where $z_1^1(0), z_2^2(0)$ are correct initial mean field states , while $z_1^2(0), z_2^1(0)$ are wrong mean field states of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

We consider the scenario of opinion evolution[21], [22]. Let's assume that a community consists of two sub-communities $\mathcal{P}_1$ and $\mathcal{P}_2$. Each agent $\mathcal{A}_{im}, 1 \le i \le N_m, 1 \le m \le M$ has a two-dimensional state, which represents satisfaction with event 1 and event 2, respectively, and its evolution is influenced by the average opinion of the whole community. Let $x_{im}(t) = (x_{im}^1(t), x_{im}^2(t))^T$, we use $x_{im}^2(t)/(x_{im}^1(t)+x_{im}^2(t))$ to represent $\mathcal{A}_{im}$'s opinion inclination on the 2nd event. Agents in different sub-communities have different ideal opinions s_1, s_2 and initial mean field states $z_1(0), z_2(0)$. The initial distribution of $\mathcal{P}_m$ is a normal distribution with $z_m(0)$ as the expectation and $0.001 I_{2\times 2}$ as the covariance matrix.

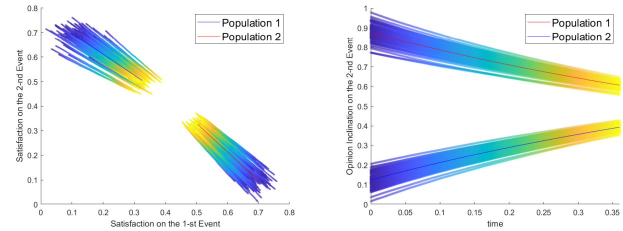

Fig. 1. Opinion evolution under correct information.

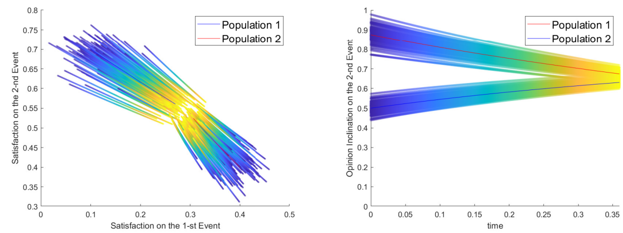

Fig. 2. Opinion evolution in population 1's prediction.

The dynamics and cost functions of $\mathcal{A}_{im}$ is given by (1) and (2), where

$$\begin{aligned} &C_m^n = 0.5I_{2\times 2}, A_m = -I_{2\times 2}, B_m = F_m = 0.5I_{2\times 2}, \\ &D_m = 0.01\sqrt{0.1}I_{2\times 2}, R_m = I_{2\times 2}, \\ &Q_{Im} = I_{2\times 2} = \bar{Q}_{Im}, Q_m^n = I_{2\times 2} = \bar{Q}_m^n, \\ &\Gamma_m^n = I_{2\times 2} = \bar{\Gamma}_m^n, \eta_m^n = 0 = \bar{\eta}_m^n, z_1(0) = (0.1, 0.7)^T, \\ &z_2(0) = (0.7, 0.1)^T, s_1 = (1, 0)^T, s_2 = (0, 1)^T. \end{aligned}$$

At $t = 0$, agents in sub-community $\mathcal{P}_m$ get the correct mean field state $z_m(0)$ of their own sub-community and the erroneous mean field state $z_m^n(0)$ of the other sub-community, and give their strategies $(u_{im}(t))_{0\le t\le T}, 1 \le i \le N_m$ for $t \in [0, T]$.

B. EC, EPn, EA, EM Situations

Let $z_1^2(0) = (0.4, 0.4)^T, z_2^1(0) = (0.4, 0.4)^T$. We simulate opinion evolutions in EC, EP1, EP2, EA, EM. The evolutionary of satisfaction on two events and opinion inclination are shown in Fig.1, Fig.2, Fig.3, Fig.4, Fig.5, respectively. The color ranges from blue to yellow, corresponding to the time from 0 to T.

It can be seen that the wrong initial mean field states of the other population leads to a significant difference among EC, EP1, EP2, EA and EM. We compare the average opinion inclinations under these situations in Fig.6.

C. Deviations

We set $z_1^2(0) - z_2(0) = k(-0.1, 0.1)^T, z_2^1(0) - z_1(0) = k(0.1, -0.1)^T$, then $\mathcal{X}_1^E(0) = k(0, 0, -0.1, 0.1)^T, \mathcal{X}_2^E(0) = k(0.1, -0.1, 0, 0)^T$. We compare the deviations $\mathcal{X}_1^E(t)$ with different k in Fig.7. The deviations between EC and EA, the deviations between EC and EM with different k are compared in Fig.8. It can be shown that the deviations have a linear relationship with k, which verifies the conclusions mentioned in Section 5.

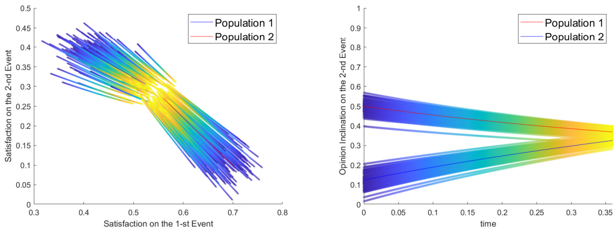

Fig. 3. Opinion evolution in population 2's prediction.

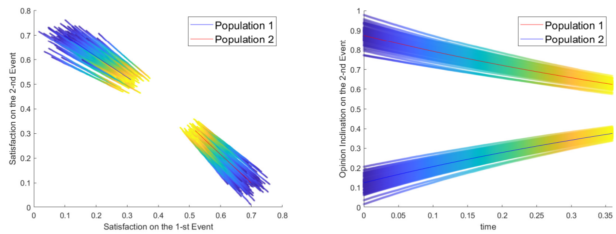

Fig. 4. Actual opinion evolution.

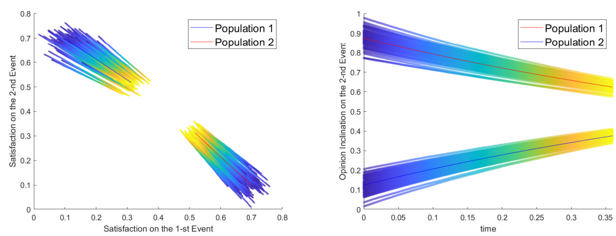

Fig. 5. Actual opinion evolution when re-game at $t_0 = 0.045$.

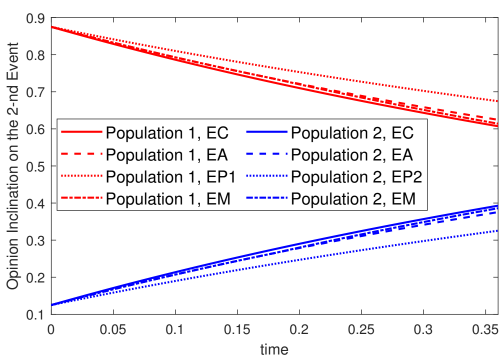

Fig. 6. Average opinion inclinations in different situations.

The deviations between EC and EM with different t_0 and the average opinion inclinations with and without re-game are shown in Fig.9. It can be seen that agents change their strategies at t_0, and the average opinion inclinations have an obvious deviation from that without re-game. The deviations have a linear relationship with k, and tend to disappear as t_0 decreases, which verifies the conclusions mentioned in Section 5.

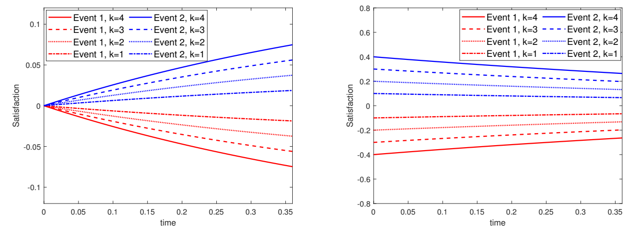

Fig. 7. From left to right, above figures show the deviations between the average opinion of population 1 under EP1 and that under EC, the deviations between the average opinion of population 2 under EP1 and that under EC.

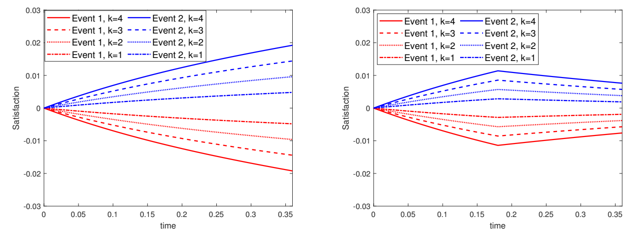

Fig. 8. From left to right, above figures show the deviations between EA and EC, EM and EC.

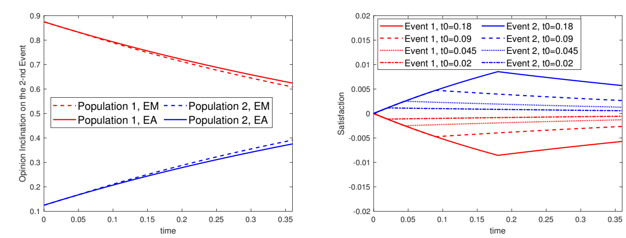

Fig. 9. From left to right, above figures show the opinion evolution with re-game at $t_0 = 0.045$ and the deviations between EM an EC with different k.

D. Results

In this section, we simulate a two-population MFG under the influence of erroneous initial mean field states information in the opinion evolution scenario. First, we simulate the opinion evolution under erroneous initial mean field states information.Then we compare EP1, EP2, EC, EM and EM, and verify the linear relationship between deviations and initial error mentioned in section 5. Finally, we simulate the opinion evolution with re-game at $t_0 = 0.045$, compare it to that without re-game, and compare the deviations between EM and EC with different t_0, observing a clear difference between the two situations and the affection of re-game time.

VII. CONCLUSIONS

We discuss the initial error affection and strategy modification in MPLQMFGs under erroneous initial distribution information are investigated. Our work presents a method of inferring unknown information using known partial information and use it to find optimal controls in MPLQMFGs. In future work, we will consider more general mean field games and more situations of erroneous parameters, as well as situations where different populations modify their strategies at different moment and for many times.

APPENDIX

Proof of Theorem 2.1: Consider a perturbation of the optimal control $u_{im}(t) = \hat{u}_{im}(t) + \theta\tilde{u}_{im}(t)$, then we have $x_{im}(t) = \hat{x}_{im}(t) + \theta\tilde{x}_{im}(t)$. $\tilde{x}_{im}(t)$ satisfies

$$\begin{aligned} d\tilde{x}_{im}(t) &= [A_m\tilde{x}_{im}(t) + B_m\tilde{u}_{im}(t)]dt \\ \tilde{x}_{im}(0) &= 0 \end{aligned} \tag{21}$$

According to Euler condition, the optimal control $\hat{u}_{im}(t)$ satisfies

$$0 = \frac{dJ(\hat{u}_{im}(t) + \theta\tilde{u}_{im}(t))}{d\theta}|_{\theta=0}$$
$$= \mathbb{E}\{\int_0^T [\tilde{x}_{im}^T(t)Q_{Im}(\hat{x}_{im}(t) - s_m) + \tilde{u}_{im}^T(t)R_m\hat{u}_{im}(t)+$$
$$\Sigma_{n=1}^M\tilde{x}_{im}^T(t)Q_m^n(\hat{x}_{im}(t) - (\Gamma_m^n z_n(t) + \eta_m^n))]dt + \tilde{x}_{im}^T(T)$$
$$\bar{Q}_{Im}(\hat{x}_{im}(T) - \bar{s}_m) + \Sigma_{n=1}^M\tilde{x}_{im}^T(T)\bar{Q}_m^n(\hat{x}_{im}(T)-$$
$$(\bar{\Gamma}_m^n z_n(T) + \bar{\eta}_m^n))\}$$

Then

$$d(p_{im}^T(t)\tilde{x}_{im}(t)) = d(p_{im}^T(t))\tilde{x}_{im}(t) + p_{im}^T(t)d(\tilde{x}_{im}(t))$$
$$= -[\tilde{x}_{im}^T(t)Q_{Im}(x_{im}(t) - s_m) - \tilde{u}_{im}^T(t)B_m^T p_{im}(t)+$$
$$\Sigma_{n=1}^M\tilde{x}_{im}^T(t)Q_m^n(x_{im}(t) - (\Gamma_m^n z_n(t) + \eta_m^n))]dt$$

When $\theta = 0$, $\hat{x}_{im}(t) = x_{im}(t), \hat{u}_{im}(t) = u_{im}(t)$. By integrating on $[0, T]$ and taking expectation on both sides of the above equation

$$\mathbb{E}[\int_0^T \tilde{u}_{im}^T(t)B_m^T\mathbb{E}[p_{im}(t)|\mathcal{F}_t^{im}] + \tilde{u}_{im}^T(t)R_m u_{im}(t)dt] = 0.$$

Due to the arbitrariness of $\tilde{u}_{im}^T(t)$, we get $u_{im}(t) = -R_m^{-1}B_m^T\mathbb{E}[p_{im}(t)|\mathcal{F}_t^{im}]$. □

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